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Theoretical Analysis and Numerical Simulation of a Weak Periodic Solution for a Parabolic Problem with Nonlinear Boundary Conditions

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Abstract. The aim of this work is to develop a numerical tool for computing the weak periodic solution for a class of parabolic equations with nonlinear boundary conditions. We formulate our problem as a minimization problem by introducing a least-squares cost function. With the help of the Lagrangian method, we calculate the gradient of the cost function. We build an iterative algorithm to simulate numerically the weak periodic solution to the considered problem. To illustrate our approach, we present some numerical examples.

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1. Introduction

Partial differential equations appear naturally in the modeling of many problems in physics, biology, economics or elsewhere, we refer the readers to see the reference [12]. In many ways, they seem to generalize ordinary differential equations to the multidimensional context. In a large majority of cases, the partial differential equations are nonlinear and one has to use the computer to solve them (numerical calculation software). When possible (linear equations), it is interesting to solve the model equations in an analytical way (by hand). In this case, the obtained solution allows to see the influence of the different parameters. At the same time, periodic partial differential equations appear frequently in various fields of applied science. Recently, enormous attention has been devoted to the study of the periodic behavior of solutions, and various methods and techniques have been developed to answer the often asked questions about the existence, uniqueness, regularity and asymptotic behavior; see for example [5, 7, 14, 13, 10, 11, 15, 17].

In this paper, we focus on a class of periodic parabolic equations with nonlinear boundary conditions modeled as follows

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = f(t,x) & \text{in } \ Q_T \\ u(0,.) = u(T,.) & \text{in } \ \Omega \\ -\dfrac{\partial u}{\partial \nu}(t,x) = \beta(t,x)u(t,x) + \gamma(t,x,u) & \text{on } \ \Sigma_T \end{cases} \tag{1}$$

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Assuming that Ω is a smooth, bounded, and regular open subset of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, and that ν represents the outward-facing unit normal vector on $\partial\Omega$, we define $T > 0$ to be the period and $Q_T =]0, T[\times\Omega$ and $\Sigma_T =]0, T[\times\partial\Omega$. We also assume that f is a measurable function that is periodic in time with period T and belongs to a specific Lebesgue space. The function $\gamma : \Sigma_T \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function, periodic in time with period T and satisfies some growth conditions and β is a nonnegative bounded function periodic in time with period T.

Our work is motivated by some recent papers on the relevant literature. To position the problem well, we propose to recall some works which are related to the special cases of (1). Related to the early literature, most of the concerned studies are devoted to the existence and stability analysis of a time-periodic solution to (1) with linear boundary conditions [14, 13, 10, 15]. Contrary, in [26] Pao studied a system of periodic equations with nonlinear boundary conditions. The author proved the existence and stability of the proposed problem by using the sub- and super-solutions method. The paper of Zhang and Lin [31] was concerned with the existence of maximal and minimal periodic solutions to quasilinear parabolic systems with nonlinear boundary conditions. They combined the method of sub- and super-solutions with monotone iterations to establish the existence of a classical periodic solution. Their work can be viewed as an application to Lotka-Volterra systems.

In [7] Badii's studied the equation (1) with specific growth assumptions on $\gamma(t, x, u)$ and bounded nonlinearity. The author combined the theory of maximal monotone operator with Schauder fixed point Theorem to obtain the existence of a bounded weak periodic solution. Recently, Alaa and al [17] investigated a class of quasilinear periodic equations with nonlinear boundary conditions which included (1). They examined the existence of a weak periodic solution when the nonlinearity has critical growth with respect to the gradient and involves a sign condition. Their approach was based on the application of Schauder's fixed point Theorem and involved the truncation method.

All the mentioned above works are devoted to the theoretical studies of parabolic periodic boundary problems. However, there are also quite a few papers that are devoted to numerical simulations for periodic solutions to parabolic boundary problems, we refer the readers to see [1, 9, 19, 23, 27, 28, 30]. To detail the discussion, let us start with the works of Hackbusch [19], who proposed a fast numerical approach to compute numerically the periodic solution to a class of linear and nonlinear parabolic equations. The author formulated the considered periodic problem as an integral equation. Thereafter, he used multigrid algorithm to construct an iterative computational algorithm for the associated discrete equations. The paper of Carasso [9] employed the least square method to compute numerically the periodic solutions for a class of parabolic equations. Lust et al. [23] introduced an innovative approach for generating periodic solutions to an ordinary differential system that allows for flexibility in determining the time period. Their iterative construction scheme offers a promising method for addressing the challenge of finding periodic solutions without prior knowledge of the period. In [28], an alternative approach is presented for addressing problem (1). The authors formulated the problem as an evolution equation in a Banach space that was deemed suitable for the purpose, and demonstrated the existence of a periodic solution through the application of semigroup theory and fixed point theorems. This approach provides a promising avenue for tackling the problem, and offers insights into the mathematical tools that can be leveraged to study similar problems. As such, this work contributes to the broader body of research on problem (1) and provides a valuable framework for future investigations. They used Newton's method to develop a numerical computation for a nonlinear heat-conduction problem. Another approach to numerical computation of periodic solutions but is limited to the case of linear parabolic equations.

Note that all the above-mentioned works are interested in numerical simulations for periodic parabolic equations with linear boundary conditions. Contrary, in [27] Pao established three monotone iterative schemes to compute numerically the periodic solutions for a discrete version of a class of nonlinear reaction-diffusion-convection equations with nonlinear boundary conditions. Their method was based on the existence of upper and lower solutions to the considered systems and involved monotone iterative method. In this work, we develop an efficient method that is able to construct numerically the periodic solution of (1).We adopt a strategy of transforming the problem (1) into a minimization problem by utilizing a cost functional of the least-squares type. It will be established that the optimization problem is well-defined when considering a specific set of admissible functions.

To solve the optimization problem, we will utilize the Lagrangian method, which allows us to explicitly calculate the derivative of the cost function using an intermediate state known as the adjoint equation. This derivative will enable us to develop an iterative algorithm for numerically solving the optimization problem.

Overall, our research provides a novel approach to solving nonlinear periodic problems and contributes to the field's theoretical and practical advancements.

The remainder of our paper is organized as follows: We begin in Section 2 by presenting the assumptions related to our problem and defining the concept of a weak periodic solution adapted to (1). In Section 3, we first transform the problem of existence for (1) into an equivalent optimization problem utilizing a least-squares cost function. Subsequently, we establish the existence of an optimal solution to this optimization problem and utilize the Lagrangian method to explicitly compute the derivative of the cost function with respect to the state variable. Section 4 is dedicated to describing the numerical technique utilized to solve the optimization problem. Finally, in Section 5, we present several numerical examples to demonstrate the effectiveness of our approach.

2. Assumptions and main result

To begin this section, we will first outline the essential assumptions required to address (1).

2.1. Assumptions. Throughout this paper, we assume that $\gamma : \Sigma_T \times \mathbb{R} \to \mathbb{R}^N$ is a Carathéodory function, periodic in time with period T such that:

(H_1) There exist constant $\alpha_1 \geq 0$ and a function $K \in L^2(\Sigma_T)$ such that for all $s \in \mathbb{R}$, we have

$$|\gamma(t,x,s)| \leq K(t,x) + \alpha_1 |s| \quad \text{a.e. in } \Sigma_T.$$

(H_2) $\gamma(t,x,s) \cdot s \geq 0 \quad$ for all $s \in \mathbb{R}$, a.e. in $(t,x) \times \Sigma_T$.

(H_3) $s \longmapsto \gamma(t,x,s)$ is nondecreasing a.e. in Σ_T.

(H_4) $s \longmapsto \gamma(.,.,s)$ is differentiable such that $\dfrac{\partial \gamma(.,.,s)}{\partial s}$ belongs to $L^\infty(\Sigma_T)$.

(H_5) β is a continuous, bounded function and periodic in time with period T, such that

$$0 < \beta_0 \leq \beta(t,x) \leq \beta_1 \quad \text{a.e. in } \Sigma_T.$$

(H_6) f is a measurable function periodic with period T and belonging to $L^2(Q_T)$.

2.2. Framework and definition. We would like to present the functional framework that pertains to our work.

We set

$$\mathcal{V}_T := L^2(0,T;H^1(\Omega))$$

and equip it the following norm

$$\|u\|_{\mathcal{V}_T} := \left(\int_{Q_T} |\nabla u|^2 + \int_{\Sigma_T} \beta(t,x)|u|^2 \right)^{\frac{1}{2}}$$

which is equivalent to the standard norm of $\mathcal{V}_T$. Furthermore, we set

$$\mathcal{V}_T^* := L^2(0,T;(H^1(\Omega))^*).$$

Where $\mathcal{V}_T^*$ the dual space of $\mathcal{V}_T$.
Using these spaces, we can define a functional space denoted

$$\mathcal{W}_T := \{u \in \mathcal{V}_T, \quad \partial_t u \in \mathcal{V}_T^*.\}$$

We will define a norm for it as follows

$$\|u\|_{\mathcal{W}_T} := \|u\|_{\mathcal{V}_T} + \|\partial_t u\|_{\mathcal{V}_T^*}.$$

From now on, the duality pairing between $(H^1(\Omega))^*$ and $H^1(\Omega)$ will be denoted by $\langle .,. \rangle$. We will now introduce the concept of a weak periodic solution, which will be utilized to solve the problem (1).

Definition 2.1. A measurable function $u : Q_T \to \mathbb{R}$ is said to be a weak periodic solution to (1) if it satisfies:

$$\begin{aligned} &u \in \mathcal{W}_T, \quad u(0,x) = u(T,x) \text{ in } L^2(\Omega), \\ &\int_0^T \langle \partial_t u, \varphi \rangle + \int_{Q_T} \nabla u \nabla \varphi + \int_{\Sigma_T} \beta(t,x) u \varphi + \int_{\Sigma_T} \gamma(t,x,u)\varphi = \int_{Q_T} f\varphi \end{aligned} \tag{2}$$

for every test function $\varphi \in \mathcal{V}_T$.

Remark 2.1. Assumptions (H_1), (H_5) and (H_6) ensure that all terms of (2) are well defined. Therefore the periodic condition makes sense in Definition (2.1) by employing the continuous embedding

$$\mathcal{W}_T \hookrightarrow \mathcal{C}([0,T];L^2(\Omega)).$$

The existence of a weak periodic solution to problem (1) can be obtained by using the theory of monotone operators [7, 17]. Here, we suggest using a cost function minimization approach to develop a numerical algorithm for simulating our periodic solution.

3. Statement of the minimization problem

In this part we will give a new formulation of the problem of existence of a weak periodic solution of (1) in a well-posed optimization problem. Consider the following least squares cost function:

$$\mathcal{J}(v) = \frac{1}{2}\int_{\Omega}(u(T,x) - v(x))^2 dx \tag{3}$$

where u is the weak solution to the initial problem

$$\begin{cases} \partial_t u(t,x) - \Delta u(t,x) = f(t,x) & \text{in } Q_T \\ u(0,.) \; = \; v(.) & \text{in } \Omega \\ -\dfrac{\partial u}{\partial \nu}(t,x) = \beta(t,x)u(t,x) + \gamma(t,x,u) & \text{on } \Sigma_T \end{cases} \tag{4}$$

We remember that there exists a unique weak solution u for problem (4) that satisfies the variational formulation below, for any $v \in L^2(\Omega)$

$$\begin{gathered} u \in \mathcal{W}_T, \quad u(0,x) = v(x) \text{ in } L^2(\Omega), \\ \int_0^T \langle \partial_t u, \varphi\rangle + \int_{Q_T} \nabla u \nabla \varphi + \int_{\Sigma_T} \beta(t,x)u\varphi + \int_{\Sigma_T} \gamma(t,x,u)\varphi = \int_{Q_T} f\varphi \end{gathered} \tag{5}$$

for all $\varphi \in \mathcal{V}_T$. For more details about the existence and uniqueness of the weak solution to (4), we refer the readers to see [6]. Therefore, we can deduce that the cost function $\mathcal{J}$ is well-defined. We introduce the minimization problem

$$\begin{cases} \text{Find } v^* \in \mathcal{U}_{ad} \\ \mathcal{J}(v^*) = \min\limits_{v\in\mathcal{U}_{ad}} \mathcal{J}(v) \end{cases} \tag{6}$$

Later, we will provide more details about the set of admissible functions, denoted by $\mathcal{U}_{ad}$. An example of this equivalence is the existence of $v^* \in \mathcal{U}ad$ such that $\mathcal{J}(v^*) = 0$, which implies that problems (5) and (6) are equivalent. This equivalence can be verified easily by observing that the cost function $\mathcal{J}$ converges to zero when u is the weak periodic solution of (1). Therefore, the minimum value of $\mathcal{J}$ on $\mathcal{U}_{ad}$ corresponds to the weak periodic solution of (1). This establishes the equivalence between the existence problem (1) and the minimization problem (6).

3.1. Existence of an optimal solution. We are interested in the existence of an optimal solution to the minimization problem (6). The choice of the set $\mathcal{U}_{ad}$ plays an important role in the well-posedness of the problem (6). Moreover, in view of (3) and (4), the right choice of the space of admissible functions is $L^2(\Omega)$, hence, to obtain a good compactness result, it is recommended to consider

$$\mathcal{U}_{ad} := \{v \in H^1(\Omega),\ \|v\|_{H^1(\Omega)} \leq C\} \tag{7}$$

where C is a strictly positive constant.

Theorem 3.1. *Under the assumptions* (H_1)-(H_6)*, the optimization problem* (6) *has at least one solution in* $\mathcal{U}_{ad}$.

Proof. From Rellich Kondrachov injection [8], we have

$$\mathcal{U}_{ad} \overset{\text{compact}}{\hookrightarrow} L^2(\Omega).$$

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$$\mathcal{J}(v_n) = \frac{1}{2}\int_\Omega (u_n(T,x) - v_n(x))^2 dx \tag{8}$$

where u_n is the unique weak solution to the problem

$$\begin{cases} \partial_t u_n - \Delta u_n = f & \text{in } Q_T \\ u_n(0,.) = v_n & \text{in } \Omega \\ -\dfrac{\partial u_n}{\partial \nu} = \beta(t,x)u_n + \gamma(t,x,u_n) & \text{on } \Sigma_T \end{cases} \tag{9}$$

Multiplying the first equation of (9) by u_n and integrating over Q_T, one obtains

$$\begin{aligned} \frac{1}{2}\int_\Omega |u_n(T)|^2 + \int_{Q_T} |\nabla u_n|^2 + \int_{\Sigma_T} \beta(t,x)|u_n|^2 + \int_{\Sigma_T} \gamma(t,x,u_n)u_n \\ = \int_{Q_T} f u_n + \frac{1}{2}\int_\Omega |v_n|^2 \end{aligned} \tag{10}$$

Thanks to the sign condition (H_2) and by applying Hölder's inequality, the relation (10) becomes

$$\|u_n\|^2_{\mathcal{V}_T} \leq \|f\|_{L^2(Q_T)}\|u_n\|_{L^2(Q_T)} + \|v_n\|^2_{L^2(Q_T)}. \tag{11}$$

As (v_n) strongly converges in $L^2(Q_T)$, it follows that (v_n) is also bounded in $L^2(Q_T)$. Additionally, we can utilize Young's inequality on the right-hand side of (11) to deduce that (u_n) is bounded in $\mathcal{V}T$. Moreover, using the equation satisfied by (u_n) and the growth conditions $(H1)$, we can establish that $(\partial_t u_n)$ is bounded in $\mathcal{V}_T^*$. By applying the Aubin compactness theorem directly (for example, see [22]), we can deduce the existence of a subsequence of (u_n), which we still denote by (u_n) for simplicity, such that

$$u_n \to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T.$$

Alternatively, according to Theorem 4.1.4 of the Trace Theorem (as described in [16]), it can be inferred that

$$u_n \to u \text{ strongly in } L^2(\Sigma_T) \text{ and a.e. in } \Sigma_T.$$

Hence, by applying the above convergences, we obtain

$$\begin{aligned} u_n &\rightharpoonup u \text{ weakly in } \mathcal{V}_T, \\ \partial_t u_n &\rightharpoonup u \text{ weakly in } \mathcal{V}_T^*, \\ \gamma(t,x,u_n) &\to \gamma(t,x,u) \text{ a.e in } \Sigma_T, \\ \gamma(t,x,u_n) &\to \gamma(t,x,u) \text{ strongly in } L^2(\Sigma_T). \end{aligned}$$

The last convergence is obtained by using lebesgue in the growth assumption (H_1) and the fact that $\gamma(t,x,u_n)$ is bounded in $L^2(\Sigma_T)$. By passing to the limit in the weak formulation of (9), one gets

$$\int_0^T \langle \partial_t u, \varphi\rangle + \int_{Q_T} \nabla u \nabla\varphi + \int_{\Sigma_T} \beta(t,x)u\varphi + \int_{\Sigma_T} \gamma(t,x,u)\varphi = \int_{Q_T} f\varphi. \tag{12}$$

Demonstration of u as a weak solution to problem 5 is established by demonstrating that u satisfies the conditions specified in (5). Conversely, assuming the unique nature of the weak solution to (5) allows us to infer that The statement that $\mathcal{J}$ is continuous on $L^2(\Omega)$ is synonymous with the expression:

$$\lim_{n\to\infty} \mathcal{J}(v_n) = \mathcal{J}(v),$$

Moreover, we can apply the theory of calculus of variations [18] to deduce the presence of an optimal solution for (6). □

3.2. Computation of the derivative of the cost function. In this section, we focus on computing the derivative of the cost function $\mathcal{J}$, which is essential for our numerical approach. To accomplish this, we will employ the Lagrangian method, which provides a quick and efficient way to compute the derivative of $\mathcal{J}$. This method involves constructing a functional $\mathcal{L}$, referred to as the Lagrangian. The Lagrangian is designed to separate the direct state variables (u) from the variable to be optimized (v) by introducing an additional equation, known as the adjoint state equation. For any $(u, p, v, \sigma) \in \mathcal{W}_T \times \mathcal{W}_T \times L^2(\Omega) \times L^2(\Omega)$, we define the Lagrangian $\mathcal{L}$ as follows

$$\begin{aligned}\mathcal{L}(u,p,v,\sigma) :=& \frac{1}{2}\int_\Omega (u(T)-v)^2 + \int_0^T \langle \partial_t u, p\rangle + \int_{Q_T} \nabla u \nabla p + \int_{\Sigma_T} \beta(t,x) up \\ &+ \int_{\Sigma_T} \gamma(t,x,u)p - \int_{Q_T} fp + \int_\Omega \sigma(u(0)-v).\end{aligned}$$

It is important to mention that the function σ will be determined subsequently to acquire the initial boundary condition for the adjoint equation. To derive the adjoint equation, we take the derivative of the Lagrangian $\mathcal{L}$ with respect to u in all directions $\varphi \in \mathcal{W}_T$.

$$\begin{aligned}\left\langle \frac{\partial \mathcal{L}}{\partial u}, \varphi \right\rangle =& \int_\Omega \varphi(T)(u(T)-v) + \int_0^T \langle \partial_t \varphi, p\rangle + \int_{Q_T} \nabla \varphi \nabla p + \int_{\Sigma_T} \beta(t,x)\varphi p \\ &+ \int_{\Sigma_T} \frac{\partial \gamma(t,x,u)}{\partial s}\varphi p + \int_\Omega \sigma \varphi(0).\end{aligned}$$

After integration by part, one obtains

$$\begin{aligned}\left\langle \frac{\partial \mathcal{L}}{\partial u}, \varphi \right\rangle =& \int_\Omega \varphi(T)(u(T)-v) - \int_0^T \langle \partial_t p, \varphi\rangle + \int_\Omega \Big(p(T)\varphi(T) - p(0)\varphi(0)\Big) \\ &- \int_{Q_T} \Delta p \varphi + \int_{\Sigma_T} \frac{\partial p}{\partial \nu}\varphi + \int_{\Sigma_T} \beta(t,x)\varphi p + \int_{\Sigma_T} \frac{\partial \gamma(t,x,u)}{\partial s}\varphi p \\ &+ \int_\Omega \sigma\varphi(0).\end{aligned} \tag{13}$$

By taking φ with compact support in (13), we get the equation

$$\partial_t p + \Delta p = 0 \text{ in } Q_T. \tag{14}$$

Then, varying the trace of the function φ yields the following nonlinear boundary condition:

$$-\frac{\partial p}{\partial \nu} = \beta(t,x)p + \frac{\partial \gamma(t,x,u)}{\partial s} p \text{ in } \Sigma_T \tag{15}$$

We still need to determine the initial condition for the adjoint state.To accomplish this, we set $\sigma = p(0)$ in (13), which yields:

$$p(T) = v - u(T) \text{ in } \Omega. \tag{16}$$

In accordance with (14)-(16), we conclude that the adjoint equation is given by the following nonlinear problem:

$$\begin{cases} \partial_t p + \Delta p = 0 & \text{in } Q_T \\ p(T) = v - u(T) & \text{in } \Omega \\ -\dfrac{\partial p}{\partial \nu} = \beta(t,x)p + \dfrac{\partial \gamma(t,x,u)}{\partial s} p & \text{in } \Sigma_T \end{cases} \tag{17}$$

Let us derive the Lagrangian $\mathcal{L}$ with respect to v, for any direction $\eta \in L^2(\Omega)$:

$$\left\langle \frac{\partial \mathcal{L}}{\partial v}, \eta \right\rangle = -\int_\Omega (u(T) - v)\eta - \int_\Omega p(0)\eta = \int_\Omega \Big(v - u(T) - p(0) \Big)\eta.$$

Furthermore, to compute the derivative of the cost function $\mathcal{J}$, we use u as the solution of the state equation (5), which gives:

$$\mathcal{L}(u, p, v, \sigma) = \mathcal{J}(v).$$

Thus, we obtain:

$$\mathcal{J}'(v) \cdot \eta = \int_\Omega \Big(v - u(T) - p(0) \Big)\eta \tag{18}$$

Here, $p(0)$ is the solution of the adjoint equation (17) at the time instant $t = 0$, and $u(T)$ is the solution of the state equation (4) at the final time T.

4. Discretisation by finite element

In this section, we assume that Ω is a bounded convex d-polyhedron. Specifically, if $d = 1$, it's a bounded interval, if $d = 2$, it's a convex polygon, and if $d = 3$, it's a convex polyhedron. We also consider a regular triangulation $\mathcal{T}_h$ of Ω with mesh size $h > 0$, such that every point in Ω is covered by at least one element in $\mathcal{T}_h$. We will use the $P1$ finite element space in our analysis.

$$V_h = \left\{ v_h \in C^0(\bar{\Omega}), v_h \quad \text{is affine on every d-simplex of} \, \mathcal{T}_h \right\}.$$

The space V_h is a finite dimensional subspace of $V = H^1(\Omega)$.
The finite element approximation of problem (6) is given by:

$$\begin{cases} \text{Find } v_h^* \in \mathcal{U}_{ad}^h \\ \mathcal{J}_h(v_h^*) = \min\limits_{v_h \in \mathcal{U}_{ad}^h} \mathcal{J}_h(v_h) \end{cases} \tag{19}$$

where $\mathcal{U}_{ad}^h := \{v_h \in V_h, \; \|v_v\|_{H^1(\Omega)} \leq C\}$ is the set of admissible functions and

$$\mathcal{J}_h(v_h) = \frac{1}{2} \int_\Omega (u_h(T,x) - v_h(x))^2 dx \tag{20}$$

with u_h being the solution to the direct initial problem:

$$\left\{\begin{array}{l}\forall t\in]0,T[,\forall\phi_h\in V_h\\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h}u_h(t,x)\phi_h(x)+\int_{\mathcal{T}_h}\nabla u_h(t,x)\nabla\phi_h(x)dx+\\ \displaystyle\int_{\partial\mathcal{T}_h}\beta u_h(t)\phi_h(s)ds+\int_{\partial\mathcal{T}_h}\gamma(t,x,u_h)\phi_h(x)dx\\ \qquad\quad=\int_{\mathcal{T}_h}f(t,x)\phi_h(x)dx\\ u_h(0,x)=v_h(x)\quad a.e.\text{ in}\quad\Omega\end{array}\right.\tag{21}$$

Consequently, we obtain the expression of the differential of $\mathcal{J}_h$ given by

$$D\mathcal{J}_h(v_h)(x)=v_h(x)-p_h(0,x)-u_h(T,x)\tag{22}$$

where p_h is a solution of the adjoint model:

$$\left\{\begin{array}{l}\forall t\in]0,T[,\forall\phi_h\in V_h\\ \dfrac{d}{dt}\displaystyle\int_{\mathcal{T}_h}p_h(t,x)\phi_h(x)-\int_{\mathcal{T}_h}\nabla p_h(t,x)\nabla\phi_h(x)dx+\\ \displaystyle\int_{\partial\mathcal{T}_h}\beta p_h(t)\phi_h(s)ds+\int_{\partial\mathcal{T}_h}\frac{\partial\gamma(t,x,u_h)}{\partial s}p_h(t,x)\phi_h(x)dx=0\\ p_h(T,x)=v_h(x)-u_h(T,x)\quad a.e.\text{ in}\quad\Omega\end{array}\right.\tag{23}$$

5. Numerical simulations

We performed numerical simulations with the software FreeFem ++ ([20]) in two spatial dimensions. Our algorithm 1 considers a bounded domain Ω of $\mathbb{R}^2$ with smooth boundary and a fixed $\mu>0$ as a step of descent.

We used an implicit method in time to solve the equation (21) and treated the nonlinear part by employing the Newton's algorithm. In the same way, we used an implicit method in time for the resolution of the linear adjoint equation (23).

Example 1: In order to illustrate our method, we computed the numerical solution obtained on the ring

$$\Omega=\left\{(x,y)\in\mathbb{R}^2: x^2+y^2<4\right\}$$

with

$$T=1,\ f(t,x,y)=(x^2+y^2+1)\times cos(\pi t)^2,\ \beta(t,x,y)=0.1,\ \text{and }\gamma(t,x,u)=0.$$

Nb vertices	162	587	1325
$h_{\min}$	0.285711	0.143948	0.0883949
$h_{\max}$	0.48218	0.250919	0.19726
J_h	$1.80748e^{-8}$	$1.79025e^{-8}$	$1.78627e^{-8}$

TABLE 1. Mesh characteristics.

Algorithm 1

Input: a mesh $\mathcal{T}_h$ which gives a triangulation of Ω_h (a polygonal approximation of Ω) and an initial estimate $u_0^0 \in V_h$ (for example a constant C_0) .
For each $1 = 0, ..., k_{max} - 1$, solve the state equation;

$$\begin{cases} \forall t \in]0, T[, \forall \phi_h \in V_h : \\ \dfrac{d}{dt} \displaystyle\int_{\mathcal{T}_h} u_h(t,x)\phi_h(x) + \int_{\mathcal{T}_h} \nabla u_h(t,x) \nabla \phi_h(x) dx + \\ \displaystyle\int_{\partial \mathcal{T}_h} \beta u_h(t) \phi_h(s) ds + \int_{\partial \mathcal{T}_h} \gamma(t,x,u_h) \phi_h(x) dx \\ \qquad = \int_{\mathcal{T}_h} f(t,x) \phi_h(x) dx. \\ u_h(0,x) = v_h(x) \quad a.e. \text{ in } \quad \Omega \end{cases}$$

Compute the value of $u_h^k(T,x)$ by solving the adjoint equation
Solve the adjoint equation

$$\begin{cases} \forall t \in]0, T[, \forall \phi_h \in V_h : \\ \dfrac{d}{dt} \displaystyle\int_{\mathcal{T}_h} p_h(t,x)\phi_h(x) - \int_{\mathcal{T}_h} \nabla p_h(t,x) \nabla \phi_h(x) dx + \\ \displaystyle\int_{\partial \mathcal{T}_h} \beta p_h(t) \phi_h(s) ds + \int_{\partial \mathcal{T}_h} \frac{\partial \gamma(t,x,u_h)}{\partial s} p_h(t,x) \phi_h(x) dx = 0 \\ p_h(T,x) = v_h(x) - u_h(T,x) \quad a.e. \text{ in } \quad \Omega \end{cases}$$

Determinate the new initial function u_0^{k+1} by computing

$$u_0^{k+1}(x) = u_0^k(x) - \mu(p_h^k(T,x) - p_h^k(0,x))$$

Compute the new value of the cost function:

$$J_h^{k+1} = J_h(u_0^{k+1})$$

Output: $u_h^{k_{max}}$, $J_h^{k_{max}}$

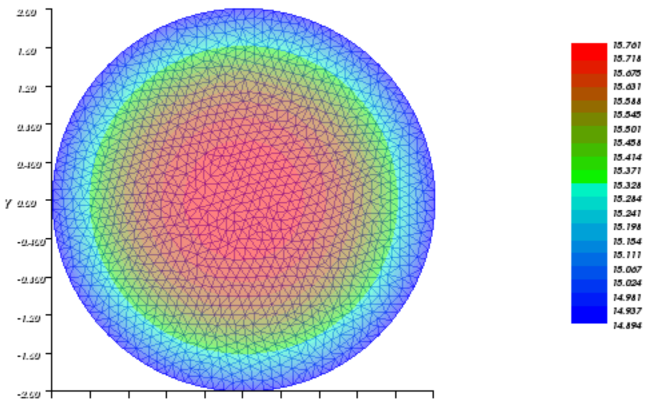

FIGURE 1. Output initial 2D Solution.

The mesh $\mathcal{T}_h$ obtained is shown in Figure 1. It has 1325 vertices. In Table 1, we present the value of $J_h = J_h^{k_{max}}$ obtained for different value of the mesh size h and for $k_{max} = 50$. The initial guess is $u_h^0 = 1$. In addition to providing the solution depicted in Figure 1, Table 1 also presents information on the number of vertices contained in the mesh $\mathcal{T}_h$, along with the minimum and maximum boundary lengths of the triangulation.

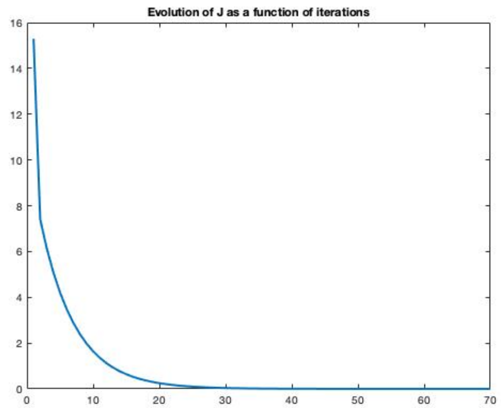

FIGURE 2. Evolution of $\mathcal{J}$ as a function of iterations.

Figure 2 shows that the value of the objective function $\mathcal{J}_h$ decreases along with the increase of iteration numbers.
Example 2: we computed the numerical solution obtained on the ring

$$\Omega = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 4\right\}$$

with

$$T = 1, \quad f(t, x, y) = (x^2 + y^2 + 1) \times \cos(\pi t)^2, \quad \beta(t, x, y) = 0.1,$$

$$\text{and} \quad \gamma(t, x, u) = \arctan(u).$$

Nb vertices	162	587	1325
$h_{\min}$	0.285711	0.143948	0.0883949
$h_{\max}$	0.48218	0.250919	0.19726
J_h	$2.09323e^{-8}$	$2.07294e^{-8}$	2.06839^{-8}

TABLE 2. Mesh characteristics.

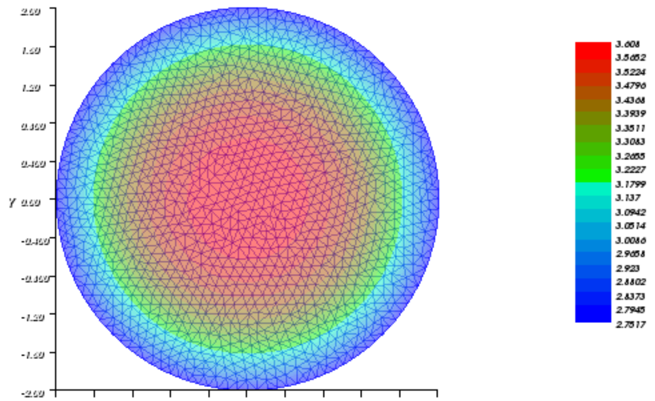

FIGURE 3. Output initial 2D Solution.

The mesh $\mathcal{T}_h$ obtained is shown in Figure 3. It has 1325 vertices. In Table 2, we present the value of $J_h = J_h^{k_{max}}$ obtained for different value of the mesh size h and for $k_{max} = 50$. The initial guess is $u_h^0 = 1$. In addition to depicting the solution shown in Figure 3, Table 2 also provides information on the number of vertices within the mesh $\mathcal{T}_h$, as well as the minimum and maximum boundary lengths of this triangulation.

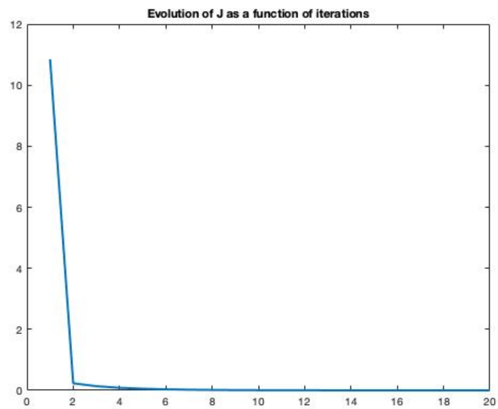

FIGURE 4. Evolution of $\mathcal{J}$ as a function of iterations.

Figure 4, shows that the value of the objective function $\mathcal{J}_h$ value decreases along with the increase of iteration numbers.

Conclusion and perspectives

Partial differential equations constitute a fascinating and ancient field of study, known for its precision, elegance, and depth of understanding. In this paper We have investigated a periodic parabolic equation with nonlinear boundary conditions and proposed a novel method to analyze and numerically simulate its weak periodic solution. Our approach utilizes a least-squares criterion to reformulate the periodic problem as an optimization problem. We have established the existence of at least one optimal solution in a suitable set of admissible functions and derived the derivative of the cost function with respect to the state variable using the Lagrangian method. We have also presented an iterative algorithm and a numerical technique to solve the optimization problem. We demonstrate the effectiveness of our approach through several numerical examples in Section 5. Our numerical results suggest that our method offers more feasibility for numerical simulations of periodic solutions to periodic parabolic equations with nonlinear boundary conditions. Additionally, our numerical findings validate the theoretical analysis presented in this work. In conclusion, our proposed method shows great potential as a numerical tool for simulating the periodic solution of partial differential equations with discontinuous coefficients.

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