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Stabilization of a Microbeam System with a Boundary or Internal Distributed Delay

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Abstract. In this paper, we consider the microbeam system with distributed delay term on the boundary or into the domain. In both cases, and thanks to a clever combination of spectral decomposition theory of Sz-Nagy-Foias [18] and frequency domain approach and under some additional and suitable assumptions, we prove the exponential stability of the total energy of our considered system.

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1. Introduction

Evolution problems appear in many applications in sciences such as finance, biology, fluid mechanics, quantum mechanics etc. In recent years, a very active research has started on the stabilization of evolution problems with delay where the delay term is known to be the cause of instability see [11] and that sometimes an arbitrarily small delay in the feedback can destabilize or improve the system performance. The study of the stability of problems with delay, even in an abstract framework, is therefore of great importance. Motivated by the results obtained in [11, 10, 19, 2, 5] for wave equations, the Rayleigh beam equation with dynamic control and more recently in [12] for a coupled system between a transport equation and an ordinary differential equation. In this paper, inspired by [13, 16], we are looking for the stability issue of a microbeam system with a boundary or internal distributed delay.
Firstly, we consider the microbeam system with boundary feedback.

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) = 0, & x \in (0,L), t>0, \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t>0, \\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) - \beta_1 u_t(x,t) - \int_{\tau_1}^{\tau_2} \beta_2(s) u_t(L,t-s)ds = 0, & t>0, \\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0, & t>0, \\ M_2 u_{xxx}(L,t) = 0, & t>0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in (0,L), \\ u_t(L,-t) = f(L,-t), & x \in (0,L),\ t \in (0,\tau_2), \end{cases} \tag{1}$$

where β_1, τ_1 and τ_2 are positive numbers with $0 \leq \tau_1 < \tau_2$, $\beta : [\tau_1, \tau_2] \to \mathbb{R}$ is a positive L^∞ function and the initial data (u_0, u_1, f_0) belong to a suitable space.

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Morever, $u(x,t)$ denotes the lateral deflection of the beam, while $\rho > 0$ is the density of the beam's material, $A > 0$ is the cross section area and L is the length. Furthermore, M_1, M_2 are positive constants determined by $M_1 = EI + GA(2l_0^2 + \frac{18}{15}l_1^2 + l_2^2)$ and $M_2 = GA(2l_0^2 + \frac{5}{4}l_1^2)$, where $E > 0$ is the Young's modulus, $I > 0$ is the area moment of inertia, and $G > 0$ is the shear modulus,while l_0, l_1, and l_2 are, respectively, the material length scale parameters associated with dilatation gradients, deviatoric stretch gradients, and rotation gradients.

Throughout this paper, we first assume that

$$\int_{\tau_1}^{\tau_2} \beta_2(s)ds < \beta_1. \tag{2}$$

We also consider the microbeam system with internal boundary distributed delay term, which takes the following form:

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) & \\ \quad + \beta_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} a(x)\beta_2(s) u_t(x,t-s)ds = 0, & x \in (0,L), t > 0 \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t > 0 \\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) = 0, & t > 0 \\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0, & t > 0 \\ M_2 u_{xxx}(L,t) = 0, & t > 0 \\ u_t(x,-t) = f(x,-t), & x \in (0,L),\ t \in (0,\tau_2) \\ u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), & x \in (0,L) \end{cases} \tag{3}$$

where $a \in L^\infty(0,1)$ is a function such that

$$a(x) \geq 0, a.e.\ \ x \in (0,L)$$

and

$$a(x) > a_0 > 0, a.e.\ \ x \in (a,b),$$

where $(a,b) \subset [0,L]$ is an open neighborhood of L.
Throughout this paper, we assume in (3) that $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a positive L^∞ function verifying

$$\|a\|_{L^\infty(0,L)} \int_{\tau_1}^{\tau_2} \beta_2(s)ds < \beta_1. \tag{4}$$

It is known that when $\beta_2 = 0$ (that is no delay occurs in the systems (1) or (3)), it has been proved in [13] that the corresponding problems are exponentially stable.

In the presence of a delay concentrated at τ, the following problems

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) & \\ +\beta_1 u_t(x,t) + \beta_2 u_t(x,t-\tau) = 0, & x \in (0,L), t > 0 \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t > 0, \\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) = 0, & t > 0, \\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0, & t > 0, \\ M_2 u_{xxx}(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in (0,L) \\ u_t(x,-t) = f(x,-t), & x \in (0,L), \ t \in (0,\tau_2), \end{cases} \tag{5}$$

and

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) = 0, & x \in (0,L), t > 0, \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t > 0, \\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) + \beta_1 u_t(L,t) + \beta_2 u_t(L,t-\tau) = 0, & t > 0, \\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0, & t > 0, \\ M_2 u_{xxx}(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in (0,L), \\ u_t(x,-t) = f(x,-t), & x \in (0,L), \ t \in (0,\tau_2), \end{cases} \tag{6}$$

are also exponentially stable, see for instance [5].

They combine multiplier method and a suitable the choice of Lyapunov function and integral inequalities.
As a reminder, both polynomial and exponential stability results with distributed delay for the wave equation with dynamical control have been investigated respectively by S. Nicaise and Christina P.[10] and Roland S. and Gilbert B. [17] under the assumption

$$\int_{\tau_1}^{\tau_2} \mu_2(s)ds < \mu_1. \tag{7}$$

To our best knowledge, no paper has taken into account the presence of interior and boundary distributed delay for the micro-beam system. This is indeed the main motivation of the present article. Of course, needless to point out that it is practically impossible to avoid the time delay in the sensors and actuators when the objective is to stabilize the micro-beam system. In this paper, staying on the one dimensional space, we look for the possible ways to stabilize the systems (1) and (3) based on frequency domain approach for exponential stability (see Huang [8] and Prüss [14]).

The paper is organized as follows: section 2 is devoted to the well-posedness results for problems (1) and (3). In section 3 we prove the strong stability of problems (1) and (3). Finally in section 4, we establish the uniform stability of the above micro-beam systems (1) and (3) .

2. Semigroup formulation

Our aim in this section is to prove the global existence and the uniqueness of the solution of the micro-beam systems (1) and (3). We will first transform the systems (1) and (3) by making the change of variables and then we use the semigroup approach to prove the existence of the corresponding micro-beam systems. In this section we will give the well posedness for the problem (1) and (3) using the semigroup theory, and establish strong stability result. We start by considering the problem with boundary feedback.

2.1. Well-posedness of the microbeam system with boundary distributed delay. In order to put the system in an abstract framework, we introduce the auxiliary variable

$$z(r,t,s) = u_t(L,t-sr), \quad r \in (0,1),\ s \in (\tau_1,\tau_2),\ t>0. \tag{8}$$

Then the problem (1) admits the following equivalent formulation

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) = 0, & x\in(0,L), t>0,\\ s z_t(r,t,s) + z_r(r,t,s) = 0, & r\in(0,1), t>0,\\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t>0,\\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) - \beta_1 u_t(L,t) - \int_{\tau_1}^{\tau_2} \beta_2(s) z(1,t,s)ds = 0, & t>0,\\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0, & t>0,\\ M_2 u_{xxx}(L,t) = 0, & t>0,\\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x\in(0,L),\\ z(x,0,t,s) = u_t(x,t), & x\in(0,L), t>0,\\ z(r,0,s) = fr(L,-rs), & s\in(0,\tau_2). \end{cases} \tag{9}$$

Setting

$$\mathcal{U} = \Big(u, u_t, z\Big)^{\mathsf{T}}.$$

Then we have

$$\mathcal{U}_t = \Big(u_t, -\frac{1}{\rho A} M_1 u_{xxxx} + \frac{1}{\rho A} M_2 u_{xxxxxx}, -s^{-1} z_\rho\Big)^{\mathsf{T}}.$$

Therefore problem (9) can be rewritten in an abstract framework:

$$\begin{cases} \mathcal{U}_t = \mathcal{A}\mathcal{U} \\ \\ \mathcal{U}(0) = (u_0, u_1, f_0(-\cdot s))^{\mathsf{T}}, \end{cases} \tag{10}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\,(u,v,z)^{\mathsf{T}} = \left(v, -\frac{1}{\rho A} M_1 u_{xxxx} + \frac{1}{\rho A} M_2 u_{xxxxxx}, -s^{-1} z_\rho\right)^{\mathsf{T}},$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} (u,v,z)^{\mathsf{T}} \in \big(H^6(0,L)\cap V\big)\times V \times L^2((\tau_1,\tau_2);H^1(0,1)),\ z(0)=v(L) \\ \\ M_1 u_{xx}(L) - M_2 u_{xxxx}(L) = 0, M_2 u_{xxx}(L) = 0 \\ \\ M_1 u_{xxx}(L) - M_2 u_{xxxxx}(L) - \beta_1 v(L) - \int_{\tau_1}^{\tau_2} \beta_2(s) z(1,s)ds = 0 \end{array} \right\},$$

where

$$V = \left\{u \in H^3(0,L), u(0) = u_x(0) = u_{xx}(0) = 0\right\}.$$

Let us define he energy space $\mathcal{H}$ as follows

$$\mathcal{H} = V \times L^2(0,L)\times L^2((\tau_1,\tau_2);L^2(0,1))$$

with the natural associated inner product given by

$$\left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ z^* \end{pmatrix} \right\rangle_{\mathcal{H}} = \int_0^L \left(\rho A v\overline{v^*} + M_1 u_{xx}\overline{u^*_{xx}} + M_2 u_{xxx}\overline{u^*_{xxx}}\right) dx + \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 z(\rho,s)\overline{z^*(\rho,s)}\, d\rho \right) ds$$

for all $U = (u,v,z)^{\mathsf{T}}, U^* = (u^*,v^*,z^*)^{\mathsf{T}} \in \mathcal{H}$.

Proposition 2.1. *Assume that (2) holds, then for any initial datum $\mathcal{U}_0 \in \mathcal{H}$, there exists a unique solution $\mathcal{U} \in C\left([0,\infty); D(\mathcal{A})\right)$ of the system (10). Furthermore, if $\mathcal{U}_0 \in D(\mathcal{A})$, then $\mathcal{U} \in C\left([0,\infty); D(\mathcal{A})\right) \cap C^1\left([0,\infty); \mathcal{H}\right)$.*

Proof. We start with the dissipativeness of $\mathcal{A}$. Let $U = (u,v,z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$. Now using the definition of the inner product and the definition of the operator, we obtain that

$$\begin{aligned}
\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ -\frac{1}{\rho A}M_1 u_{xxxx} + \frac{1}{\rho A}M_2 u_{xxxxxx} \\ -s^{-1}z_\rho \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} \\
&= \rho A \int_0^L \left(-\frac{1}{\rho A}M_1 u_{xxxx} + \frac{1}{\rho A}M_2 u_{xxxxxx}\right)\overline{v}\, dx \\
&\quad + \int_0^L M_1 v_{xx}\overline{u_{xx}} + M_2 v_{xxx}\overline{u_{xxx}}\, dx - \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 z_\rho(\rho,s)\overline{z(\rho,s)}\, d\rho \right) ds \\
&= \int_0^L (-M_1 u_{xxxx} + M_2 u_{xxxxxx})\overline{v}\, dx + \int_0^L M_1 v_{xx}\overline{u_{xx}} + M_2 v_{xxx}\overline{u_{xxx}}\, dx \\
&\quad - \int_{\tau_1}^{\tau_2} \left(\beta_2(s) \int_0^1 z_\rho(\rho,s)\overline{z(\rho,s)}\, d\rho \right) ds.
\end{aligned}$$

Integrating by parts and using the boundary conditions, we get

$$\begin{aligned}
\Re\left\langle \mathcal{A}\begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u \\ v \\ z \end{pmatrix} \right\rangle_{\mathcal{H}} &= -\beta_1|v(L)|^2 - \Re\left(\left(\int_{\tau_1}^{\tau_2} \beta_2(s)\overline{v(L)}z(1,s)ds\right)\right) \\
&\quad -\frac{1}{2}\int_{\tau_1}^{\tau_2} \beta_2(s)\left|z(1,s)\right|^2 ds + \frac{1}{2}\int_{\tau_1}^{\tau_2} \beta_2(s)\left|z(0,s)\right|^2 ds \\
&\le -\beta_1\left|v(L)\right|^2 + \frac{1}{2}\int_{\tau_1}^{\tau_2} \beta_2(s)\left|z(1,s)\right|^2 ds + \frac{1}{2}\left|v(L)\right|^2 \int_{\tau_1}^{\tau_2} \beta_2(s)ds \\
&\quad -\frac{1}{2}\int_{\tau_1}^{\tau_2} \beta_2(s)\left|z(1,s)\right|^2 ds + \frac{1}{2}\left|v(L)\right|^2 \int_{\tau_1}^{\tau_2} \beta_2(s)ds \\
&\le -\beta_1\left|v(L)\right|^2 + \frac{1}{2}\left|v(L)\right|^2 \int_{\tau_1}^{\tau_2} \beta_2(s)ds + \frac{1}{2}\left|v(L)\right|^2 \int_{\tau_1}^{\tau_2} \beta_2(s)ds \\
&\quad -\frac{1}{2}\int_{\tau_1}^{\tau_2} \beta_2(s)\left|z(1,s)\right|^2 ds + \frac{1}{2}\int_{\tau_1}^{\tau_2} \beta_2(s)\left|z(1,s)\right|^2 ds \\
&\le \left(-\beta_1 + \int_{\tau_1}^{\tau_2} \beta_2(s)ds\right)\left|v(L)\right|^2. \qquad (11)
\end{aligned}$$

Now the relation (2) allows to conclude that $\mathcal{A}$ is dissipative.
In order to use the Lumer Philips theorem, we need to prove that the operator $\lambda I - \mathcal{A}$ is surjective for at least one $\lambda > 0$.

For $(f, g, h)^{\mathsf{T}} \in \mathcal{H}$, we look for $(u, v, z)^{\mathsf{T}} \in \mathcal{D}(\mathcal{A})$ solution of

$$\begin{cases} \lambda u - v = f & \text{in } (0, L) \\ \lambda v + \dfrac{1}{\rho A} M_1 u_{xxxx} - \dfrac{1}{\rho A} M_2 u_{xxxxxx} = g & \text{in } (0, L) \\ \lambda z + s^{-1} z_r = k & \text{in } (0, 1). \end{cases} \tag{12}$$

Suppose that we have found u with the appropriate regularity. Then $v = \lambda u - f$ and we can determine immediately z by solving the system

$$\begin{cases} s^{-1} z_r + \lambda z = k & \text{in } (0, 1) \\ z(0) = v(L). \end{cases} \tag{13}$$

We obtain

$$z(r, s) = v(L) e^{-\lambda s r} + s e^{-\lambda s r} \int_0^r k(\sigma, s) e^{\lambda s \sigma} \, d\sigma. \tag{14}$$

$$z(1, s) = v(L) e^{-\lambda s} + s e^{-\lambda s} \int_0^1 k(\sigma, s) e^{\lambda s \sigma} \, d\sigma. \tag{15}$$

The function u verifies now

$$\begin{cases} \dfrac{1}{\rho A} M_1 u_{xxxx} - \dfrac{1}{\rho A} M_2 u_{xxxxxx} + \lambda^2 u = g + \lambda f \text{ in } (0, L) \\ u(0) = u_x(0) = u_{xx}(0) = 0 \\ M_1 u_{xxx}(L) - M_2 u_{xxxxx}(L) - \beta_1 v(L) - \displaystyle\int_{\tau_1}^{\tau_2} \beta_2(s) z(1, s) ds = 0 \qquad t \in (0, +\infty) \\ M_1 u_{xx}(L) - M_2 u_{xxxx}(L) = 0 \\ M_2 u_{xxx}(L) = 0 \end{cases} \tag{16}$$

By using Lax-Milgram's Lemma, the problem (16) admits a unique weak solution. Indeed multiplying the first equation by $v \in V$ and by integrating formally by parts we get

$$a(u, v) = F(v), \forall\, v \in V, \tag{17}$$

where the bilinear and continuous form a is given by

$$a(u, w) = \int_0^L \Big(\frac{1}{\rho A} M_1 u_{xx} w_{xx} + \frac{1}{\rho A} M_2 u_{xxx} w_{xvx} + \lambda^2 u w \Big) dx + \left(\beta_1 + \int_{\tau_1}^{\tau_2} \beta_2(s) e^{-\lambda s} ds \right) \lambda u(L) w(L), \ \forall\, (u, w) \in V \times V$$

while the linear form F is

$$F(w) = \int_0^L (g + \lambda f) w \, dx - \left(\int_{\tau_1}^{\tau_2} \int_0^1 s \beta_2(s) k(\sigma, s) e^{\lambda s (\sigma - 1)} d\sigma ds \right) w(L), \quad \forall\, w \in V.$$

Since a is clearly strongly coercive on V and F is continuous on V, by Lax-Milgram's Lemma, problem (16) admits a unique solution $u \in V$. By taking test functions

$v \in \mathcal{D}(0;L)$, we recover the first equation of (16). This guarantees that u belongs to $H^6(0,L)$.

We conclude that the operator $\mathcal{A}$ is m-dissipative on $\mathcal{H}$ and it generates a $\mathcal{C}_0$ semigroup of contractions in $\mathcal{H}$, under Lumer-Phillips theorem. So, we have found $(u,v,z)^T \in \mathcal{D}(\mathcal{A})$ which verifies (16). □

2.2. Well-posedness of the microbeam system with internal distributed delay. Now, we consider the problem with internal feedback, namely the system (3). Let us set

$$z(x,r,t,s) = u_t(x,t-sr), \quad x \in (0,L), r \in (0,1),\ s \in (\tau_1,\tau_2),\ t > 0. \tag{18}$$

The problem (3) is now equivalent to

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) & \\ \quad + \beta_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} a(x)\beta_2(s) z(x,1,t,s)ds = 0, & x \in (0,L), t > 0 \\ s z_t(x,r,t,s) + z_r(x,r,t,s) = 0, & x \in (0,L), r \in (0,1),\ t > 0, \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0, & t > 0, \\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) = 0, & t > 0, \\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0, & t > 0, \\ M_2 u_{xxx}(L,t) = 0, & t > 0, \\ u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), & x \in (0,L) \\ z(x,0,t,s) = u_t(x,t), & x \in (0,L), t > 0, s \in (\tau_1,\tau_2), \\ z(x,r,0,s) = f_0(x,-rs), & x \in (0,L), s \in (0,\tau_2),\ r \in (0,1). \end{cases} \tag{19}$$

Setting

$$\mathcal{U} = \Big(u, u_t, z\Big)^{\mathsf{T}}.$$

Then we have

$$\begin{aligned} \mathcal{U}_t = & \Big(u_t, -\frac{1}{\rho A} M_1 u_{xxxx}(x,t) + \frac{1}{\rho A} M_2 u_{xxxxxx}(x,t) - \beta_1 u_t(x,t) \\ & - \int_{\tau_1}^{\tau_2} a(x)\beta_2(s) z(x,1,t,s)ds, -s^{-1} z_\rho\Big)^{\mathsf{T}}. \end{aligned}$$

Therefore problem (19) can be rewritten in an abstract framework:

$$\begin{cases} \mathcal{U}_t = \widehat{\mathcal{A}}\mathcal{U} \\ \mathcal{U}(0) = (u_0, u_1, f_0(-\cdot s))^{\mathsf{T}}, \end{cases} \tag{20}$$

where the operator $\widehat{\mathcal{A}}$ is defined by

$$\begin{aligned} \widehat{\mathcal{A}}(u,v,\eta,z)^{\mathsf{T}} = & \Big(v, -\frac{1}{\rho A} M_1 u_{xxxx} + \frac{1}{\rho A} M_2 u_{xxxxxx} - \frac{\beta_1}{\rho A} v \\ & - \frac{1}{\rho A}\int_{\tau_1}^{\tau_2} a(x)\beta_2(s) z(x,1,s)ds, -s^{-1} z_\rho\Big)^{\mathsf{T}}, \end{aligned}$$

with domain

$$\mathcal{D}(\widehat{\mathcal{A}}) = \left\{ \begin{array}{c} (u,v,z)^{\mathsf{T}} \in \left(H^6(0,L)\cap V\right) \times V \times L^2((\tau_1,\tau_2);H^1(0,1)),\ z(0)=v, \\ \\ M_1 u_{xx}(L) - M_2 u_{xxxx}(L) = 0, M_2 u_{xxx}(L) = 0, \\ \\ M_1 u_{xxx}(L) - M_2 u_{xxxxx}(L) = 0. \end{array} \right\},$$

where

$$V = \left\{ u \in H^3(0,L), u(0) = u_x(0) = u_{xx}(0) = 0 \right\}.$$

Let us now introduce the Hilbert space

$$\widehat{\mathcal{H}} = V \times L^2(0,L) \times L^2((\tau_1,\tau_2);L^2(0,1))$$

with the natural associated inner product

$$\begin{aligned} \left\langle \begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ z^* \end{pmatrix} \right\rangle_{\widehat{\mathcal{H}}} &= \int_0^L \left(\rho A v\overline{v^*} + M_1 u_{xx}\overline{u^*_{xx}} + M_2 u_{xxx}\overline{u^*_{xxx}}\right) dx \\ &+ \int_0^L a(x) \int_{\tau_1}^{\tau_2} \left(s\beta_2(s) \int_0^1 z(r,s)\overline{z^*(r,s)}\, dr \right) ds dx. \end{aligned}$$

Arguing analogously to the case of boundary feedback under the assumption (4), we show that the operator $\widehat{\mathcal{A}}$ defined above is m-dissipative on the energy space $\widehat{\mathcal{H}}$ and therefore generates a $\mathcal{C}_0$ semigroup of contractions in $\widehat{\mathcal{H}}$. In others words, for the sake of brevity in this paper , we omit the proof of the following proposition because the proof is very similar to the procedure for problem (10) .

Proposition 2.2. *Assume that (4) holds, then for any initial datum $\mathcal{U}_0 \in \widehat{\mathcal{H}}$, there exists a unique solution $\mathcal{U} \in C\left([0,\infty);D(\widehat{\mathcal{A}})\right)$ of the system (20). Furthermore, if $\mathcal{U}_0 \in D(\widehat{\mathcal{A}})$, then $\mathcal{U} \in C\left([0,\infty);D(\widehat{\mathcal{A}}))\right) \cap\ C^1([0,\infty);\mathcal{H})$.*

3. Strong stability

In this section we establish strong stability results and the main results of this subsection are the following.

Theorem 3.1. *The C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ is strongly stable on the energy space $\mathcal{H}$, that is for any $U_0 \in \mathcal{H}$,*

$$\lim_{t\longrightarrow\infty} \left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} = 0.$$

Proof of Theorem 3.1. We use the spectral decomposition theory of Sz-Nagy-Foias and Foguel [18, 5, 7]. Following this theory, since the resolvent of $\mathcal{A}$ is compact, it suffices to establish that $\mathcal{A}$ has no eigenvalue on the imaginary axis. For our purpose, it is easy to prove that the resolvent of the operator $\mathcal{A}$ defined in (10) is compact. We are ready now to achieve the proof of Theorem 3.1 with the following lemma.

Lemma 3.2. *There is no eigenvalue of $\mathcal{A}$ on the imaginary axis, that is*

$$i\mathbb{R} \subset \rho(\mathcal{A}).$$

Proof. By contradiction argument, we assume that there exists at least one $i\lambda \in \sigma(\mathcal{A})$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ on the imaginary axis. Let $U = (u, v, \eta, z)^{\mathsf{T}} \in D(\mathcal{A})$ be the corresponding normalized eigenvector, that is verifying $\|U\| = 1$ and

$$\mathcal{A}(u, v, z)^{\mathsf{T}} = i\lambda(u, v, z)^{\mathsf{T}}, \tag{21}$$

In the case of the microbeam system with boundary distributed delay, the system (21) is equivalent to

$$\begin{cases} v - i\lambda u = 0 \quad \text{in } (0, L) \\ -\dfrac{1}{\rho A} M_1 u_{xxxx} + \dfrac{1}{\rho A} M_2 u_{xxxxxx} - i\lambda v = 0 \quad \text{in } (0, L) \\ -s^{-1} z_r - i\lambda z = 0 \quad \text{in } (0, 1). \end{cases} \tag{22}$$

Recalling the dissipativity of $\mathcal{A}$ and basing on the result (11), it follows from (21) that

$$0 = \Re \left\langle \mathcal{A}(u, v, z)^{\mathsf{T}}, (u, v, z)^{\mathsf{T}} \right\rangle_{\mathcal{H}} \quad \leq \quad |v(L)|^2 \left(-\beta_1 + \int_{\tau_1}^{\tau_2} \beta_2(s) ds \right) \leq 0 \tag{23}$$

and we conclude that $v(L) = 0$.
Owing to the expression of z given in (14), we deduce that $z = 0$.

Now (22) becomes

$$\begin{cases} v - i\lambda u = 0 \quad \text{in } (0, L), \\ -\dfrac{1}{\rho A} M_1 u_{xxxx} + \dfrac{1}{\rho A} M_2 u_{xxxxxx} - i\lambda v = 0 \quad \text{in } (0, L). \end{cases} \tag{24}$$

From the first equation of (24) we deduce that

$$u(L) = 0.$$

Setting $v = i\lambda u$, it remains to find $u \in V$ which verifies

$$\begin{cases} -\dfrac{1}{\rho A} M_1 u_{xxxx} + \dfrac{1}{\rho A} M_2 u_{xxxxxx} + \lambda^2 u = 0 \quad \text{in } \ (0, L), \\ u(L) = 0, \ M_2 u_{xxx}(L) = 0, \\ M_1 u_{xx}(L) = M_2 u_{xxxx}(L) \ , M_1 u_{xxx}(L) - M_2 u_{xxxxx}(L) = 0, \\ u(0) = \ u_x(0) = \ u_{xx}(0) = 0. \end{cases} \tag{25}$$

Therefore, from the general theory of ordinary differential equations, we deduce that

$$u = 0, \quad \text{on } (0, L). \tag{26}$$

Now it follows that $(u, v, z)^{\mathsf{T}} = (0, 0, 0)^{\mathsf{T}}$ which contradicts the fact that $\|U\|_{\mathcal{H}} = 1$. We conclude that $\mathcal{A}$ has no eigenvalue on the imaginary axis. □

As the conditions of the spectral decomposition theory of Sz-Nagy-Foias and Foguel are full satisfied, the proof of Theorem 3.1 is thus completed. □

Theorem 3.3. *The C_0-semigroup $\left(e^{t\widehat{\mathcal{A}}}\right)_{t \geq 0}$ is strongly stable on the energy space $\widehat{\mathcal{H}}$, that is for any $U_0 \in \widehat{\mathcal{H}}$,*

$$\lim_{t \longrightarrow \infty} \left\| e^{t\widehat{\mathcal{A}}} U_0 \right\|_{\widehat{\mathcal{H}}} = 0.$$

Proof of Theorem 3.3. We omit the proof since it is analogous to the proof of Theorem 3.1. □

4. Exponential stability

In this section, under some assumptions we want to prove exponential stability result for the micro-beam systems (1) and (3). Our futur computations are based on frequency domain approach for exponential stability (see Huang [8] and Prüss [14]), more precisely on the below result.

Lemma 4.1. *A C_0-semigroup $\left(e^{t\mathcal{A}}\right)_{t\geq 0}$ of contractions on a Hilbert space $\mathcal{H}$ is exponentially stable, namely satisfies*

$$\left\|e^{t\mathcal{A}}U_0\right\|_{\mathcal{H}} \leq Ce^{-\omega t}\|U_0\|_{\mathcal{H}} \quad \forall\, U_0 \in \mathcal{H},\ \forall\, t \geq 0, \tag{27}$$

for some positive constants C and ω if and only if

$$\rho(\mathcal{A}) \supset \{i\beta,\ \beta \in \mathbb{R}\} \equiv i\mathbb{R} \tag{28}$$

and

$$\sup_{\beta\in\mathbb{R}} \left\|(i\beta - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{29}$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

4.1. Exponential stability of the microbeam system with boundary distributed delay.

The main result of current section is the following.

Theorem 4.2. *Assume that $(u_0, u_1, f_0)^\top \in \mathcal{D}(\mathcal{A})$. Then, the system (1) is exponentially stable in the energy space $\mathcal{H}$.*

Proof. As the condition (28) is guaranteed by Lemma 3.2, it suffices now to check the condition (29) in other words, the boundedness of the resolvent on the imaginary axis. For that, we will establish that for any $\lambda \in \mathbb{R}$ and $F = (f, g, h)^\top \in \mathcal{H}$, the solution $U = (u, v, z)^\top \in \mathcal{D}(\mathcal{A})$ of

$$(i\lambda I - \mathcal{A})\, U = F \tag{30}$$

satisfies

$$\|U\|_{\mathcal{H}} \leq C\|F\|_{\mathcal{H}} \tag{31}$$

where C is a positive constant (independent of λ and F).
Note that problem (1) without delay (corresponding to $\beta_2 = 0$) is the following one

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) = 0 \text{ in } (0,L)\times(0,+\infty) \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0 \qquad t \in (0,+\infty) \\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) - \beta_1 u_t(L,t) = 0 \qquad t \in (0,+\infty) \\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0 \qquad t \in (0,+\infty) \\ M_2 u_{xxx}(L,t) = 0 \qquad t \in (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad x \in (0,L). \end{cases} \tag{32}$$

This problem (32) is well-posed throughout its equivalent abstract formulation

$$\mathcal{A}_0\,(u,v)^{\mathsf{T}} = \left(v, -\frac{1}{\rho A}M_1 u_{xxxx} + \frac{1}{\rho A}M_2 u_{xxxxxx}\right)^{\mathsf{T}} \tag{33}$$

in the following Hilbert space

$$\mathcal{H}_0 = V \times L^2(0,L) \tag{34}$$

with domain

$$\mathcal{D}(\mathcal{A}_0) = \left\{ \begin{array}{c} (u,v)^{\mathsf{T}} \in \left(H^6(0,L)\cap V\right)\times V, \\ M_1 u_{xx}(L) - M_2 u_{xxxx}(L) = 0,\, M_2 u_{xxx}(L) = 0, \\ \\ M_1 u_{xxx}(L) - M_2 u_{xxxxx}(L) - \beta_1 v(L) = 0. \end{array} \right\} \tag{35}$$

and where $\mathcal{H}_0$ is endowed with the norm

$$\left\|(u,v)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 = M_1\|u_{xx}\|_{L^2(0,L)}^2 + \rho A\|v\|_{L^2(0,L)}^2 + M_2\|u_{xxx}\|_{L^2(0,L)}^2. \tag{36}$$

The system (32) has been studied in [16] by Vatankhah et al. where they proved that the operator $\mathcal{A}_0$ of the problem without delay generates an exponentially stable semigroup. So, according to this study we have $i\mathbb{R} \subset \rho(\mathcal{A}_0)$ and there exist a constant $C_0 > 0$ such that

$$\left\|(i\xi - \mathcal{A}_0)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_0)} \le C_0, \quad \forall\, \xi \in \mathbb{R}. \tag{37}$$

The relation (37) implies that the solution $U^* = (u^*, v^*)^{\mathsf{T}} \in \mathcal{D}\left(\mathcal{A}_0\right)$ of

$$(i\lambda I - \mathcal{A}_0)\begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{38}$$

verifies

$$\left\|(u^*,v^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0} \le C_0 \left\|(u,v)^{\mathsf{T}}\right\|_{\mathcal{H}_0}. \tag{39}$$

Also, the system (38) can be rewritten as

$$\left\{ \begin{array}{l} i\lambda u^* - v^* = u \\ i\lambda v^* + \frac{1}{\rho A}M_1 u^*_{xxxx} - \frac{1}{\rho A}M_2 u^*_{xxxxxx} = v \end{array} \right. . \tag{40}$$

However, we have

$$\begin{aligned}
\left\langle (i\lambda I - \mathcal{A})\begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \alpha z \end{pmatrix} \right\rangle_{\mathcal{H}} &= \int_0^L M_1\,(i\lambda u - v)_{xx}\,\overline{u^*_{xx}}dx + \int_0^L M_2\,(i\lambda u - v)_{xxx}\,\overline{u^*_{xxx}}dx \\
&\quad + \int_0^L \rho A\left(i\lambda v + \frac{1}{\rho A}M_1 u_{xxxx} - \frac{1}{\rho A}M_2 u_{xxxxxx}\right)\overline{v^*}dx \\
&\quad + \alpha\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 \left(i\lambda z + s^{-1}z_r\right)\overline{z}dr\right)ds.
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
\int_0^L M_1\,(i\lambda u - v)_{xx}\,\overline{u^*_{xx}}dx &= \int_0^L M_1 u_{xx}\overline{(-i\lambda u^*)}_{xx}dx - \int_0^L M_1 v\overline{u^*_{xxxx}}dx \\
&\quad - M_1 v_x(L)\overline{u^*_{xx}(L)} + M_1 v(L)\overline{u^*_{xxx}(L)},
\end{aligned}$$

$$
\begin{aligned}
\int_0^L M_2\left(i\lambda u-v\right)_{xxx}\overline{u^*_{xxx}}dx=&\int_0^L M_2u_{xxx}\overline{(-i\lambda u^*)}_{xxx}dx+\int_0^L M_2v\overline{u^*_{xxxxxx}}dx\\
&-M_2v_{xx}(L)\overline{u^*_{xxx}(L)}+M_2v_x(L)\overline{u^*_{xxxx}(L)}-M_2v(L)\overline{u^*_{xxxxx}(L)},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^L\rho A\left(i\lambda v+\frac{1}{\rho A}M_1u_{xxxx}-\frac{1}{\rho A}M_2u_{xxxxxx}\right)\overline{v^*}dx=&\int_0^L\rho Ai\lambda v\overline{v^*}dx+M_1u_{xxx}(L)\overline{v^*}(L)\\
&-M_1u_{xx}(L)\overline{v^*_x}(L)+\int_0^L M_1u_{xx}\overline{v^*_{xx}}dx-M_2u_{xxxxx}(L)\overline{v^*}(L)\\
&+M_2u_{xxxx}(L)\overline{v^*_x}(L)-M_2u_{xxx}(L)\overline{v^*_{xx}}(L)+\int_0^L M_1u_{xx}\overline{v^*_{xx}}dx
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1\left(i\lambda z+s^{-1}z_r\right)\overline{z}dr\right)ds=&i\lambda\alpha\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|z|^2dr\right)ds\\
&+\alpha\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)ds.
\end{aligned}
$$

Adding the last integrals and using the relations (40) and (36) , we obtain that

$$
\begin{aligned}
\left\langle(i\lambda I-\mathcal{A})\begin{pmatrix}u\\v\\z\end{pmatrix},\begin{pmatrix}u^*\\v^*\\\alpha z\end{pmatrix}\right\rangle_{\mathcal{H}}=&-\left\|(u,v)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0}+2\beta_1v(L)\overline{v^*(L)}+\int_{\tau_1}^{\tau_2}\beta_2(s)z(\cdot,1,s)\overline{v^*(L)}ds\\
&+i\lambda\alpha\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|z|^2dr\right)dsd\Gamma\\
&+\alpha\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)ds. \qquad (41)
\end{aligned}
$$

In the sequel we set $\alpha=-\dfrac{1}{\varepsilon}$. Then recalling (30) and taking the real part in (41), we obtain

$$
\begin{aligned}
\left\|(u,v)^{\mathsf{T}}\right\|^2_{\mathcal{H}_0}=&-\Re\left\langle F,\begin{pmatrix}u^*\\v^*\\-\frac{1}{\varepsilon}z\end{pmatrix}\right\rangle_{\mathcal{H}}+\Re\left(2\beta_1v(L)\overline{v^*(L)}\right)+\Re\left(\int_{\tau_1}^{\tau_2}\beta_2(s)z(\cdot,1,s)\overline{v^*(L)}ds\right)\\
&-\Re\left(\frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)ds\right). \qquad (42)
\end{aligned}
$$

Using (39) and the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
-\Re\left\langle F,\begin{pmatrix}u^*\\v^*\\-\frac{1}{\varepsilon}z\end{pmatrix}\right\rangle_{\mathcal{H}}\leq&\ \|F\|_{\mathcal{H}}\left\|(u^*,v^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}\left\|(0,0,z)^{\mathsf{T}}\right\|_{\mathcal{H}}\\
\leq&\ \|F\|_{\mathcal{H}}\left\|(u^*,v^*)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}\left\|(u,v,z)^{\mathsf{T}}\right\|_{\mathcal{H}}\\
\leq&\ C_0\|F\|_{\mathcal{H}}\left\|(u,v)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \qquad (43)
\end{aligned}
$$

Applying the Young's inequality one obtains

$$\Re\left(2\beta_1 v(L)\overline{v^*}(L)\right) \leq \frac{2\beta_1^2}{\varepsilon}|v(L)|^2 + \varepsilon|v^*(L)|^2, \quad \text{with } \varepsilon > 0. \tag{44}$$

From the dissipativeness of $\mathcal{A}$, we deduce using (30) and the Cauchy-Schwarz inequality that

$$\left(\beta_1 - \int_{\tau_1}^{\tau_2} \beta_2(s)ds\right)|v(L)|^2 \leq \Re\left\langle (i\lambda I - \mathcal{A})\,U, U\right\rangle_{\mathcal{H}} \leq \|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}\,. \tag{45}$$

Note further that (39) and the dissipativeness of $\mathcal{A}_0$ directly yield

$$\beta_1\,|v^*(L)|^2 \leq \Re\left\langle (i\lambda I - \mathcal{A}_0)\,U^*, U^*\right\rangle_{\mathcal{H}_0} \leq \left\|(u,v)^\top\right\|_{\mathcal{H}_0}\|U^*\|_{\mathcal{H}_0} \leq C_0\left\|(u,v)^\top\right\|_{\mathcal{H}_0}^2. \tag{46}$$

Consequently using (45) and (46) in (44), we get

$$\Re\left(2\beta_1 v(L)\overline{v^*(L)}\right) \leq \frac{2\beta_1^2}{\varepsilon\left(\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}} + \frac{\varepsilon C_0}{\beta_1}\left\|(u,v)^\top\right\|_{\mathcal{H}_0}^2. \tag{47}$$

Thanks to the Young's inequality, we get

$$\Re\left(\int_{\tau_1}^{\tau_2}\beta_2(s)\overline{v^*(L)}z(\cdot,1,s)ds\right) \leq \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,|z(\cdot,1,s)|^2\,ds + \varepsilon\int_{\tau_1}^{\tau_2}\beta_2(s)\,|v^*(L)|^2\,ds.$$

That is using (46)

$$\begin{aligned}\Re\left(\int_{\tau_1}^{\tau_2}\beta_2(s)\overline{v^*}z(\cdot,1,s)ds\right) &\leq \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,|z(\cdot,1,s)|^2\,ds \\ &+ \frac{\varepsilon C_0\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1}\left\|(u,v)^\top\right\|_{\mathcal{H}_0}^2.\end{aligned} \tag{48}$$

Furthermore, we have

$$\begin{aligned}-\frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_\rho\overline{z}d\rho\right)ds &= -\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,|z(\cdot,1,s)|^2\,ds \\ &+ \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,|v^*(L)|^2\,ds.\end{aligned}$$

Using (45), one can write

$$\begin{aligned}-\Re\left(\frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)ds\right) &\leq -\frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\beta_2(s)\,|z(\cdot,1,s)|^2\,ds \\ &+ \frac{\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}}\end{aligned} \tag{49}$$

Now adding (43), (47), (49) and (48) one gets

$$
\begin{aligned}
&\left\|(u,v)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 \\
&\leq C_0\left\|F\right\|_{\mathcal{H}}\left\|(u,v)^{\mathsf{T}}\right\|_{\mathcal{H}_0}+\frac{1}{\varepsilon}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}+\frac{2\beta_1^2}{\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}} \\
&\quad+\frac{\varepsilon C_0}{\beta_1}\left(1+\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)\left\|(u,v)^{\top}\right\|_{\mathcal{H}_0}^2+\frac{\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}.
\end{aligned}
$$

At this level we chose ε sufficiently small such that $\varepsilon \ll \dfrac{\beta_1}{C_0\left(1+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}$ to obtain

$$
\left\|(u,v)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2 \leq \left(C_0+\frac{1}{\varepsilon}+\frac{4\beta_1^2+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}. \tag{50}
$$

Since $\left\|(u,v,z)^{\mathsf{T}}\right\|_{\mathcal{H}}^2=\left\|(u,v)^{\mathsf{T}}\right\|_{\mathcal{H}_0}^2+\displaystyle\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1\left|z(\cdot,r,s)\right|^2dr\right)ds$, we deduce that

$$
\left\|U\right\|_{\mathcal{H}}^2 \leq \left(C_0+\frac{1}{\varepsilon}+\frac{4\beta_1^2+\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1-\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\left\|F\right\|_{\mathcal{H}}\left\|U\right\|_{\mathcal{H}}+\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1\left|z(\cdot,r,s)\right|^2dr\right)ds. \tag{51}
$$

Now we need a best estimation for $\displaystyle\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1\left|z(\cdot,r,s)\right|^2dr\right)ds$.

Following (30) and solving the next Cauchy problem (52)

$$
\begin{cases} s^{-1}z_r+i\lambda z=l \\ z(\cdot,0,s)=v(L) \end{cases} \tag{52}
$$

we obtain

$$
z(\cdot,r,s)=v(L)e^{-i\lambda sr}+s\int_0^r e^{-i\lambda s(r-\sigma)}l(\cdot,\sigma,s)\,d\sigma,\quad \forall\, r\in(0,1). \tag{53}
$$

Using the triangular inequality, it follows from (53) that

$$
\left|z(\cdot,r,s)\right|\leq\left|v(L)\right|+s\int_0^\rho\left|l(\cdot,\sigma,s)\right|d\sigma,\quad \forall\, r\in(0,1),
$$

which leads to

$$
\left|z(\cdot,r,s)\right|^2\leq\left|v(L)\right|^2+s^2\left(\int_0^\rho\left|l(\cdot,\sigma,s)\right|d\sigma\right)^2+2\left|v(L)\right|s\left(\int_0^\rho\left|l(\cdot,\sigma,s)\right|d\sigma\right),\ \forall\, r\in(0,1). \tag{54}
$$

On the one hand, by Cauchy-Schwarz's inequality we obtain

$$\begin{aligned}\left(\int_0^\rho |l(\cdot,\sigma,s)|\, d\sigma\right)^2 &\leq \left(\int_0^\rho |l(\cdot,\sigma,s)|^2\, d\sigma\right)\left(\int_0^\rho d\sigma\right)\\ &\leq \int_0^\rho |l(\cdot,\sigma,s)|^2\, d\sigma;\end{aligned}$$

that is

$$\left(\int_0^\rho |l(\cdot,\sigma,s)|\, d\sigma\right)^2 \leq \int_0^\rho |l(\cdot,\sigma,s)|^2\, d\sigma. \tag{55}$$

On the other hand, Young's inequality guarantees that

$$2\,|v(L)|\, s\left(\int_0^\rho |l(\cdot,\sigma,s)|\, d\sigma\right) \leq |v(L)|^2 + s^2\left(\int_0^\rho |l(\cdot,\sigma,s)|\, d\sigma\right)^2. \tag{56}$$

Combining (54), (55) and (56) it follows that

$$|z(\cdot,r,s)|^2 \leq 2\,|v(L)|^2 + 2s^2\int_0^\rho |l(\cdot,\sigma,s)|^2\, d\sigma. \tag{57}$$

Integrating (57) on $\times(\tau_1,\tau_2)\times(0,1)$ yields

$$\begin{aligned}&\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |z(\cdot,r,s)|^2 dr\right) ds\\ &\leq 2\int_{\tau_1}^{\tau_2} s\beta_2(s)\,|v(L)|^2\, ds + 2\int_{\tau_1}^{\tau_2} s^3\beta_2(s)\int_0^1 |l(\cdot,r,s)|^2\, dsdr\\ &\leq 2\tau_2\int_{\tau_1}^{\tau_2}\beta_2(s)ds\int_{\Gamma_N}|v|^2 + 2\tau_2^2\int_{\tau_1}^{\tau_2} s\beta_2(s)\int_0^1 |l(\cdot,r,s)|^2\, dsdr.\end{aligned}$$

Then using (45) and the $\mathcal{H}$-norm definition, the above relation can be rewritten as

$$\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |z(\cdot,r,s)|^2 dr\right) ds \leq \left(\frac{2\tau_2\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}\right)\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}} + 2\tau_2^2\,\|F\|_{\mathcal{H}}^2. \tag{58}$$

Putting (58) in (82), it follows that

$$\|U\|_{\mathcal{H}}^2 \leq \left(C_0 + \frac{1}{\varepsilon} + \frac{4\beta_1^2 + (1+4\varepsilon\tau_2)\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}} + 2\tau_2^2\,\|F\|_{\mathcal{H}}^2$$

that is

$$\|U\|_{\mathcal{H}}^2 \leq C_\varepsilon\,\|F\|_{\mathcal{H}}\,\|U\|_{\mathcal{H}} + 2\tau_2^2\,\|F\|_{\mathcal{H}}^2 \tag{59}$$

where C_ε is a positive constant which doesn't depend on λ. More precisely,

$$C_\varepsilon = C_0 + \frac{1}{\varepsilon} + \frac{4\beta_1^2 + (1+4\varepsilon\tau_2)\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}. \tag{60}$$

Applying Young's inequality to (59) it follows that

$$\|U\|_{\mathcal{H}}^2 \leq \frac{C_\varepsilon}{2\varepsilon'}\|F\|_{\mathcal{H}}^2 + \frac{\varepsilon' C_\varepsilon}{2}\|U\|_{\mathcal{H}}^2 + 2\tau_2^2\|F\|_{\mathcal{H}}^2, \quad \text{with } \varepsilon' > 0. \tag{61}$$

One can choose ε' small enough such that $\dfrac{\varepsilon' C_\varepsilon}{2} < 1$. Consequently, (61) becomes

$$\|U\|_{\mathcal{H}}^2 \leq C_{\varepsilon\varepsilon'}\|F\|_{\mathcal{H}}^2, \tag{62}$$

where one sets

$$C_{\varepsilon\varepsilon'} = \frac{\dfrac{C_\varepsilon}{2\varepsilon'} + 2\tau_2^2}{1 - \dfrac{\varepsilon' C_\varepsilon}{2}}. \tag{63}$$

Finally (62) directly leads to (31) with

$$C = \sqrt{C_{\varepsilon\varepsilon'}}. \tag{64}$$

That means the resolvent of $\mathcal{A}$ is uniformly bounded on the imaginary axis. The proof of theorem 4.3 is thus completed. □

4.2. Exponential stability of the microbeam system with internal distributed delay.

Theorem 4.3. *Assume that $(u_0, u_1, f_0)^\top \in \mathcal{D}(\widehat{\mathcal{A}})$. Then, the system (3) is exponentially stable in the energy space $\widehat{\mathcal{H}}$.*

Proof. As the condition (28) is guaranteed by Lemma 3.2, it suffices now to check the condition (29) in other words, the boundedness of the resolvent on the imaginary axis. For that, we will establish that for any $\lambda \in \mathbb{R}$ and $F = (f, g, h)^\top \in \widehat{\mathcal{H}}$, the solution $U = (u, v, z)^\top \in \mathcal{D}(\widehat{\mathcal{A}})$ of

$$\left(i\lambda I - \widehat{\mathcal{A}}\right)U = F \tag{65}$$

satisfies

$$\|U\|_{\mathcal{H}} \leq C\|F\|_{\widehat{\mathcal{H}}} \tag{66}$$

where C is a positive constant (independent of λ and F).
Problem (3) without delay (corresponding to $\beta_2 = 0$) is the following one

$$\begin{cases} \rho A u_{tt}(x,t) + M_1 u_{xxxx}(x,t) - M_2 u_{xxxxxx}(x,t) + \beta_1 u_t(x,t) = 0 \text{ in } (0,L)\times(0,+\infty) \\ u(0,t) = u_x(0,t) = u_{xx}(0,t) = 0 \qquad t \in (0,+\infty) \\ M_1 u_{xxx}(L,t) - M_2 u_{xxxxx}(L,t) = 0 \qquad t \in (0,+\infty) \\ M_1 u_{xx}(L,t) - M_2 u_{xxxx}(L,t) = 0 \qquad t \in (0,+\infty) \\ M_2 u_{xxx}(L,t) = 0 \qquad t \in (0,+\infty) \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \qquad x \in (0,L) \end{cases} \tag{67}$$

System (67) is well-posed and equivalent to the abstract formulation

$$\widehat{\mathcal{A}}_0\,(u,v)^{\mathsf{T}} = \left(v, -\frac{1}{\rho A}M_1 u_{xxxx} + \frac{1}{\rho A}M_2 u_{xxxxxx} - \frac{\beta_1}{\rho A}v\right)^{\mathsf{T}} \tag{68}$$

with domain

$$\mathcal{D}(\widehat{\mathcal{A}}_0) = \left\{ \begin{array}{c} (u,v)^{\mathsf{T}} \in \left(H^6(0,L)\cap V\right)\times V, \\ \\ M_1 u_{xx}(L) - M_2 u_{xxxx}(L) = 0, M_2 u_{xxx}(L) = 0, \\ \\ M_1 u_{xxx}(L) - M_2 u_{xxxxx}(L) = 0 \end{array} \right\}. \tag{69}$$

in the Hilbert space

$$\widehat{\mathcal{H}}_0 = V \times L^2(0,L) \tag{70}$$

endowed with the norm

$$\left\|(u,v)^{\mathsf{T}}\right\|^2_{\widehat{\mathcal{H}}_0} = M_1\|u_{xx}\|^2_{L^2(0,L)} + \rho A\|v\|^2_{L^2(0,L)} + M_2\|u_{xxx}\|^2_{L^2(0,L)}. \tag{71}$$

The system (67) has been studied in [16] by Vatankhah et al. where they proved that the operator $\widehat{\mathcal{A}}_0$ of the problem (67) of the problem without delay generates an exponentially stable semigroup. So, according to this study we have $i\mathbb{R} \subset \rho(\widehat{\mathcal{A}}_0)$ and there exists a constant $C_0 > 0$ such that

$$\left\|\left(i\xi - \widehat{\mathcal{A}}_0\right)^{-1}\right\|_{\mathcal{L}(\widehat{\mathcal{H}}_0)} \le C_0, \quad \forall\, \xi \in \mathbb{R}. \tag{72}$$

The relation (72) implies that the solution $U^* = (u^*, v^*)^{\mathsf{T}} \in \mathcal{D}\left(\widehat{\mathcal{A}}_0\right)$ of

$$(i\lambda I - \widehat{\mathcal{A}}_0)\begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{73}$$

verifies

$$\left\|(u^*, v^*)^{\mathsf{T}}\right\|_{\widehat{\mathcal{H}}_0} \le C_0 \left\|(u,v)^{\mathsf{T}}\right\|_{\widehat{\mathcal{H}}_0}. \tag{74}$$

Also, the system (73) can be rewritten as

$$\left\{ \begin{array}{l} i\lambda u^* - v^* = u \\ \\ i\lambda v^* + \frac{1}{\rho A}M_1 u^*_{xxxx} - \frac{1}{\rho A}M_2 u^*_{xxxxxx} + \frac{\beta_1}{\rho A}v^* = v \end{array} \right. . \tag{75}$$

We have

$$\begin{aligned} \left\langle (i\lambda I - \widehat{\mathcal{A}})\begin{pmatrix} u \\ v \\ z \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \\ \alpha z \end{pmatrix} \right\rangle_{\widehat{\mathcal{H}}} &= \int_0^L M_1\,(i\lambda u - v)_{xx}\,\overline{u^*_{xx}}dx + \int_0^L M_2\,(i\lambda u - v)_{xxx}\,\overline{u^*_{xxx}}dx \\ &+ \int_0^L \rho A\left(i\lambda v + \frac{1}{\rho A}M_1 u_{xxxx} - \frac{1}{\rho A}M_2 u_{xxxxxx} + \frac{\beta_1}{\rho A}v\right)\overline{v^*}dx \\ &+ \int_0^L \left(\int_{\tau_1}^{\tau_2} a(x)\beta_2(s)z(x,1,s)ds\right)\overline{v^*}dx \\ &+ \alpha\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 \left(i\lambda z + s^{-1}z_r\right)\bar{z}dr\right)dsdx. \end{aligned}$$

Integrating by parts, we get

$$\int_0^L M_1\left(i\lambda u-v\right)_{xx}\overline{u^*_{xx}}dx=\int_0^L M_1u_{xx}\overline{(-i\lambda u^*)}_{xx}dx-\int_0^L M_1v\overline{u^*_{xxxx}}dx$$
$$-M_1v_x(L)\overline{u^*_{xx}(L)}+M_1v(L)\overline{u^*_{xxx}(L)},$$

$$\int_0^L M_2\left(i\lambda u-v\right)_{xxx}\overline{u^*_{xxx}}dx=\int_0^L M_2u_{xxx}\overline{(-i\lambda u^*)}_{xxx}dx+\int_0^L M_2v\overline{u^*_{xxxxxx}}dx$$
$$-M_2v_{xx}(L)\overline{u^*_{xxx}(L)}+M_2v_x(L)\overline{u^*_{xxxx}(L)}-M_2v(L)\overline{u^*_{xxxxx}(L)},$$

$$\int_0^L \rho A\left(i\lambda v+\frac{1}{\rho A}M_1u_{xxxx}-\frac{1}{\rho A}M_2u_{xxxxxx}\right)\overline{v^*}dx$$
$$=\int_0^L \rho Ai\lambda v\overline{v^*}dx+M_1u_{xxx}(L)\overline{v^*}(L)-M_1u_{xx}(L)\overline{v^*_x}(L)+\int_0^L M_1u_{xx}\overline{v^*_{xx}}dx$$
$$-M_2u_{xxxxx}(L)\overline{v^*}(L)+M_2u_{xxxx}(L)\overline{v^*_x}(L)-M_2u_{xxx}(L)\overline{v^*_{xx}}(L)+\int_0^L M_1u_{xx}\overline{v^*_{xx}}dx$$

and

$$\alpha\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1\left(i\lambda z+s^{-1}z_r\right)\overline{z}dr\right)dsdx$$
$$=i\lambda\alpha\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|z|^2dr\right)dsdx+\alpha\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)dsdx.$$

Adding the last integrals and using the (75) and (71) , we obtain that

$$\begin{aligned}\left\langle (i\lambda I-\widehat{\mathcal{A}})\begin{pmatrix}u\\ v\\ z\end{pmatrix},\begin{pmatrix}u^*\\ v^*\\ \alpha z\end{pmatrix}\right\rangle_{\widehat{\mathcal{H}}} &= -\left\|(u,v)^{\mathsf{T}}\right\|^2_{\widehat{\mathcal{A}}_0}+\int_0^L\left(\int_{\tau_1}^{\tau_2}a(x)\beta_2(s)z(x,1,s)ds\right)\overline{v^*}dx\\ &+ i\lambda\alpha\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1|z|^2dr\right)dsdx\\ &+ \alpha\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)dsdx. \end{aligned}\tag{76}$$

In the sequel we set $\alpha=-\dfrac{1}{\varepsilon}$. Now taking the real part in (76), we obtain

$$\begin{aligned}\left\|(u,v)^{\mathsf{T}}\right\|^2_{\widehat{\mathcal{H}}_0} &= -\Re\left\langle F,\begin{pmatrix}u^*\\ v^*\\ -\frac{1}{\varepsilon}z\end{pmatrix}\right\rangle_{\widehat{\mathcal{H}}}+\Re\left(\int_0^L\left(\int_{\tau_1}^{\tau_2}a(x)\beta_2(s)z(x,1,s)ds\right)\overline{v^*}dx\right)\\ &- \Re\left(\frac{1}{\varepsilon}\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)dsdx\right). \end{aligned}\tag{77}$$

Using Cauchy-Schwarz's inequality we have

$$-\Re\left\langle F,\begin{pmatrix}u^*\\ v^*\\ -\frac{1}{\varepsilon}z\end{pmatrix}\right\rangle_{\widehat{\mathcal{H}}}\leq C_0\left\|F\right\|_{\widehat{\mathcal{H}}}\left\|(u,v)^{\mathsf{T}}\right\|_{\widehat{\mathcal{H}}_0}+\frac{1}{\varepsilon}\left\|F\right\|_{\widehat{\mathcal{H}}}\left\|U\right\|_{\widehat{\mathcal{H}}}.\tag{78}$$

Using (74) and Young's inequality we obtain

$$\Re\left(\int_0^L \left(\int_{\tau_1}^{\tau_2} a(x)\beta_2(s)z(x,1,s)ds\right)\overline{v^*}dx\right)$$
$$\leq \frac{1}{2\varepsilon}\int_0^L\int_{\tau_1}^{\tau_2} a(x)\beta_2(s)\left|z(x,1,s)\right|^2 dsdx + \frac{\varepsilon C_0 \left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1}\left\|(u,v)^\top\right\|^2_{\mathcal{H}_0}. \tag{79}$$

Using the dissipativeness of $\widehat{\mathcal{A}}_0$, one can write

$$\Re\left(-\frac{1}{\varepsilon}\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(\beta_2(s)\int_0^1 z_r\overline{z}dr\right)dsdx\right)$$
$$\leq -\frac{1}{2\varepsilon}\int_0^L\int_{\tau_1}^{\tau_2} a(x)\beta_2(s)\left|z(x,1,s)\right|^2 dsdx + \frac{\left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\widehat{\mathcal{H}}}\left\|U\right\|_{\widehat{\mathcal{H}}}. \tag{80}$$

Now adding (78), (79) and (80) one gets

$$\begin{aligned}\left\|(u,v)^\top\right\|^2_{\widehat{\mathcal{H}}_0} &\leq C_0\left\|F\right\|_{\mathcal{H}}\left\|(u,v)^\top\right\|_{\widehat{\mathcal{H}}_0} + \frac{1}{\varepsilon}\left\|F\right\|_{\widehat{\mathcal{H}}}\left\|U\right\|_{\widehat{\mathcal{H}}}\\ &+ \frac{2\varepsilon C_0\left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1}\left\|(u,v)^\top\right\|^2_{\widehat{\mathcal{H}}_0}\\ &+ \frac{\left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\left\|F\right\|_{\widehat{\mathcal{H}}}\left\|U\right\|_{\widehat{\mathcal{H}}}.\end{aligned}$$

For ε sufficiently small that is $\varepsilon \ll \dfrac{2\varepsilon C_0\left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1}$, we obtain

$$\left\|(u,v)^\top\right\|^2_{\widehat{\mathcal{H}}_0} \leq \left(C_0 + \frac{1}{\varepsilon} + \frac{\left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \left\|a\right\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\left\|F\right\|_{\widehat{\mathcal{H}}}\left\|U\right\|_{\widehat{\mathcal{H}}}. \tag{81}$$

Since $\left\|(u,v,z)^{\mathsf{T}}\right\|_{\widehat{\mathcal{H}}}^2 = \left\|(u,v)^{\mathsf{T}}\right\|_{\widehat{\mathcal{H}}_0}^2 + \int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |z(x,r,s)|^2 dr\right) dsdx,$ we deduce that

$$\begin{aligned}\|U\|_{\widehat{\mathcal{H}}}^2 &\leq \left(C_0 + \frac{1}{\varepsilon} + \frac{\|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\|F\|_{\widehat{\mathcal{H}}}\|U\|_{\widehat{\mathcal{H}}} \\ &+ \int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |z(x,r,s)|^2 dr\right) dsdx. \end{aligned} \tag{82}$$

Now we need a best estimation for $\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |z(x,r,s)|^2 dr\right) dsdx.$ Solving the next Cauchy problem

$$\begin{cases} s^{-1}z_r + i\lambda z = h \\ z(\cdot,0,s) = v(x) \end{cases} \tag{83}$$

we obtain

$$z(\cdot,r,s) = v(x)e^{-i\lambda sr} + s\int_0^r e^{-i\lambda s(r-\sigma)}h(\cdot,\sigma,s)\,d\sigma, \quad \forall\, r \in (0,1)\,. \tag{84}$$

Arguing analogously to the previous case, we can show that

$$\begin{aligned}&\int_0^L a(x)\int_{\tau_1}^{\tau_2}\left(s\beta_2(s)\int_0^1 |z(\cdot,r,s)|^2 dr\right) dsdx \\ &\leq \left(\frac{2\tau_2\|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{\beta_1 - \|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}\right)\|F\|_{\widehat{\mathcal{H}}}\|U\|_{\widehat{\mathcal{H}}} + 2\tau_2^2\|F\|_{\widehat{\mathcal{H}}}^2\,. \end{aligned} \tag{85}$$

Putting (85) in (82), it follows that

$$\|U\|_{\widehat{\mathcal{H}}}^2 \leq \left(C_0 + \frac{1}{\varepsilon} + \frac{(1+4\varepsilon\tau_2)\|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}\right)\|F\|_{\widehat{\mathcal{H}}}\|U\|_{\widehat{\mathcal{H}}} + 2\tau_2^2\|F\|_{\widehat{\mathcal{H}}}^2$$

that is

$$\|U\|_{\widehat{\mathcal{H}}}^2 \leq C_\varepsilon\|F\|_{\widehat{\mathcal{H}}}\|U\|_{\widehat{\mathcal{H}}} + 2\tau_2^2\|F\|_{\widehat{\mathcal{H}}}^2\,, \tag{86}$$

where C_ε is a positive constant which doesn't depend on λ. More precisely,

$$C_\varepsilon = C_0 + \frac{1}{\varepsilon} + \frac{(1+4\varepsilon\tau_2)\|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds}{2\varepsilon\left(\beta_1 - \|a\|_{L^\infty(0,L)}\displaystyle\int_{\tau_1}^{\tau_2}\beta_2(s)ds\right)}. \tag{87}$$

Applying Young's inequality to (86) it follows that

$$\|U\|_{\widehat{\mathcal{H}}}^2 \leq \frac{C_\varepsilon}{2\varepsilon'}\|F\|_{\widehat{\mathcal{H}}}^2 + \frac{\varepsilon' C_\varepsilon}{2}\|U\|_{\widehat{\mathcal{H}}}^2 + 2\tau_2^2\|F\|_{\widehat{\mathcal{H}}}^2\,, \quad \text{with } \varepsilon' > 0. \tag{88}$$

One can choose ε' small enough such that $\dfrac{\varepsilon' C_\varepsilon}{2} < 1$. Consequently, (88) becomes

$$\|U\|^2_{\widehat{\mathcal{H}}} \leq C_{\varepsilon\varepsilon'} \|F\|^2_{\widehat{\mathcal{H}}}, \tag{89}$$

where one sets

$$C_{\varepsilon\varepsilon'} = \frac{\dfrac{C_\varepsilon}{2\varepsilon'} + 2\tau_2^2}{1 - \dfrac{\varepsilon' C_\varepsilon}{2}}. \tag{90}$$

Finally (89) directly leads to (66) with

$$C = \sqrt{C_{\varepsilon\varepsilon'}}. \tag{91}$$

That means the resolvent of $\widehat{\mathcal{A}}$ is uniformly bounded on the imaginary axis. The proof of Theorem 4.3 is thus completed. □

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