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Stability of a Schrödinger Equation with Internal Fractional Damping

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Abstract. In this paper, we are concerned with the stabilization of a linear Schrödinger equation in an n-dimensional open bounded domain under Dirichlet boundary conditions with an internal fractional damping. We reformulate the system into an augmented model and prove the well-posedness of it by using semigroup method. Based on a general criteria of Arendt-Batty, we show that the system is strongly stable. By combining frequency domain method and multiplier techniques, we establish an optimal polynomial energy decay rate.

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1. Introduction

We consider the following multidimensional Schrödinger equation

$$\begin{cases} iy_t(x,t) + a\Delta y(x,t) + i\gamma \partial^{\alpha,\eta} y(x,t) = 0 & \text{in } \Omega \times [0,\infty), \\ y(x,t) = 0 & \text{on } \partial\Omega \times [0,\infty), \\ y(x,0) = y_0(x) & \text{on } \Omega, \\ , & \end{cases} \tag{P}$$

where Ω is a bounded domain of $\mathbb{R}^n$,$n \in \mathbb{N}^*$, with a regular boundary $\partial\Omega$, a and γ are two positive constants. Moreover y_0 is the initial data belong to an appropriate functional space. The term $\partial^{\alpha,\eta}$ stands for the generalized Caputo's fractional derivative of order α with respect to the time variable (see [10]), which is defined by

$$\partial^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} w(s)\, ds, \quad 0 < \alpha < 1, \quad \eta \geq 0, \tag{1}$$

where Γ denotes the Gamma function.

In [13], Machtyngier and Zuazua have shown that the $L^2(\Omega)$-energy of the solution of (P), with $\alpha = 1$, decays exponentially to zero. Their proof relies on an observability inequality established previously by the first author in [9].

Recently, A. Guesmia et al. [9] studied the well-posedness and stability for two linear Schrödinger equations in n-dimensional open bounded domain under Dirichlet

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boundary conditions with an infinite memory, the system is defined by

$$\begin{cases} iy_t(x,t) + a\Delta y(x,t) - i\displaystyle\int_0^\infty f(s)\Delta y(x,t-s)\,ds = 0 & \text{in } \Omega\times[0,\infty),\\ y(x,t)=0 & \text{on } \partial\Omega\times[0,\infty),\\ y(x,-t)=y_0(x,t) & \text{on } \Omega\times[0,\infty),\end{cases}$$

and

$$\begin{cases} iy_t(x,t) + a\Delta y(x,t) + i\displaystyle\int_0^\infty f(s) y(x,t-s)\,ds = 0 & \text{in } \Omega\times[0,\infty),\\ y(x,t)=0 & \text{on } \partial\Omega\times[0,\infty),\\ y(x,-t)=y_0(x,t) & \text{on } \Omega\times[0,\infty).\end{cases}$$

Under the conditions

$$f\in C^2(\mathbb{R}_+),\ \ f(0)>0 \text{ and } \lim_{s\to\infty} f(s)=0,$$

they establish the well-posedness (existence, uniqueness and smoothness of solutions) in the sense of semigroup theory. Then, a decay estimate depending on the smoothness of initial data and the arbitrarily growth at infinity of the relaxation function f is established for each equation. The proofs are based on the semigroup approach, the multipliers method and some arguments devised in [11] and [12].

Very recently, in [1], Ammari et al., studied the wave equation with internal fractional damping. The system considered is as follows:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \gamma\partial_t^{\alpha,\eta}u(x,t) = 0 & \text{in } \Omega\times[0,+\infty),\\ u(x,t)=0 & \text{on } \partial\Omega\times[0,+\infty),\\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) & \text{on } \Omega,\end{cases}$$

where

$$\partial_t^{\alpha,\eta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)\,ds, \quad 0<\alpha<1, \quad \eta\geq 0. \tag{2}$$

The authors proved that the energy decays polynomially as $t^{-2/(1-\alpha)}$.

Recently, A. Benaissa and S. Rafa [5] studied the well-possedness and asymptotic stability of a similar wave equation with general boundary condition of diffusive type, that is

$$\begin{cases} \partial_{tt}u - \Delta_x u = 0 & x\in\Omega, t>0\\ u=0 & x\in\Gamma_0, t>0\\ \partial_\nu u = -\zeta\displaystyle\int_{-\infty}^{+\infty}\mu(\vartheta)\phi(x,\vartheta,t)\,d\vartheta & x\in\Gamma_1, t>0\\ \partial_t\phi(x,\vartheta,t) + (\vartheta^2+\eta)\phi(x,\vartheta,t) - \partial_t u\mu(\vartheta) = 0 & x\in\Gamma_1, \vartheta\in\mathbb{R}, t>0\\ u=u_0(x), \quad u_t=u_1(x) & x\in\Omega, t=0\\ \phi(x,\xi,0)=\phi_0(x,\xi) & x\in\Gamma_1, \xi\in\mathbb{R}.\end{cases} \tag{3}$$

The authors showed a general decay rate result of the system, from which the usual damping of fractional derivative type is a special case.

Let us mention here that the main approach used in [9], is based on the introducing of suitable Lyapunov functionals. Unfortunately, this method does not seem to be applicable in the case of a singular memory term.

The main result of this article is to obtain an accurate and optimal estimate of the energy decay for a fractional damping.

This work is organized as follows.In Section 2, we reformulate the system (P) into an augmented system (P'). In section 3, We prove the existence and uniqueness of our problem by Semigroup Theory.In section **4**,we study the asymptotic stability of the above model. The proof strongly relies on spectral analysis,we prove the strong stability by the theorem of Arendt and Batty [2]. Then we base on the J. Pruss theorem [18] to show that the stability is non-exponential. Finally, we establishe the optimality of the polynomial decay of the energy which will depend on the parameter α by the theorem of A. Borichev and Y. Tomilov [4].

2. Preliminary results

2.1. Augmented model. This section is concerned with the reformulation of the model (P) into an augmented system. For that, we need the following claims.

Theorem 2.1 (see [15]). *Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\alpha-1)/2}, \quad -\infty < \xi < +\infty,\ 0 < \alpha < 1. \tag{4}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t\theta(x,\xi) + \xi^2\theta(x,\xi) + \eta\theta(x,\xi) - U(t)\mu(\xi) = 0, \quad -\infty < \xi < +\infty, \eta \geq 0, t > 0, \tag{5}$$

$$\theta(x,\xi,0) = \theta_0(x,\xi), \tag{6}$$

$$O(t) = (\pi)^{-1}\sin(\alpha\pi)\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi)\,d\xi \tag{7}$$

is given by

$$O(t) = I^{1-\alpha,\eta}U(t), \tag{8}$$

where

$$[I^{\alpha,\eta}f](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}f(\tau)\,d\tau.$$

Here, taking the input $U(x,t) = y(x,t)$, then combining (1) with (8), we obtain

$$O(t) = I^{1-\alpha,\eta}y(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}y(x,s)\,ds = \partial^{\alpha,\eta}y(x,t).$$

By substituting this equality into Theorem 2.1, we get

$$\begin{cases} \partial_t\theta(x,\xi,t) + (\xi^2+\eta)\theta(x,\xi,t) - U(t)\mu(\xi) = 0, & (x,\xi,t) \in \Omega\times\mathbb{R}\times\mathbb{R}^+, \\ \theta(x,\xi,0) = 0, & (x,\xi) \in \Omega\times\mathbb{R}, \\ \partial^{\alpha,\eta}y(x,t) - (\pi)^{-1}\sin(\alpha\pi)\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi,t)\,d\xi = 0, & (x,\xi,t) \in \Omega\times\mathbb{R}^+. \end{cases} \tag{9}$$

From the representation (9), system (P) can be written as an augmented model

$$\begin{cases} y_t(x,t) - ai\Delta y(x,t) + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi,t)d\xi dx = 0 & x\in\Omega, t>0, \\ \partial_t\theta(x,\xi,t) + (\xi^2+\eta)\theta(x,\xi,t) - \mu(\xi)y = 0 & x\in\Omega, \xi\in\mathbb{R}, t>0, \\ y(x,t) = 0 & x\in\partial\Omega, t>0, \\ y(x,0) = y_0(x) & x\in\Omega, \\ \theta(x,\xi,0) = 0 & x\in\Omega, \xi\in\mathbb{R}, \end{cases} \tag{P'}$$

with $\widetilde{\gamma} = \gamma\pi^{-1}\sin(\alpha\pi)$.

The energy of system (P') is given by

$$\mathcal{E}(t) = \frac{1}{2}\|y\|^2_{L^2(\Omega)} + \frac{\widetilde{\gamma}}{2}\|\theta\|^2_{L^2(\Omega\times(-\infty,+\infty))}. \tag{10}$$

Lemma 2.2. *Let (y,θ) be a regular solution of the problem (P'), then the energy $\mathcal{E}(t)$ satisfies*

$$\frac{d}{dt}\mathcal{E}(t) = -\widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(x,\xi,t)|^2 d\xi dx \le 0. \tag{11}$$

Proof. A simple computation gives the result. □

We shall need the following Lemma in all Sections.

Lemma 2.3 (see [6]). *If $\lambda \in D_\eta = \mathbb{C}\backslash]-\infty,-\eta]$ then*

$$\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+\eta+\xi^2}\,d\xi = \frac{\pi}{\sin\alpha\pi}(\lambda+\eta)^{\alpha-1}.$$

3. The well-posedness of the problem

In this section, we prove an existence and uniqueness result of our problem based on the semigroup theory. For that, we write problem (P') in a matrix form by posing $\Psi = (y,\theta)^T$, then problem (P') takes the form

$$\begin{cases} \partial_t\Psi = \mathcal{A}\Psi, \\ \Psi(0) = \Psi_0 = (y_0,\theta_0)^T, \end{cases} \tag{12}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Psi = \left(ai\Delta y - \widetilde{\gamma}\int_{-\infty}^{+\infty}\mu(\xi)\theta(x,\xi)d\xi, -(\xi^2+\eta)\theta(x,\xi)+\mu(\xi)y\right)^T, \tag{13}$$

with domain

$$D(\mathcal{A}) = \left\{\begin{array}{l} (y,\theta) \text{ in } \mathcal{H} : y \in H^2(\Omega)\cap H_0^1(\Omega), \\ ai\Delta y - \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\mu(\xi)\theta d\xi \in L^2(\Omega), \\ -\left(\xi^2+\eta\right)\theta + y(x)\mu(\xi) \in L^2(\Omega\times(-\infty,+\infty)) \\ |\xi|\theta \in L^2(\Omega\times(-\infty,+\infty)) \end{array}\right\}, \tag{14}$$

where

$$\mathcal{H} = L^2(\Omega)\times L^2(\Omega\times(-\infty,+\infty)),$$

equipped with the inner product

$$\langle\Psi,\tilde{\Psi}\rangle_{\mathcal{H}} = \int_\Omega y\bar{\tilde{y}}\,dx + \zeta\int_\Omega\int_{-\infty}^{+\infty}\theta\bar{\tilde{\theta}}d\xi dx,$$

for $\Psi = (y,\theta), \tilde{\Psi} = (\tilde{y},\tilde{\theta}) \in \mathcal{H}$.

The main result in this section is given by the following theorem.

Theorem 3.1. (1) *If $\Psi_0 \in D(\mathcal{A})$, then system (12) has a unique strong solution*

$$\Psi \in C^0(\mathbb{R}_+, D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

(2) *If $\Psi_0 \in \mathcal{H}$, then system (12) has a unique weak solution*

$$\Psi \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof. We use the semigroup approach. According to Lumer-Philips theorem, it suffices to show that $\mathcal{A}$ is a maximal monotone operator.

First, we prove that $\mathcal{A}$ is monotone. For any $\Psi \in D(\mathcal{A})$, and using the inner product and integration by parts, we easily arrive at

$$\Re\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} = -\widetilde{\gamma}\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2\, d\xi dx \le 0. \tag{15}$$

Hence $\mathcal{A}$ is monotone. Secondly, we prove that the operator $I-\mathcal{A}$ is surjective. Given $F=(f_1,f_2)^T \in \mathcal{H}$, we prove that there exists $\Psi \in D(\mathcal{A})$ satisfying

$$\Psi - \mathcal{A}\Psi = F, \tag{16}$$

this means

$$\begin{cases} y - ai\Delta y + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \mu(\xi)\theta(x,\xi)d\xi = f_1, \\ \theta + (\xi^2+\eta)\theta(x,\xi) - \mu(\xi)y = f_2. \end{cases} \tag{17}$$

Using $(17)_2$, we get

$$\theta(x,\xi) = \frac{f_2(x,\xi)+\mu(\xi)y}{\xi^2+\eta+1}. \tag{18}$$

By substituting (18) into $(17)_1$, we get

$$y - ai\Delta y + A_0 y = f_1 - \widetilde{\gamma}\int_{-\infty}^{+\infty}\frac{f_2(x,\xi)}{\xi^2+\eta+1}\mu(\xi)d\xi, \tag{19}$$

where

$$A_0 = \widetilde{\gamma}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\xi^2+\eta+1}d\xi.$$

The variational formulation corresponding to (19) is

$$\mathcal{B}(y,w) = \mathcal{L}(w), \tag{20}$$

where $\mathcal{B} : H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{C}$ is the sesquilinear form given by

$$\mathcal{B}(y,w) \;=\; \int_\Omega (y - ai\Delta y + A_0 y)\,\overline{w}dx$$

and $\mathcal{L} : H_0^1(\Omega) \to \mathbb{C}$ is the antilinear functional defined by

$$\mathcal{L}(w) \;=\; \int_\Omega f_1\overline{w}\,dx - \widetilde{\gamma}\int_\Omega \overline{w}\int_{-\infty}^{+\infty}\frac{f_2(x,\xi)}{\xi^2+\eta+1}\mu(\xi)d\xi dx.$$

It is not hard to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Owing to the Lax-Milgram theorem, we deduce that for all $w \in H_0^1(\Omega)$ the problem (20) admits a unique solution $y \in H_0^1(\Omega)$. Applying the classical elliptic regularity, it follows that $y \in H^2(\Omega)$. Therefore, the operator $I-\mathcal{A}$ is surjective. □

4. Asymptotic behavior

4.1. Strong stability of the system. To prove that the semigroup $(e^{t\mathcal{A}})_{t\geq 0}$ is strongly asymptotically stable, we shall apply a version of the Arendt-Batty and Lyubich-Vu for Hilbert spaces [2], [17].

Theorem 4.1 ([2], [17]). *Let $\mathcal{A}$ be the generator of a uniformly bounded C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If:*
(i) *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*
(ii) *The intersection of the spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set,*
then the semigroup $\{S(t)\}_{t\geq 0}$ is asymptotically stable, i.e, $\|S(t)z\|_{\mathcal{H}} \to 0$ as $t \to \infty$ for any $z \in \mathcal{H}$.

Our next main result in this part is the following theorem.

Theorem 4.2. *The C_0-semigroup $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $\Psi_0 \in \mathcal{H}$, the solution of (12) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}}\Psi_0\|_{\mathcal{H}} = 0.$$

Proof. For the proof of Theorem 4.2, we need the following two lemmas.

Lemma 4.3. *For all $\lambda \in \mathbb{R}$, we have $i\lambda I - \mathcal{A}$ is injective, that is*

$$ker(i\lambda I - \mathcal{A}) = \{0\}.$$

Proof. Let $\lambda \in \mathbb{R}$ such that $i\lambda$ is an eigenvalue of the operator $\mathcal{A}$ and let $\Psi = (y, \theta) \in D(\mathcal{A})$ be a corresponding eigenvector such that

$$\mathcal{A}\Psi = i\lambda\Psi. \tag{21}$$

Equivalently,

$$\begin{cases} i\lambda y - ai\Delta y + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi = 0, \\ i\lambda\theta + (\xi^2+\eta)\theta(x,\xi) - \mu(\xi)y = 0. \end{cases} \tag{22}$$

From (15) and (21), we get

$$0 = \Re\langle \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} = -\widetilde{\gamma}\int_{\Omega}\int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2\, d\xi dx. \tag{23}$$

It's clear that

$$\theta(x,\xi) = 0 \quad \text{a.e. in} \quad \Omega\times(-\infty,+\infty). \tag{24}$$

By substituting (24) into $(22)_2$, we get

$$y = 0, \tag{25}$$

that is $\Psi = 0$. □

Lemma 4.4. *If $\eta > 0$ and $\lambda \in \mathbb{R}$ or $\eta = 0$ and $\lambda \in \mathbb{R}^*$, then $i\lambda I - \mathcal{A}$ is surjective.*

Proof. **Case 1: $\lambda \neq 0$.** Let $F = (f_1, f_2)^T \in \mathcal{H}$ be given. We look for $\Psi = (y, \theta)^T \in D(\mathcal{A})$ solving

$$(i\lambda I - \mathcal{A})\Psi = F. \tag{26}$$

Equivalently, we have

$$\begin{cases} i\lambda y - ai\Delta y + \widetilde{\gamma} \displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi = f_1(x), \\ i\lambda\theta + (\xi^2 + \eta)\theta(\xi, x) - \mu(\xi)y = f_2(x,\xi). \end{cases} \tag{27}$$

By $(27)_2$ we can find θ as

$$\theta(x,\xi) = \frac{f_2(x,\xi) + \mu(\xi)y}{\xi^2 + \eta + i\lambda}. \tag{28}$$

Using (28) to obtain

$$i\lambda y - ai\Delta y + \widetilde{\gamma} \int_{-\infty}^{+\infty} \frac{f_2(x,\xi) + \mu(\xi)y}{\xi^2 + \eta + i\lambda}\mu(\xi)d\xi = f_1(x). \tag{29}$$

Solving system (29) is equivalent to finding $y \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\begin{aligned} i\lambda \int_\Omega y\overline{w}dx - ai \int_\Omega \Delta y\overline{w}dx + \widetilde{\gamma} \int_\Omega y \int_{-\infty}^{+\infty} \frac{\mu(\xi)^2}{\xi^2 + \eta + i\lambda} d\xi\overline{w}dx = \int_\Omega f_1\overline{w}\, dx \\ -\widetilde{\gamma} \int_\Omega \int_{-\infty}^{+\infty} \frac{f_2(x,\xi)}{\xi^2 + \eta + i\lambda}\mu(\xi)d\xi\overline{w}dx \end{aligned} \tag{30}$$

For all $w \in H_0^1(\Omega)$.

Consequently, problem (30) is equivalent to the problem

$$\widetilde{\mathcal{B}}(y, w) = \widetilde{\mathcal{L}}(w) \tag{31}$$

where the sesquilinear form

$$\widetilde{\mathcal{B}} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$$

and the antilinear form

$$\widetilde{\mathcal{L}} : H_0^1(\Omega) \to \mathbb{C}$$

are defined by

$$\widetilde{\mathcal{B}}(y, w) = i\lambda \int_\Omega y\overline{w}dx - ai \int_\Omega \Delta y\overline{w}dx + \widetilde{\gamma} \int_\Omega y \int_{-\infty}^{+\infty} \frac{\mu(\xi)^2}{\xi^2 + \eta + i\lambda} d\xi\overline{w}dx$$

and

$$\widetilde{\mathcal{L}}(w) = \int_\Omega f_1\overline{w}\, dx - \widetilde{\gamma} \int_\Omega \int_{-\infty}^{+\infty} \frac{f_2(x,\xi)}{\xi^2 + \eta + i\lambda}\mu(\xi)d\xi\overline{w}dx$$

It is not hard to verify that $\widetilde{\mathcal{B}}$ is continuous and coercive, and $\widetilde{\mathcal{L}}$ is continuous. Owing to the Lax-Milgram theorem, we deduce that for all $w \in H_0^1(\Omega)$ the problem (31) admits a unique solution $y \in H_0^1(\Omega)$. Applying the classical elliptic regularity, it follows that $y \in H^2(\Omega)$. Therefore, the operator $i\lambda I - \mathcal{A}$ is surjective.

Case 2: $\lambda = 0$ and $\eta \neq 0$.

The system (27) is reduced to the following

$$\begin{cases} -ai\Delta y + \widetilde{\gamma} \displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi = f_1(x), \\ (\xi^2 + \eta)\theta(\xi, x) - \mu(\xi)y = f_2(x,\xi). \end{cases} \tag{32}$$

By $(32)_2$ we can find θ as

$$\theta(x,\xi)=\frac{f_2(x,\xi)+\mu(\xi)y}{\xi^2+\eta}. \tag{33}$$

Using (33)to obtain

$$-ai\Delta y+\widetilde{\gamma}\int_{-\infty}^{+\infty}\frac{f_2(x,\xi)+\mu(\xi)y}{\xi^2+\eta}\mu(\xi)d\xi=f_1(x). \tag{34}$$

Solving system (34) is equivalent to finding $y\in H^2(\Omega)\cap H_0^1(\Omega)$ such that

$$-ai\int_\Omega\Delta y\overline{w}dx+\widetilde{\gamma}\int_\Omega y\int_{-\infty}^{+\infty}\frac{\mu(\xi)^2}{\xi^2+\eta}d\xi\overline{w}dx=\int_\Omega f_1\overline{w}\,dx-\widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\frac{f_2(x,\xi)}{\xi^2+\eta}\mu(\xi)d\xi\overline{w}dx, \tag{35}$$

for all $w\in H_0^1(\Omega)$.
Consequently, problem (35) is equivalent to the problem

$$\widetilde{\mathcal{B}}(y,w)=\widetilde{\mathcal{L}}(w), \tag{36}$$

where the sesquilinear form

$$\widetilde{\mathcal{B}}:H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{C}$$

and the antilinear form

$$\widetilde{\mathcal{L}}:H_0^1(\Omega)\to\mathbb{C}$$

are defined by

$$\widetilde{\mathcal{B}}(y,w)\quad=\quad-ai\int_\Omega\Delta y\overline{w}dx+\widetilde{\gamma}\int_\Omega y\int_{-\infty}^{+\infty}\frac{\mu(\xi)^2}{\xi^2+\eta}d\xi\overline{w}dx$$

and

$$\widetilde{\mathcal{L}}(w)=\int_\Omega f_1\overline{w}\,dx-\widetilde{\gamma}\int_\Omega\overline{w}\int_{-\infty}^{+\infty}\frac{f_2(x,\xi)}{\xi^2+\eta}\mu(\xi)d\xi dx.$$

It is not hard to verify that $\widetilde{\mathcal{B}}$ is continuous and coercive, and $\widetilde{\mathcal{L}}$ is continuous. Owing to the Lax-Milgram theorem, we deduce that for all $w\in H_0^1(\Omega)$ the problem (36) admits a unique solution $y\in H_0^1(\Omega)$. Applying the classical elliptic regularity, it follows that $y\in H^2(\Omega)$. Therefore, the operator $i\lambda I-\mathcal{A}$ is surjective. □

Lemma 4.5. *Assume that $\eta=0$. Then, the operator $-\mathcal{A}$ is not invertible and consequently $0\in\sigma(\mathcal{A})$*

Proof. First, let $y_n\in L^2(\Omega)$ be an eigenfunction of the following problem

$$\begin{cases}\Delta y_n=-\beta_n^2y_n & \text{in }\Omega,\\ y_n=0 & \text{on }\partial\Omega,\end{cases}$$

such that

$$\|y_n\|^2_{L^2(\Omega)}=\int_\Omega|y_n|^2\,dx.$$

Next, define the vector $F=(y_n,0)\in\mathcal{H}$. Assume that there exists $\Psi=(y,\theta)\in D(\mathcal{A})$ such that

$$-\mathcal{A}U=F.$$

It follows that

$$|\xi|^2\theta-\mu(\xi)y=0\quad\text{on}\quad\Omega. \tag{37}$$

From (37), we deduce that $\theta(x,\xi) = |\xi|^{\frac{2\alpha-5}{2}} y \notin L^2(\Omega \times (-\infty,+\infty))$. So, the assumption of the existence of Ψ is false and consequently, the operator $-\mathcal{A}$ is not invertible. □

Following a general criteria of Arendt-Batty see [2], the C_0- semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable, if $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable and no eigenvalue of $\mathcal{A}$ lies on the imaginary axis. First, using Lemma4.3, we directly deduce that $\mathcal{A}$ has non pure imaginary Next, using Lemmas 4.4 and 4.5, we conclude, with the help of the closed graph theorem of Banach, that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\emptyset\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. The proof is thus completed. □

4.2. Lack of exponential stability.

Theorem 4.6 ([18]). *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on Hilbert space X. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta : \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{38}$$

and

$$\overline{\lim}_{|\beta|\to\infty} \left\|(i\beta I - \mathcal{A})^{-1}\right\|_{L(X)} < \infty. \tag{39}$$

Our main result in this part is the following theorem.

Theorem 4.7. *The semigroup generated by the operator $\mathcal{A}$ is not exponentially stable.*

Proof. Let $-\beta_n^2 = (i\beta_n)^2$ be a sequence of eigenvalues corresponding to the sequence of normalized eigenfunctions y_n of the operator Δ such that

$$|\beta_n| \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

and

$$\begin{cases} \Delta y_n = -\beta_n^2 y_n & \text{in } \Omega, \\ y_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Our aim is to prove, under some conditions, that if $i\beta_n$ satisfies Eq.(38) then Eq.(39) does not hold. In other words we want to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the Schördinger system (P) from being exponentially stable. Indeed we first compute the characteristic equation that gives the eigenvalues of $\mathcal{A}$. λ be an eigenvalue of $\mathcal{A}$ with associated eigenvector $\Psi = (y,\theta)^T$. Then $\mathcal{A}\Psi = \lambda\Psi$ is equivalent to

$$\begin{cases} \lambda y - ai\Delta y + \widetilde{\gamma} \displaystyle\int_{-\infty}^{+\infty} \theta(x,\xi)\mu(\xi)d\xi = 0, \\ \lambda\theta(x,\xi) + (\xi^2+\eta)\theta(x,\xi) - \mu(\xi)y = 0. \end{cases} \tag{40}$$

By $(40)_2$ we can find θ as

$$\theta(x,\xi) = \frac{\mu(\xi)y}{\xi^2+\eta+\lambda}. \tag{41}$$

By replacing (41) in (40) we get

$$\lambda y - ai\Delta y - \widetilde{\gamma} \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)y}{\xi^2+\eta+\lambda} d\xi = 0. \tag{42}$$

Then

$$i\Delta y_n = -i\beta_n^2 y_n = y_n\left(\frac{\lambda}{a} + \frac{\widetilde{\gamma}}{a}\int_{-\infty}^{+\infty} \frac{\mu^2(\xi)y_n}{\xi^2+\eta+\lambda} d\xi\right). \tag{43}$$

That is

$$-i\beta_n^2=\frac{\lambda}{a}+\frac{\gamma}{a}(\lambda+\eta)^{\alpha-1}. \tag{44}$$

Hence

$$\frac{\lambda}{a}+i\beta_n^2+\frac{\gamma}{a\lambda^{1-\alpha}}+o(\frac{1}{\lambda^{1-\alpha}})=0 \tag{45}$$

and

$$\lambda_n=-ai\beta_n^2+\varepsilon_n+o(\frac{1}{\lambda^{1-\alpha}}). \tag{46}$$

Form equation and (45) and (46) we get

$$\varepsilon_n+\frac{\gamma}{(-ai\beta_n^2+\varepsilon_k)^{1-\alpha}}+o(\frac{1}{\beta_n^{2(1-\alpha)}})=0. \tag{47}$$

Then

$$\varepsilon_n+\frac{\gamma}{(-ai\beta_n^2)^{1-\alpha}(1-\frac{\varepsilon_k}{-ai\beta_n^2})^{1-\alpha}}+o(\frac{1}{\beta_n^{2(1-\alpha)}})=0. \tag{48}$$

We deduce that

$$\varepsilon_n=-\frac{\gamma}{(-ai\beta_n^2)^{1-\alpha}}+o(\frac{1}{\beta_n^{2(1-\alpha)}}). \tag{49}$$

Using (46) and (49), we obtain

$$\lambda_n=-ai\beta_n^2-\frac{\gamma}{a^{1-\alpha}\beta_n^{2(1-\alpha)}}\left(\cos(1-\alpha)\frac{\pi}{2}+i\sin(1-\alpha)\frac{\pi}{2}\right)+o(\frac{1}{\beta_n^{2(1-\alpha)}}). \tag{50}$$

From (50) we have in that case $\beta_n^{2(1-\alpha)}\Re\lambda_n\sim\beta$, with

$$\beta=-\frac{\gamma}{a^{1-\alpha}}\cos(1-\alpha)\frac{\pi}{2}. \tag{51}$$

The operator $\mathcal{A}$ has a non exponential decaying branche of eigenvalues. Thus the proof is complete. □

4.3. Residual spectrum of $\mathcal{A}$.

Lemma 4.8. *Let $\mathcal{A}$ be defined by Eq.(13). Then*

$$\mathcal{A}^*\begin{pmatrix} y \\ \theta \end{pmatrix}=\begin{pmatrix} ai\Delta y+\tilde{\gamma}\int_{-\infty}^{+\infty}\mu(\xi)\theta d\xi \\ -\left(\xi^2+\eta\right)\theta-y(x)\mu(\xi) \end{pmatrix}, \tag{52}$$

with

$$D(\mathcal{A}^*)=\left\{\begin{array}{l} (y,\theta)\ in\ \mathcal{H}: y\in H^2(\Omega)\cap H_0^1(\Omega), \\ ai\Delta y+\tilde{\gamma}\int_{-\infty}^{+\infty}\mu(\xi)\theta d\xi\in L^2(\Omega) \\ -\left(\xi^2+\eta\right)\theta-y(x)\mu(\xi)\in L^2(\Omega\times(-\infty,+\infty)) \\ |\xi|\theta\in L^2(\Omega\times(-\infty,+\infty)) \end{array}\right\}.$$

Proof. Let $\Psi=(y,\theta)^T$ and $\tilde{\Psi}=(y_1,\theta_1)^T$.We have $<\mathcal{A}\Psi,\tilde{\Psi}>_{\mathcal{H}}=<\Psi,\mathcal{A}^*\tilde{\Psi}>_{\mathcal{H}}$. Indeed,

$$\begin{aligned}
<\mathcal{A}\Psi,\tilde{\Psi}>_{\mathcal{H}} &= \int_\Omega\left(ai\Delta y-\widetilde{\gamma}\int_{-\infty}^{+\infty}\mu(\xi)\theta d\xi\right)\overline{y_1}dx \\
&+ \widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\left[-\left(\xi^2+\eta\right)\theta+y(x)\mu(\xi)\right]\overline{\theta_1}d\xi dx \\
&= -\int_\Omega ai\nabla y\nabla\overline{y_1}dx-\widetilde{\gamma}\int_\Omega\overline{y_1}(x)\int_{-\infty}^{+\infty}\mu(\xi)\theta d\xi dx \\
&- \widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}\left(\xi^2+\eta\right)\theta\overline{\theta_1}d\xi dx+\widetilde{\gamma}\int_\Omega y(x)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\theta_1}d\xi dx \\
&= \int_\Omega ai\Delta\overline{y_1}y(x)dx+\widetilde{\gamma}\int_\Omega y(x)\int_{-\infty}^{+\infty}\mu(\xi)\overline{\theta_1}d\xi dx \\
&- \widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)\overline{\theta_1}\theta d\xi dx-\widetilde{\gamma}\int_\Omega\overline{y_1}\int_{-\infty}^{+\infty}\mu(\xi)\theta d\xi dx \\
&= \int_\Omega[ai\Delta\overline{y_1}+\widetilde{\gamma}\int_{-\infty}^{+\infty}\mu(\xi)\overline{\theta_1}d\xi]y(x)dx \\
&- \widetilde{\gamma}\int_\Omega\int_{-\infty}^{+\infty}[(\xi^2+\eta)\overline{\theta_1}+\mu(\xi)\overline{y_1}]\theta d\xi dx \\
&= <\Psi,\mathcal{A}^*\tilde{\Psi}>_{\mathcal{H}}.
\end{aligned}$$

□

Theorem 4.9. *$\sigma_r(\mathcal{A})=\emptyset$, where $\sigma_r(\mathcal{A})$ denotes the set of residual spectrum of $\mathcal{A}$.*

Proof. Since $\lambda\in\sigma_r(\mathcal{A}),\overline{\lambda}\in\sigma_p(\mathcal{A}^*)$ the proof will be accomplished if we can show that $\sigma_p(\mathcal{A})=\sigma_p(\mathcal{A}^*)$. This is because obviously the eigenvalues of $\mathcal{A}$ are symmetric on the real axis. From (52), the eigenvalue problem $\mathcal{A}^*Z=\lambda Z$ for $\lambda\in\mathbb{C}$ and $0\neq Z=(y,\theta)\in D(\mathcal{A}^*)$ we have

$$\left\{\begin{array}{l}\lambda y-ai\Delta y-\widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\theta(x,\xi)\mu(\xi)d\xi=0,\\ \lambda\theta+(\xi^2+\eta)\theta(x,\xi)+\mu(\xi)y=0.\end{array}\right. \tag{53}$$

Furthermore, by (53) we can find θ as

$$\theta(x,\xi)=-\frac{\mu(\xi)y}{\xi^2+\eta+\lambda}. \tag{54}$$

By replacing (54) in (53) we obtain

$$\lambda y-ai\Delta y+\widetilde{\gamma}\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)y}{\xi^2+\eta+\lambda}d\xi=0 \tag{55}$$

System (53)-(55) is the same as (42). Hence $\mathcal{A}^*$ has the same eigenvalues with $\mathcal{A}$. The proof is complete. □

Remark 4.1. From Lemmas 4.3 and 4.5 and according to the theorem 4.9 we deduce that the zero element belongs to the continuous spectrum.

4.4. Polynomial stability $\eta \neq 0$. The aim of this section is to prove the polynomial stability of the system (P'). Our main results in this section is the following theorem.

Theorem 4.10. *The semigroup $S_{\mathcal{A}}(t)_{t\geq 0}$ is polynomially stable and*

$$E(t) = \|S_{\mathcal{A}}(t)\Psi_0\|^2_{\mathcal{H}} \leq \frac{1}{t^{\frac{2}{(1-\alpha)}}}\|\Psi_0\|^2_{D(\mathcal{A})}. \tag{56}$$

Moreover, the rate of energy decay $t^{2/(1-\alpha)}$ is optimal for general initial data in $D(\mathcal{A})$.

To prove them, let us first recall the following necessary and sufficient condition on the polynomial stability of semigroup proposed by Borichev-Tomilov in [4].

Theorem 4.11. *Assume that $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions $(e^{\mathcal{A}t})_{t\geq 0}$ on a Hilbert space $\mathcal{H}$. If*

$$i\mathbb{R} \subset \rho(\mathcal{A}), \tag{57}$$

then for a fixed $l > 0$, the following conditions are equivalent:

$$\lim_{s\in\mathbb{R}} \sup_{|s|\to\infty} \frac{1}{|s|^l}\|(isI-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{58}$$

$$\|e^{\mathcal{A}t}\Psi_0\|^2_{\mathcal{H}} \leq \frac{c}{t^{\frac{2}{l}}}\|\Psi_0\|^2_{D(\mathcal{A})},\ \Psi_0 \in D(\mathcal{A}), \quad \text{for some } C > 0. \tag{59}$$

Proof. According to Theorem 4.11, to prove Theorem 4.10, we need to prove that (57) and (58) hold, where $l = 1-\alpha$. As condition (57) is already proved in Theorem 4.2, we only need to prove condition (58). Here, we use a contradiction argument. Namely, suppose that (58) is false, then there exists a sequence $\lambda_n \in \mathbb{R}, n \in \mathbb{N}$ such that $\lambda_n \to +\infty$ as $n \to +\infty$, and a sequence $\Psi_n = (y_n, \theta_n) \in D(\mathcal{A})$ such that

$$\|\Psi_n\| = 1 \tag{60}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^l}\|(i\lambda_n I - \mathcal{A})^{-1}\|_{\mathcal{L}(H)} = 0. \tag{61}$$

For simplification, we denote λ_n by λ and Ψ_n by $\Psi = (y, \theta)$ and

$$F_n = \lambda_n^l(i\lambda_n I - A)\Psi_n = (f_{1_n}, f_{2_n}),$$

by

$$F = \lambda^l(i\lambda I - A)\Psi = (f_1, f_2).$$

By (61), we obtain

$$\begin{cases} i\lambda y - ai\Delta y + \widetilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\theta(x,\xi,t)\mu d\xi = \frac{f_1}{\lambda^l} \to 0, \\ i\lambda\theta + (\xi^2+\eta)\theta - \mu(\xi)y = \dfrac{f_2}{\lambda^l} \to 0. \end{cases} \tag{62}$$

We are going to derive from (61) that $\|\Psi\|_{\mathcal{H}} = o(1)$, for this end, we have to collect a number of results.

Lemma 4.12. *Under (62) we have*

$$\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\eta)|\theta(x,\xi)|^2\, d\xi dx = \frac{o(1)}{\lambda^l}, \tag{63}$$

and

$$\int_\Omega \int_{-\infty}^{+\infty} |\theta(x,\xi)|^2 \, d\xi dx = \frac{o(1)}{\lambda^l}, \tag{64}$$

Proof. From (15) and (61), we have

$$\Re\langle i\lambda\Psi - \mathcal{A}\Psi, \Psi\rangle_{\mathcal{H}} = \widetilde{\gamma} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2 \, d\xi dx = \frac{o(1)}{\lambda^l},$$

which implies (63).
The estimation (64) is a consequence of

$$\begin{aligned} \int_\Omega \int_{-\infty}^{+\infty} |\theta(x,\xi)|^2 \, d\xi dx &\leq \frac{1}{\eta} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2 \, d\xi dx \\ &\leq \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2 \, d\xi dx. \end{aligned}$$

□

Lemma 4.13. *We have*

$$\int_\Omega |y(x)|^2 dx = \frac{o(1)}{\lambda^{l+2\alpha-2}}.$$

Proof. From $(62)_3$, we have

$$i\lambda\theta + (\xi^2+\eta)\theta - \frac{f_2}{\lambda^l} = y(x)\mu(\xi), \quad \text{on} \quad \Omega.$$

Multiplying it by $(i\gamma+\xi^2+\eta)^{-2}|\xi|$, we get

$$(i\gamma+\xi^2+\eta)^{-2}|\xi|y(x) = (i\gamma+\xi^2+\eta)^{-1}|\xi|\theta - (i\gamma+\xi^2+\eta)^{-2}|\xi|\frac{f_2}{\lambda^l}, \forall x \in \Omega. \tag{65}$$

By taking absolute values of both sides of (65), integrating over$]-\infty,+\infty[$ with respect to the variable ξ, applying Cauchy-Schwarz's inequality , we get

$$\mathcal{S}|y(x)| \leq \mathcal{U}\left(\int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2 \, d\xi\right)^{\frac{1}{2}} + \mathcal{V}\left(\int_{-\infty}^{+\infty} \left|\frac{f_2}{\lambda^l}\right|^2 d\xi\right)^{\frac{1}{2}}. \tag{66}$$

Here

$$\mathcal{S} = |\int_{-\infty}^{+\infty} (\lambda+\xi^2+\eta)^{-2}|\xi|\mu(\xi) \, d\xi| = \frac{|1-2\alpha|}{4} \frac{\pi}{|\sin\frac{(2\alpha+3)}{4}\pi|}|i\lambda+\eta|^{\frac{(2\alpha-5)}{4}},$$

$$\mathcal{U} = \left(\int_{-\infty}^{+\infty} |i\lambda+\xi^2+\eta|^{-2} \, d\xi\right)^{\frac{1}{2}} \leq \sqrt{2}(\frac{\pi}{2})^{1/2}||\lambda|+\eta|^{-\frac{3}{4}},$$

$$\mathcal{V} = \left(\int_{-\infty}^{+\infty} (|i\lambda+\xi^2+\eta|)^{-4}|\xi|^2 \, d\xi\right)^{\frac{1}{2}} \leq 2\left(\frac{\pi}{16}||\lambda|+\eta|^{-\frac{5}{2}}\right)^{1/2}.$$

By using Young's inequality and integrating (66), over Ω, we get

$$\int_\Omega |y(x)|^2 dx \leq \frac{2\mathcal{U}^2}{\mathcal{S}^2} \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\eta)|\theta(x,\xi)|^2 \, d\xi dx + \frac{2\mathcal{V}^2}{\mathcal{S}^2} \int_\Omega \int_{-\infty}^{+\infty} \left|\frac{f_2}{\lambda^l}\right|^2 d\xi dx.$$

It is easy to verify

$$\mathcal{S}^2 = O(|\lambda|^{\frac{2\alpha-5}{2}}), \quad \mathcal{V}^2 = O(|\lambda|^{-\frac{5}{2}}), \mathcal{U}^2 = O(|\lambda|^{-\frac{3}{2}})$$

and using (62) and (63), we obtain

$$\int_\Omega |y(x)|^2 dx = \frac{o(1)}{\lambda^{\alpha-1+l}} + \frac{o(1)}{\lambda^{\alpha+2l}} = \frac{o(1)}{\lambda^{\alpha-1+l}}. \tag{67}$$

□

As

$$\begin{aligned} \|\Psi\|^2 &= \int_\Omega |y(x)|^2 dx + \widetilde{\gamma} \int_\Omega \int_{-\infty}^{+\infty} |\theta(x,\xi)|^2 \, d\xi dx \\ &= \frac{o(1)}{\lambda^{\alpha-1+l}} + \frac{o(1)}{\lambda^{l}}, \end{aligned} \tag{68}$$

then taking $l = 1-\alpha$, we deduce that $\|\Psi\| = o(1)$ which contradicts (60), consequently condition (58) holds. This implies, from Lemma 4.11, the energy decay estimation (56). Besides, we prove that the decay rate is optimal. Indeed, the decay rate is consistent with the asymptotic expansion of eigenvalues which shows a behavior of the real part like $k^{(1-\alpha)}$. The proof is thus complete. □

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