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A note on a algorithm studying the uniform controllability of a class of semidiscrete hyperbolic problems

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Abstract. We propose an algorithm which is based on the the technique introduced in [23]. The aim of the algorithm is to study, in a simple way, the approximation of the controls for a class of hyperbolic problems. It is well-known that, the finite-difference semi-discrete scheme for the approximation of controls can leads to high frequency numerical spurious oscillations which gives a loss of the uniform (with respect to the mesh-size) controllability property of the semi-discrete model. It is also known that an appropriate filtration of the high eigenfrequencies of the discrete initial data enable us to restore the uniform controllability property of the whole solution. But, the methods used to prove such results are very constructive and use difficult and fine computations. As an example, which proves the effectiveness of our algorithm, we consider the case of the semidiscrete one dimensional wave equation. In this particular case, we are able to prove the uniform controllability, where the initial data are filtered in a range which contains as many modes as possibles, taking into account previous results obtained in literature (see [18]).

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1. Introduction

In the las decades the approximation of controls of PDE's becomes and important field of research, due to the necessity of understanding and implementing smart devices for controlling phenomena in real life which are described by hyperbolic equations. Unfortunately, even if we consider the simplest numerical approximation of such equations, we cannot assure the convergence of the approximate controls. This lack of convergence is motivated by of the non-uniform (with respect to the mesh-size) controllability property of the semi-discrete model when we are dealing with natural settings initial data.

This is the case of the finite-difference semi-discrete scheme for the approximation of boundary controls of the one-dimensional wave equation. It is proved that in this case we have high frequency numerical spurious oscillations which lead to a loss of the uniform (with respect to the mesh-size) controllability property of the semi-discrete model. In [18, 22] it is considered an appropriate filtration of the high eigenfrequencies of the discrete initial data and in this case the uniform controllability property of the whole solution is obtained. The strategy for restoring the uniform controllability property starting with the equivalence between a control problem and a problem of moments. The next step consist of constructing explicit solutions for the problem of

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moments by using biorthogonal sequences. The last step is to define and evaluate a sequence of discrete controls in order to prove the convergence to a control of the continuous problem. Similar problems and more details can be found in [1, 16, 28, 29, 30].

Moreover, we emphasise that another important purpose it is also to allow filtrations which contain as many modes as possible which will consistently help to improve the precision of the approximation. In this context, we refer to [2, 12, 13, 16, 17, 31].

We starts by recalling the well-known problem which study the boundary exact controllability property for the one-dimensional (1-D) linear wave equation: for any $T \geq 2$ and $(w^0, w^1) \in L^2(0,1) \times H^{-1}(0,1)$ we can assure the existence of a control function $u \in L^2(0,T)$ such that the solution of the wave equation

$$\begin{cases} w_{tt}(t,x) - w_{xx}(t,x) = 0 & x \in (0,1), t > 0, \\ w(t,0) = 0, w(t,1) = u(t) & t > 0, \\ w(0,x) = w^0(x), w'(0,x) = w^1(x) & x \in (0,1), \end{cases} \tag{1}$$

verifies

$$w(T,\cdot) = w'(T,\cdot) = 0.$$

Let $N \in \mathbb{N}^*$ and $h = \frac{1}{N+1}$. For any $T > 0$, we consider the following semi-discrete space approximation of the wave equation given by

$$\begin{cases} w_j'(t) - \frac{w_{j+1}(t) - 2w_j(t) + w_{j-1}(t)}{h^2} = 0 & 1 \leq j \leq N,\ t > 0, \\ w_0(t) = 0 & t \in (0,T), \\ w_{N+1}(t) = u_h(t) & t \in (0,T), \\ w_j(0) = w_j^0, \quad w_j'(0) = w_j^1 & 1 \leq j \leq N. \end{cases} \tag{2}$$

Note that, (2) represents a system of N linear equations with N unknowns $w_1, w_2, \ldots w_N$. Since the quantity $w_j(t)$ approximates $w(t, x_j)$, the solution of (1) at time t and in the point $x_j = jh$, for each $j \in \{0, 1, ..., N+1\}$, we shall choose the classical discretization by points

$$w_j^0 = w^0(jh), \quad w_j^1 = w^1(jh) \qquad (1 \leq j \leq N). \tag{3}$$

Given $T \geq 2$, $h > 0$ and $((w_j^0, w_j^1))_{1 \leq j \leq N} \in \mathbb{C}^{2N}$, it is also well known that there exists a control function $u_h \in C^0([0,T])$ such that the solution of the equation (2) verifies

$$w_j(T) = w_j'(T) = 0 \qquad (1 \leq j \leq N).$$

Our aim is to simplify the study of the existence of a uniformly bounded sequence of controls $(u_h)_{h>0}$ (with respect to the mesh size h) for the problem (2) and then to prove that this family (or a subfamily) converges to a control of the continuous problem (1). Moreover, our general aim is to construct an algorithm which can be used for many other similar problems.

We recall that the discretization of the wave equation (with finite-differences schemes but also finite-element schemes) is known for a long time (see [14] and [15]) to lead to high-frequency spurious solutions generated by the discretization process that make the discrete controls diverge when the mesh-size goes to zero.

In order to restore the uniform controllability property we refer to different strategies proposed in literature:

- a Tychonov regularization of the HUM cost functional (see [14, 31]);

- a change of the numerical scheme (mixed finite elements [5], vanishing viscosity [3, 20, 24], different types of finite difference schemes [27]), non-uniform meshes ([10, 11]) and an approximation of discrete controls [7].

- a filtering technique (see [20]) which are successfully used in [6, 18, 19, 25] for treating the cases of wave or beam equation, having the strategy to consider the control acting only on low-frequency part of the solution or, finally, to cut the furious frequencies (range of filtration) of the initial conditions.

This last possibility is considered in the present paper. Note that, in our case, even if the initial condition is filtered, the control will excite all frequencies and this creates a lot of technical difficulties. For more details about filtered spaces and resolvent estimates, the interested reader is referred [8, 9, 26].

The main result of our paper is to introduce an algorithm which simplify the methodology used to study the existence of a uniformly bounded sequence of controls $(u_h)_{h>0}$ (with respect to the mesh size h) of problems which can be described as in (an example is given in (2)) which converges to a control of the continuous problem of the general form (8) (an example is given in (1)).

Our approach is based on the technique introduced in [23], when an appropriate filtration of the high eigenfrequencies of the discrete initial data enable us to restore the uniform controllability property of the whole solution.

Algorithm1. Let us consider two operators $\mathcal{A}_h$ and $\mathcal{A}$, which have the property that the families of eigenvalues are purely imaginary, i. e. have the form $(i\lambda_{hn})_{1\leq|n|\leq N}$, respectively $(i\lambda_n)_{n\in\mathbb{Z}^*}$. The steps of the algorithm are presented in the following sentences.

(1) We define a Weierstrass product P_m such that

$$P_m(\lambda_{hn}) = \delta_{mn} \qquad (1 \leq |m|,\, |n| \leq N);$$

(2) We evaluate the Weierstrass product P_m on the real axis, i. e. we get an increasing and one to one function $\Phi_{hm} : [0,\infty) \to [0,\infty)$ such that

$$|P_m(x)| \leq \exp(\Phi_{hm}(x)) \qquad (x \in \mathbb{R});$$

(3) We are looking for a range N_{hm} such that

$$\int_{\phi_{hm}^{-1}(N_{hm})}^{\infty} \frac{\phi'_{hm}(t)}{t-\lambda_{hm}}\, dt < \infty \ \text{ and } \ \phi_{hm}^{-1}(N_{hm}) \geq |\lambda_{hm}|, \ \ (1 \leq |m| \leq N);$$

(4) Let $M = \max\{1 \leq m \leq N |\ N_{hm} = \mathcal{O}(1)\}$ be the range of filtration. Then, there exists $T_0 > 0$ such that for any $T > T_0$ and any initial data filtered in the range M there exists a uniformly bounded sequence of controls $(u_h)_h$ in $L^2(0,T)$ for problem of type (16) (see the case of problem (2));

(5) The family $(u_h)_h$ has a subfamily which is weakly convergent to a control $u \in L^2(0,T)$ for the continuous problem of type (8) (see the case of problem (1)).

Remark 1. Note that, in the above algorithm we need only to get accurate estimates for the Weierstrass product P_m in order to find the function Ψ_{hm}. Then, the choose of the two ranges N_{hm} and M has a clear statement, for which the uniform controllability property holds. We have considered only the case of problems of type (16) and (8), for which the associated operators $\mathcal{A}_h$ and $\mathcal{A}$ have purely imaginary eigenvalues of the form $(i\lambda_{hn})_{1\leq|n|\leq N}$, respectively $(i\lambda_n)_{n\in\mathbb{Z}^*}$.

We mention also that the choice of an appropriate approximation $((w_j^0, w_j^1))_{1\leq j\leq N}$ for the initial datum (w_0, w_1) is very important if we want to ensure the existence of a bounded sequence of controls. This is why we consider in the convergence result a filtered version of the initial condition (w^0, w^1) given by

$$(w_M^0, w_M^1) = \sum_{|n|\leqslant M, n\neq 0} a_n \Phi^n(x), \tag{4}$$

where $M \in \mathbb{N}^*$ (M will depend on the mesh size h in what follows) and Φ^n are the eigenvectors of an operator $\mathcal{A}$ (see for example (10)).

On the other hand, we say nothing about a multiplier M_m, since we apply the methodology used in [23] in the conditions imposed at Step 3, we can guarantee the existence of a multiplier, i.e. $M_{hm} : \mathbb{C} \to \mathbb{C}$ an entire function of exponential type independent of m and h, which belongs to L^2 on the real axis and verifiy

$$M_{hm}(\lambda_{hm}) = 1,\ |M_{hm}(x)| \leq \exp\left(-\phi_{hm}(x) + N_{hm} + 1\right) \qquad (x \in \mathbb{R},\ 1 \leq |m| \leq N).$$

The structure of the paper is as follows: Section 2 is devoted to some preliminaries on continuous and discrete moment problems and spectral analysis concerning the controllability of the controllability and discrete wave equation. Section 3 is devoted to the construction and evaluation of a biorthogonal sequence of the family of exponential functions. Section 4 is devoted to the uniform controllability and convergence results. Finally, Section 5 is devoted to some conclusions concerning the algorithm for constructing uniformly bounded sequences for families of exponential functions with no real parts and other further extensions.

2. Preliminaries on spectral properties and moment problems

In this section we introduce some notations and we recall some well-known properties concerning the boundary null-controllability problem for the linear wave continuous and discrete equations. In this sense, following the technique used in [22] and [23] we transform the controllability problems into a equivalent problems, via variational results (see Lemma 2.1).

2.1. The continuous moment problem. We start this section by presenting the continous problem and how spectral estimates are used to obtain the moment problem. To do that we need the following variational result from [21, 22].

Lemma 2.1. *Let $T > 0$ and $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in L^2(0,1) \times H^{-1}(0,1)$. The function $u \in L^2(0,T)$ is a control for (1) in time T if and only if, we have*

$$\int_0^T u(t)\overline{v_x}(t,1)dt = -\int_0^1 w^0(x)\overline{v_t}(0,x)dx + \langle w^1, v(0,\cdot)\rangle_{H^{-1}\times H_0^1}, \tag{5}$$

for every $\begin{pmatrix} v^0 \\ v^1 \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$, where $\begin{pmatrix} v \\ v_t \end{pmatrix} \in H_0^1(0,1) \times L^2(0,1)$ is the solution of the adjoint backward problem

$$\begin{cases} v_{tt}(t,x) - v_{xx}(t,x) = 0 & t \in (0,T),\ x \in (0,1) \\ v(t,0) = v(t,1) = 0 & t \in (0,T) \\ v(T,x) = v^0(x) & x \in (0,1) \\ v_t(T,x) = v^1(x) & x \in (0,1), \end{cases} \tag{6}$$

and $\langle \cdot, \cdot \rangle_{H^{-1},H_0^1}$ *denotes the duality product between the spaces* $H^{-1}(0,1)$ *and* $H_0^1(0,1)$*.*

Let $L : \mathcal{D}(L) \to L^2(0,1)$ be the unbounded operator in $L^2(0,1)$ (with the properties that $(\mathcal{D}(L), L)$ is maximal monotone) defined by

$$\begin{array}{l} \mathcal{D}(L) = H^2(0,1) \cap H_0^1(0,1), \\ Lu = -u_{xx}, \qquad (u \in \mathcal{D}(L)). \end{array} \tag{7}$$

Note that L is a skew-adjoint operator in $L^2(0,1)$ with compact resolvent where that the eigenvalues and the corresponding eigenfunctions of L are given by

$$\nu_n = n^2\pi^2, \qquad \varphi^n = \sqrt{2}\sin(n\pi x) \qquad (n \in \mathbb{N}^*),$$

and the eigenfunctions $(\varphi^n)_{n\in\mathbb{N}^*}$ form an orthonormal basis in $L^2(0,1)$.

By denoting $Z = \begin{pmatrix} v \\ v_t \end{pmatrix}$, (6) is equivalent with

$$\begin{cases} Z_t(t) + \mathcal{A}Z(t) = 0 & t \in (0,T) \\ Z(T) = Z^0 = \begin{pmatrix} v^0 \\ v^1 \end{pmatrix}, \end{cases} \tag{8}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H_0^1(0,1) \times L^2(0,1)$ is the skew adjoint operator defined by

$$\begin{array}{l} \mathcal{D}(\mathcal{A}) = H^2(0,1) \cap H_0^1(0,1) \times H_0^1(0,1), \\ \mathcal{A} = \begin{pmatrix} 0 & -1 \\ -\partial_{xx} & 0 \end{pmatrix}. \end{array} \tag{9}$$

The main spectral properties of the operator $(D(\mathcal{A}), \mathcal{A})$ are described in the following sentences.

Lemma 2.2. *The eigenvalues of the operator* $(D(\mathcal{A}), \mathcal{A})$ *are given by* $(i\lambda_n)_{n\in\mathbb{Z}^*}$*, where*

$$\lambda_n = n\pi \quad (n \in \mathbb{Z}^*), \tag{10}$$

and the corresponding eigenfunctions are given by

$$\Phi^n = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i\lambda_n \end{pmatrix} \varphi^{|n|} \qquad (n \in \mathbb{Z}^*). \tag{11}$$

Moreover, the families $(\Phi^n)_{n\in\mathbb{Z}^*}$ *and* $\left(\frac{1}{\sqrt{\nu_{|n|}}}\Phi^n\right)_{n\in\mathbb{Z}^*}$ *form an orthonormal basis in* $L^2(0,1) \times H^{-1}(0,1)$ *and* $H_0^1(0,1) \times L^2(0,1)$*, respectively.*

The following result gives us the moment problem associated with (1), as is presented in [22].

Theorem 2.3. *Problem* (1) *is null-controllable in time* $T > 2$ *if and only if, for any initial data* $\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} \in L^2(0,1) \times H^{-1}(0,1)$ *with the Fourier expansion*

$$\begin{pmatrix} w^0 \\ w^1 \end{pmatrix} = \sum_{n\in\mathbb{Z}^*} \beta_n^0 \Phi^n, \tag{12}$$

there exists $u \in L^2(0,T)$ *such that*

$$\int_0^T u(t)e^{-i\lambda_n t}dt = \frac{(-1)^{|n|+1}}{\sqrt{\nu_{|n|}}}\lambda_n \beta_n^0 \qquad (n \in \mathbb{Z}^*). \tag{13}$$

2.2. The discrete moment problem. In a similar way, as in the continuous case, by considering the semi-discrete space approximations of system (1), using again discrete spectral properties, we obtain the corresponding moment problem. In this section and the remaining part of the paper we study the semi-discrete control problem (2).

In the following sentences we recall some well known facts about the spectral properties of our problem. Let us consider the corresponding homogeneous adjoint problem

$$\begin{cases} v_j''(t) - \frac{v_{j+1}(t)-2v_j(t)+v_{j-1}(t)}{h^2} = 0 & 1 \le j \le N,\ t>0, \\ v_0(t) = 0 & t \in (0,T), \\ v_{N+1}(t) = 0 & t \in (0,T), \\ v_j'(T) = v_j^0, \quad v_j'(T) = v_j^1 & 1 \le j \le N. \end{cases} \tag{14}$$

We define the matrix $A_h \in \mathcal{M}_{N\times N}(\mathbb{R})$ as follows:

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 2 & -1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The adjoint problem (14) can be rewritten in a matricial form as follows:

$$\begin{cases} V_h''(t) + A_h V_h(t) = 0,\ t>0, \\ V_h(T) = V_h^0, \quad V_h'(T) = V_h^1, \end{cases} \tag{15}$$

where $V_h(t) = (v_1(t), ..., v_N(t))^T \in \mathbb{C}^N$ and the initial data is

$$\begin{pmatrix} V_h^0 \\ V_h^1 \end{pmatrix} = \begin{pmatrix} (v_j^0)_{1\le j\le N} \\ (v_j^1)_{1\le j\le N} \end{pmatrix} \in \mathbb{C}^{2N}.$$

Now, if we set $Z_h(t) = \begin{pmatrix} V_h(t) \\ V_h'(t) \end{pmatrix}$ and $Z_h^0 = \begin{pmatrix} V_h^0 \\ V_h^1 \end{pmatrix}$, then (15) has the following equivalent vectorial form

$$\begin{cases} Z_h'(t) + \mathcal{A}_h Z_h(t) = 0,\ t>0, \\ Z_h(T) = Z_h^0, \end{cases} \tag{16}$$

where the operator $\mathcal{A}_h$ is given by $\mathcal{A}_h = \begin{pmatrix} 0 & -I_N \\ A_h & 0 \end{pmatrix}$ and I_N is the identity matrix of size N. It is known that the eigenvalues of $\mathcal{A}_h$ are given by the family $(i\,\lambda_{hn})_{1\le|n|\le N}$, where

$$\lambda_{hn} = \frac{2}{h}\sin\left(\frac{n\pi h}{2}\right) \qquad (1 \le |n| \le N), \tag{17}$$

and the corresponding eigenvectors $(\Phi_h^n)_{1\le|n|\le N}$ which forms an orthonormal basis in $\mathbb{C}^{2N}$ are given by

$$\Phi_h^n = \begin{pmatrix} \frac{1}{i\lambda_{hn}}\varphi_h^n \\ -\varphi_h^n \end{pmatrix} \qquad (1 \le |n| \le N), \tag{18}$$

where

$$(\varphi_h^n)_{1\le|n|\le N} = \begin{pmatrix} \sin(n\pi h) \\ \sin(2n\pi h) \\ \dots \\ \sin(n\pi h N) \end{pmatrix} \in \mathbb{C}^N$$

are the eigenvectors of A_h.

Based on [18, 22] we transform our controllability problem into a moment problem.

Proposition 2.4. *For any $T > 0$ system (2) is null-controllable at time T if, and only if, for any initial data $(w_j^0, w_j^1)_{1\le j\le N} = \sum_{1\le|n|\le N} a_n \Phi_h^n$ there exists $u_h \in C^0([0,T],\mathbb{C})$ which verifies*

$$\int_0^T u_h(t)\, e^{-i\lambda_{hn}t}\, dt = \frac{(-1)^n h}{\sin(n\pi h)} a_n, \quad (1 \le |n| \le N). \tag{19}$$

We recall that a sequence $(\theta_m)_{1\le|m|\le N} \subset L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ is *biorthogonal to the family of exponential functions* $\left(e^{\lambda_{hn}t}\right)_{1\le|n|\le N}$ *in* $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_m(t) e^{\overline{\lambda_{hn}}t} dt = \delta_{mn} \qquad (1 \le |m|, |n| \le N)\,. \tag{20}$$

We consider the following filtered version of the initial condition (w^0, w^1) given by

$$(w_M^0, w_M^1) \;=\; \sum_{|n|\leqslant M, n\neq 0} a_n \Phi^n(x), \tag{21}$$

where $M \in \mathbb{N}^*$ (M will depend on the mesh size h in what follows).

We mention also that the choice of an appropriate approximation $((w_j^0, w_j^1))_{1\le j\le N}$ for the initial data (w_M^0, w_M^1) is important in order to ensure the existence of a bounded sequence of controls. The next step is to chose the initial data of the semi-discrete problem (2) as an approximation of $(w_M^0, w_M^1))$ in such way to ensure the boundedness of the sequence of controls $(u_h)_{h>0}$.

Hence, we consider as initial data for the semi-discrete equation (2) of the form

$$((w_j^0, w_j^1))_{1\le j\le N} = \sum_{1\le|n|\le M} a_n(h)\Phi_h^n, \tag{22}$$

where $a_n(h)$ is related to a_n by the following relations (see [20, Page 758]):

$$a_n(h) := \begin{cases} 0, & |n| > M, \\ \frac{1}{2}\left(\frac{\lambda_{hn}}{n\pi} + 1\right) a_n + \frac{1}{2}\left(\frac{\lambda_{hn}}{n\pi} - 1\right) a_{-n}, & |n| \leqslant M. \end{cases} \tag{23}$$

Taking into account the above ideas, if $(\theta_m)_{1\le|m|\le N}$ is a biorthogonal sequence to the family of exponential functions $(e^{i\lambda_{hn}t})_{1\le|n|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ we can construct a control thanks to the following formula (see [22, Page 758]):

$$u_h(t) = \sum_{1\le|n|\le M} \frac{(-1)^{n+1}h}{\sin(n\pi h)} e^{-i\lambda_{hn}\frac{T}{2}} a_n(h)\theta_n\left(t - \frac{T}{2}\right). \tag{24}$$

3. The construction and estimates of the sequence of biorthogonals

In this section we construct and evaluate a biorthogonal sequence $(\theta_m)_{1\le|m|\le N}$ to the family $\left(e^{i\lambda_{hn}t}\right)_{1\le|n|\le N}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$. Then we are able to estimate the norm of v_h from (24). In order to do that, we proceed as follows:

- we define a family $(\Psi_{hm})_{1\le|m|\le N}$ which verifies:
  - $\Psi_{hm}(\lambda_{hn})=\delta_{mn}$;
  - are entire functions of exponential type in L^2 on the real axis;
- Ψ_{hm} is obtained as a product between a Weierstrass product P_m and a multiplier M_{hm} (see the proof of Theorem 3.4 below);
- P_m is defined in the simplest way to get a function with zeros on each eigenvalue λ_{hn}.
- M_{hm} is introduced in [18, Proposition 2.2.], in order to ensure the L^2 property of Ψ_{hm} on the real axis;
- The inverse Fourier transforms of $(\Psi_{hm})_{1\le|m|\le N}$ and Paley-Wiener's Theorem gives us the possibility to define a biorthogonal family $(\theta_{hm})_{1\le|m|\le N}$;
- Finally, Plancherel's Theorem allow to estimate the norms $\|\theta_{hm}\|_{L^2(\mathbb{R})}$.

Firstly, we define the Weierstrass product P_m, with the property that $P_m(\lambda_{hn})=\delta_{mn}$ and we recall the estimate of the product P_m on the real axis obtained in [18]. In the sequel, $C>0$ denotes an absolute constant (independent on N and h).

For every $1\le|m|\le N$, we define the function

$$P_m(z)=\prod_{\substack{1\le|n|\le N\\ n\ne m}}\left(\frac{z}{\lambda_{hn}}-1\right)\prod_{\substack{1\le|n|\le N\\ n\ne m}}\frac{\lambda_{hn}}{\lambda_{hm}-\lambda_{hn}}\qquad(z\in\mathbb{C}).\tag{25}$$

The exponential type of P_m is treated to [18, Lemma 2.4 and proof of Proposition 2.2], hence we have a constant $L_0>0$ which quantify the exponential type. Moreover, concerning the behavior of the product P_m on the real axis, we have an optimal estimate in [18, Proposition 2.2.] and it is shown that for every $x\in\mathbb{R}$, there exists a constant $C>0$ such that for any $1\le|m|\le N$, we have that

$$|P_m(x)|\le C\exp\left(\phi(x)\right)\qquad(x\in\mathbb{R}),\tag{26}$$

where

$$\phi(x)=\begin{cases}0 & \left(|x|<\frac{2}{h}\right)\\ \frac{2}{h}\ln\left(\frac{|x|h}{2}+\sqrt{\frac{x^2h^2}{4}-1}\right) & \left(|x|\ge\frac{2}{h}\right).\end{cases}\tag{27}$$

Secondly, we introduce another entire function, called multiplier, with rapid decay on the real axis to compensate the grow of the product P_m on the real axis. We use the technique introduced in [23]. To begin with, let us consider $\varphi:[0,\infty)\to[0,\infty)$ a continuous, increasing and onto function. We define the following sequence of real numbers $(a_n)_{n\ge1}$ by

$$\varphi(ea_n)=n\qquad(n\ge1).\tag{28}$$

We recall a useful result from [23] (for more details, see [25, Lemma 4.4]).

Lemma 3.1. *Let $0\le y<x$. Then*

$$\sum_{j=A(y)+1}^{A(x)}\ln\left(\frac{a_j}{x}\right)=-\int_{a_{A(y)}}^{x}\frac{A(u)-A(y)}{u}du,\tag{29}$$

where $A(u) = \#\{a_n \leq u\} = [\varphi(eu)]$ *for any* $u \geq 0$.

For $1 \leq m \leq N$, let us consider the function

$$\phi_{hm}(x) = \begin{cases} 0 & \text{if } 0 < x \leq \frac{2}{h}, \\ \frac{2}{h} \ln\left(\frac{xh}{2} + \sqrt{\frac{x^2h^2}{4} - 1}\right) & \text{if } \frac{2}{h} < x, \end{cases}$$

and let $\varphi : [0, \infty) \to [0, \infty)$ be defined by $\varphi(x) = \phi_{hm}(x + \lambda_{hm})$. The function φ is continuous, increasing and onto and we construct the sequence of real numbers $(a_n)_{n\geq 1}$ by (28). The following technical result is useful to evaluate the type of our multiplier function in Proposition 3.3 below. Note that the symbol $\mathcal{O}$ denotes the Bachman-Landau notation.

Lemma 3.2. *Let* $\delta \in (0, 1)$, $1 \leq m \leq \delta N$ *and* $N_{hm} = \phi_{hm}(\frac{2}{h} + \mathcal{O}(h))$. *There exists a constant* $L > 0$, *which is independent of* h, *such that* $\displaystyle\sum_{n \geq N_{hm}+1} \frac{1}{a_n} \leq L < \infty$.

Proof. We remark that the sequence $(a_n)_{n\geq 1}$ verifies

$$\begin{aligned} \frac{1}{e} \sum_{n=N_{hm}+1}^{\infty} \frac{1}{a_n} &= \sum_{n=N_{hm}+1}^{\infty} \frac{1}{\phi_{hm}^{-1}(n) - \lambda_{hm}} \\ &\leq \int_{N_{hm}}^{\infty} \frac{\mathrm{d}t}{\phi_{hm}^{-1}(t) - \lambda_{hm}} = \int_{\phi_{hm}^{-1}(N_{hm})}^{\infty} \frac{\phi'_{hm}(t)}{t - \lambda_{hm}} \mathrm{d}t := I. \end{aligned}$$

Taking into account that $\phi_{hm}^{-1}(N_{hm}) = \frac{2}{h} + \mathcal{O}(h)$ it follows that

$$I = \int_{\phi_{hm}^{-1}(N_{hm})}^{\infty} \frac{1}{(t - \lambda_{hm})\sqrt{\frac{t^2h^2}{4} - 1}} \mathrm{d}t = \int_{\frac{2}{h}+\mathcal{O}(h)}^{\infty} \frac{1}{(t - \lambda_{hm})\sqrt{\frac{t^2h^2}{4} - 1}} \mathrm{d}t.$$

If we consider the change of variable given by $\sqrt{\frac{t^2h^2}{4} - 1} = u + \frac{th}{2}$, we infer that

$$I = \int_{-1+\mathcal{O}(h)}^{0} \frac{2}{u^2 + h\lambda_{hm}u + 1} \mathrm{d}u = \int_{-1+\mathcal{O}(h)}^{0} \frac{2}{\left(u + \frac{h\lambda_{hm}}{2}\right)^2 + \left(\sqrt{1 - \frac{h^2\lambda_{hm}^2}{4}}\right)^2} \mathrm{d}u$$

$$= \frac{1}{\sqrt{1 - \frac{h^2\lambda_{hm}^2}{4}}} \left(\arctan \frac{\frac{h\lambda_{hm}}{2}}{\sqrt{1 - \frac{h^2\lambda_{hm}^2}{4}}} - \arctan \frac{-1 + \mathcal{O}(h) + \frac{h\lambda_{hm}}{2}}{\sqrt{1 - \frac{h^2\lambda_{hm}^2}{4}}} \right) \leq \frac{\pi}{\sqrt{1 - \frac{h^2\lambda_{hm}^2}{4}}}.$$

Finally, since $\delta \in (0, 1)$ and $1 \leq m \leq \delta N$ we deduce that $L = \frac{e\pi}{\cos\left(\frac{m\pi h}{2}\right)} \leq \frac{e\pi}{\cos\left(\frac{\pi\delta}{2}\right)} < \infty$. □

The following result gives us the multiplier we need in order to construct the biorthogonal sequence.

Proposition 3.3. *Let* $\delta \in (0, 1)$. *For each* $1 \leq m \leq \delta N$ *there exists a function* $M_{hm} : \mathbb{C} \to \mathbb{C}$ *with the properties:*

(1) M_{hm} *is an entire function of exponential type independent of* m *and* h, *which belongs to* L^2 *on the real axis and verifies* $M_{hm}(\lambda_{hm}) = 1$.

(2) $|M_{hm}(x)| \leq \exp\left(-\phi_{hm}(x) + N_{hm} + 1\right)$ *for all* $x \in \mathbb{R}$, *where* N_{hm} *is given by*

$$N_{hm} = \left[\phi_{hm}\left(\frac{2}{h} + \mathcal{O}(h)\right)\right] + 1 = \mathcal{O}(1). \tag{30}$$

Proof. Using the same argument as in [23] we define $M_{hm} : \mathbb{C} \to \mathbb{C}$ as follows

$$M_{hm}(z) = \prod_{n=N_{hm}}^{\infty} \frac{\sin\left(\frac{z-\lambda_{hm}}{a_n}\right)}{\frac{z-\lambda_{hm}}{a_n}}, \tag{31}$$

where the sequence $(a_n)_{n\geq 1}$ is defined in (28).

Based on the following estimate

$$\prod_{n=N_{hm}}^{\infty} \left| \frac{\sin\left(\frac{z-\lambda_{hm}}{a_n}\right)}{\frac{z-\lambda_{hm}}{a_n}} \right| \leq e^{|z-\lambda_{hm}| \sum\limits_{n=N_{hm}}^{\infty} \frac{1}{a_n}} \leq e^{L|z-\lambda_{hm}|},$$

and using Lemma 3.2 we get that M_{hm} is an entire function of exponential type L. Moreover, M_{hm} belongs to $L^2(\mathbb{R})$ and $M_{hm}(\lambda_{hm}) = 1$.

In order to prove the second property of the function M_{hm}, using Lemma 3.1 we proceed exactly as in [23] and the proof is complete. □

3.1. The biorthogonal sequence. Taking into account the estimates from the previous sections we are able to construct a biorthogonal sequence $(\theta_m)_{1\leq|m|\leq N}$ to the family $\left(e^{i\lambda_{hn}t}\right)_{1\leq|n|\leq N}$.

Theorem 3.4. *There exist* $h_0, T_0 > 0$ *such that for any* $T > T_0$ *and* $h \in (0, h_0)$ *there exists a biorthogonal sequence* $(\theta_{hm})_{1\leq|m|\leq N}$ *to the family of exponentials* $\left(e^{i\lambda_{hn}t}\right)_{1\leq|n|\leq N}$ *in* $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ *and the following estimate holds*

$$\left\| \sum_{1\leq|m|\leq N} \beta_m \theta_{hm} \right\|^2_{L^2\left(-\frac{T}{2},\frac{T}{2}\right)} \leq C \sum_{1\leq|m|\leq N} |\beta_m|^2 e^{2N_{hm}}, \tag{32}$$

where C *is a positive constant independent of* m *and* h *and* $(\beta_m)_{1\leq|m|\leq N}$ *is any finite sequence.*

Proof. Firstly, we consider P_m and M_{hm} the functions from (25) and (31), respectively. In the case $m < 0$ we define $M_{hm}(z) = M_{-hm}(-z)$. Let $1 \leq |m| \leq N$ and let us consider the function

$$\Psi_{hm}(z) = P(z)M_{hm}(z)\frac{\sin\delta(z-\lambda_{hm})}{\delta(z-\lambda_{hm})} \quad (z \in \mathbb{C}). \tag{33}$$

Note that the function Ψ_{hm} verifies

- $\Psi_{hm}(\lambda_{hn}) = \delta_{nm} \qquad (1 \leq |n|, |m| \leq N)$;
- Ψ_{hm} is entire function of exponential type L, independent of h and m;
- $\Psi_{hm} \in L^2(\mathbb{R})$.

Let us consider the Fourier transform of Ψ_{hm} defined as

$$\zeta_{hm}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \Psi_{hm}(x)e^{ixt}dx, \tag{34}$$

and from Paley-Wienner Theorem it follows that $\zeta_{hm}(t)$ has compact support in $(-L, L)$, it belongs to $L^2(-L, L)$, and

$$\int_{-L}^{L} \zeta_{hm}(t)e^{i\lambda_{hm}t}dt = \Psi_{hm}(\lambda_{hm}) = \delta_{nm} \qquad (1 \le |n|, |m| \le N).$$

Hence $(\zeta_{hm})_{1\le|m|\le N}$ is a biorthogonal sequence to $\left(e^{i\lambda_{hn}t}\right)_{1\le|n|\le N}$ in $L^2(-L, L)$. Moreover, from Plancherel's Theorem we have that the following estimate holds

$$\|\zeta_{hm}\|_{L^2} \le Ce^{N_{hm}}. \tag{35}$$

Let $T_0 = 2(L + L_0)$ and $T > T_0$. The proof of the existence of a new biorthogonal in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ verifying (32) is similar to [24, Theorem 3.2]. Hence, using the same strategy we can construct easily a biorthogonal sequence $(\theta_{hm})_{m\in\mathbb{Z}^*}$ to the family $\left(e^{i\lambda_n t}\right)_{n\in\mathbb{Z}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ and (32) is proved. □

4. Uniform controllability results

The aim of this section is to show that a control for the continuous system (1) may be obtained as a limit of controls of the corresponding semidiscrete problems (2), when the initial data are filtered in the range δN, for any $\delta \in (0, 1)$. The main idea is to construct controls u_h by using the biorthogonal sequence $(\theta_m)_{1\le|m|\le N}$ from Theorem 3.4 and to evaluate their norms with the aid of estimate (32).

Using Proposition 3.3, for any for any $\delta \in (0, 1)$ and $1 \le m \le \delta N$ we can choose

$$N_{hm} = \left[\phi_{hm}\left(\frac{2}{h} + \mathcal{O}(h)\right)\right] + 1 = \mathcal{O}(1),$$

hence, we define the range of filtration as $M = \max\{1 \le m \le N |\ N_{hm} = \mathcal{O}(1)\} = \delta N$.

Let T_0 the constant appearing in Theorem 3.4.

Theorem 4.1. *Let $T > T_0$. There exists $h_0 > 0$ such that for any $((w_j^0, w_j^1))_{1\le j\le N} \in \mathbb{C}^{2N}$ of the form*

$$((w_j^0, w_j^1))_{1\le j\le N} = \sum_{1\le|n|\le M} a_n(h)\Phi_h^n, \tag{36}$$

with $(a_n(h))_{1\le|n|\le M}$ uniformly bounded in l^2, there exists a control $u_h \in L^2(0, T)$ for problem (2) such that the family $(u_h)_h$ verifies

$$\|u_h\|_{L^2(0,T)} \le C \qquad (0 < h < h_0), \tag{37}$$

where $C > 0$ is a constant independent of h.

Proof. Let $(\theta_n)_{1\le|n|\le M}$ be the biorthogonal given by Theorem 3.4. Let $u_h \in L^2(0, T)$ be the control given by Proposition 2.4 defined in (24). From (24) and the estimates for the norm of $(\theta_n)_{1\le|n|\le N}$ given by (32), we have

$$\int_0^T |u_h(t)|^2 dt = \int_0^T \left| \sum_{1\le|n|\le M} \frac{(-1)^{n+1}h}{\sin(n\pi h)} e^{-i\lambda_{hn}\frac{T}{2}} a_n(h)\theta_n\left(t - \frac{T}{2}\right)\right|^2 dt$$

$$\leq \sum_{1\leq|n|\leq M} \frac{h^2}{|\sin(n\pi h)|^2}|a_n(h)|^2 e^{2N_{hm}} \leq C \sum_{1\leq|n|\leq M} |a_n(h)|^2 < \infty$$

where C is a constant independent of h. Hence, we deduce that (37) holds. □

The results from Theorem 4.1 gives the conditions which ensure the boundedness of the sequence of discrete controls, hence we pass to study the convergence properties.

Theorem 4.2. *Let $T > T_0$ and let (w_M^0, w_M^1) be given by (21). Suppose that the discrete initial data $((w_j^0, w_j^1))_{1\leq j\leq N} \in \mathbb{C}^{2N}$ verify (22) - (23) and*

$$(a_n(h))_n \rightharpoonup (a_n)_n \text{ in } \ell^2 \text{ when } h \to 0. \tag{38}$$

Then the family of controls $(u_h)_h \subset L^2(0,T)$ for problem (2), given by Theorem 4.1, has a subfamily which is weakly convergent to a control $u \in L^2(0,T)$ for the continuous problem (1).

Proof. Since the family of controls given by Theorem 4.1 is bounded in $L^2(0,T)$ there exists a subfamily, denoted in the same way, which converges weakly to a function u from $L^2(0,T)$. We prove that u is control for the continuous problem (1), i.e verifies (13).

Let $n \in \mathbb{Z}^*$. From (10) and (17) we deduce that

$$\lambda_{hn} \to \lambda_n \text{ when } \quad h \to 0. \tag{39}$$

Taking into account that, for each $n \in \mathbb{Z}$, we have

$$e^{i\lambda_{hn}t} \to e^{i\lambda_n t} \text{ in } L^2(0,T) \text{ when } \quad h \to 0 \tag{40}$$

and

$$\frac{h}{\sin(n\pi h)} a_n(h) \to \frac{1}{n\pi} a_n \text{ in } L^2(0,T) \text{ when } \quad h \to 0, \tag{41}$$

by passing to limit in (19), as h tends to zero, and using (38), (39), (40) and (41) we obtain that u verifies the continuous moment problem (13). Consequently, u is a control for problem (1) and the proof ends. □

5. Conclusions and further results

In the present paper we prove that our algorithm is efficient and offers a simplified and powerful tool in order to prove the uniform controllability results for hyperbolic type problems. More precisely, based on Algorithm 1 we are able to re-obtain the best optimal range of filtration in literature δN, as is revealed in [18].

Moreover, using Algorithm 1, even the range of filtration $\sqrt{N}$ obtained in [22] can be derived. We remark that the estimates of P_m has a strongly influence on the range on filtration. In this case, the estimate on real axis of P_m gives the function

$$\Phi_{hm}(x) = \begin{cases} hx^2 & \text{if } |x| \leq \frac{1}{h}, \\ \sqrt{\frac{x}{h}} & \text{if } |x| > \frac{1}{h}, \end{cases} \qquad (1 \leq m \leq N).$$

and the following remark holds.

Remark 2. For any $1 \leq m \leq \sqrt{N}$, $\lambda_{hm} < \frac{1}{\sqrt{h}}$ and $N_{hm} = \phi_{hm}\left(\sqrt{N}\right)$ we have $\sum_{n \geq N_{hm}+1} \frac{1}{a_n} < \infty$. This fact is motivated by the following estimates

$$\frac{1}{e} \sum_{n=N_{hm}+1}^{\infty} \frac{1}{a_n} \leq \int_{\sqrt{N}}^{\frac{1}{h}} \frac{\phi'_{hm}(t)}{t-\lambda_{hm}} \mathrm{d}t + \int_{\frac{1}{h}}^{\infty} \frac{\phi'_{hm}(t)}{t-\lambda_{hm}} \mathrm{d}t$$
$$= \underbrace{\int_{\sqrt{N}}^{\frac{1}{h}} \frac{2ht}{t-\lambda_{hm}} \mathrm{d}t}_{:=I_1} + \underbrace{\int_{\frac{1}{h}}^{\infty} \frac{1}{2\sqrt{ht}(t-\lambda_{hm})} \mathrm{d}t}_{:=I_2}$$

where

$$I_1 = 2h\left(\frac{1}{h} - \mathcal{O}\left(\frac{1}{\sqrt{h}}\right)\right) + 2h\lambda_{hm} \ln \frac{\frac{1}{h} - \lambda_{hm}}{\mathcal{O}\left(\frac{1}{\sqrt{h}}\right) - \lambda_{hm}} = \mathcal{O}(1)$$

and

$$I_2 \leq \int_{\frac{1}{h}}^{\infty} \frac{1}{t\sqrt{ht}} \mathrm{d}t = \mathcal{O}(1).$$

Hence, using Algorithm 1, the range of filtration can be chosen in the range $\sqrt{N}$.

Taking into account the results presented in this paper we can conclude that our proposed algorithm simplify the methods used for the range of filtration (depends only on the estimates we have on the real axis for the Weierstrass product), when we deal with a large class of uniform controllability problems. This also confirm that the results presented in this paper represent an important step in order to treat very general problems.

In this sense, let us mention further perspectives and open questions related to our work. In this paper, we are doing an important step for a more long-term remaining open question consists in finding the filtering scale in a general setting. This scale should depend on the defect of uniform discrete spectral gap and on the asymptotics of the discrete/continuous eigenvalues. The present study is very efficient in order to reach this goal.

Moreover, we intend to extend our algorithm even in the case when do not know exactly the eigenvectors and eigenvalues of our discrete operator (localization of the discrete eigenvectors/eigenvalues), especially for complex eigenvalues with nonzero real parts. This is the case studied in [21] or, for problems with vanishing viscosity [3, 4, 24].

On the other hand, a very natural question is to ask if we can extend Algorithm 1 for non-uniform mesh grids. We need to generalise our technique based on a deeper understanding of the phenomena under study, being again in the difficult situation when we do not know exactly the eigenvectors and eigenvalues of our discrete operator.

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