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Almost Lacunary Statistical and Strongly Almost Lacunary Convergence of order (β, γ) of Sequences of Fuzzy Numbers

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Abstract. The main purpose of this article is to introduce the concepts of almost lacunary statistical convergence and strongly almost lacunary convergence of order (β, γ) of sequences of fuzzy numbers with respect to an Orlicz function. We give some relations between strongly almost lacunary convergence and almost lacunary statistical convergence of order (β, γ) of sequences of fuzzy numbers, where β and γ are two fixed real numbers such that $0 < \beta \leq \gamma \leq 1$.

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1. Introduction, definitions and preliminaries

The concept of fuzzy set was introduced by Zadeh [30]. Matloka [23] introduced sequences of fuzzy numbers and provided that every convergent sequence of fuzzy numbers is bounded. Recently, sequences of fuzzy numbers have been discussed by Altın [1], Altınok *et al.* [2], Aytar and Pehlivan [3], Başarır et al. [4], Braha and Et [5], Işık and Et [20], Nuray [25] and many others.

The idea of statistical convergence was introduced by Fast [12] and the notion was linked with summability theory by Çınar et al. [6], Connor [8], Et *et al.* ([9],[10],[11],[26]), Fridy [14], Gadjiev and Orhan [16], Işık et al. ([17],[18],[19]), Mohiuddine *et al.* [24], Şengül ([28],[29]). Recently, the notion was generalized by Çolak [7].

A fuzzy set u on $\mathbb{R}$ is called a fuzzy number if it has the following properties:

i) u is normal,

ii) u is fuzzy convex,

iii) u is upper semicontinuous,

iv) $\operatorname{supp} u = cl\{x \in \mathbb{R} : u(x) > 0\}$ is compact, where cl denoted the closure of the enclosed set.

$\alpha-$level set $[u]^\alpha$ of a fuzzy number u is defined by

$$[u]^\alpha = \begin{cases} \{x \in \mathbb{R} : u(x) \geq \alpha\}, & \text{if } \alpha \in (0,1] \\ \operatorname{supp} u, & \text{if } \alpha = 0 \end{cases}.$$

It is clear that u is a fuzzy number if and only if $[u]^\alpha$ is a closed interval for each $\alpha \in [0,1]$ and $[u]^1 \neq \emptyset$. We denote space of all fuzzy numbers by $L(\mathbb{R})$.

Let u and v be two fuzzy numbers, then we calculate the distance between u and v by

$$d(u,v) = \sup_{0 \leq \alpha \leq 1} d_H\left([u]^\alpha, [v]^\alpha\right),$$

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where d_H is the Hausdorff metric defined by

$$d_H\left([u]^{\alpha},[v]^{\alpha}\right)=\max\left\{\left|\underline{u}^{\alpha}-\underline{v}^{\alpha}\right|,\left|\overline{u}^{\alpha}-\overline{v}^{\alpha}\right|\right\}.$$

It can be shown that $(L(\mathbb{R}),d)$ is a complete metric space.

A sequence $X=(X_k)$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of all positive integers into $L(\mathbb{R})$. A sequence $X=(X_k)$ is said to be bounded if the set $\{X_k : k\in\mathbb{N}\}$ is bounded. A sequence $X=(X_k)$ is said to be convergent if there exists a positive integer k_0 such that $d\left(X_k,X_0\right)<\varepsilon$ for $k>k_0$, for every $\varepsilon>0$. By $w^{\mathcal{F}},\ell_{\infty}^{\mathcal{F}}$ and $c^{\mathcal{F}}$ we denote the set of *all*, *bounded* and *convergent* sequences of fuzzy numbers, respectively [23].

An Orlicz function is a function $N:[0,\infty)\to[0,\infty)$, which is continuous, non decreasing and convex with $N(0)=0$, $N(x)>0$ for $x>0$ and $N(x)\to\infty$ as $x\to\infty$.

By a lacunary sequence we mean an increasing integer sequence $\theta=(k_r)$ of non-negative integers such that $k_0=0$ and $h_r=(k_r-k_{r-1})\to\infty$ as $r\to\infty$. The intervals determined by θ will be denoted by $I_r=(k_{r-1},k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1=k_1$ for convenience. In recent years, lacunary sequences have been studied in ([13],[15],[25],[26],[27]).

The space $\hat{c}$ was introduced by Lorentz [21] and Maddox [22] has defined x to be strongly almost convergent to a number L if $\lim\limits_{m\to\infty}\frac{1}{m}\sum\limits_{k=1}^{m}\left|x_{k+n}-L\right|=0$, uniformlyin n.

Let $\theta=(k_r)$ be a lacunary sequence, β and γ are two fixed real numbers such that $0<\beta\leq\gamma\leq 1$, lacunary (β,γ) −density of the set $E\subset\mathbb{N}$ is defined by

$$\delta_{\theta}^{(\beta,\gamma)}(E)=\lim_{r\to\infty}\frac{1}{h_r^{\beta}}\left|\{k\in I_r : k\in E\}\right|^{\gamma}.$$

In cases of $\gamma=1,\beta=1$ and $\theta=(2^r)$, the lacunary (β,γ) −density reduces the natural density.

In this article, we study the concepts of almost lacunary statistical convergence and strongly almost lacunary convergence of order (β,γ) of sequences of fuzzy numbers with respect to an Orlicz function and examine some properties of almost lacunary statistical convergence and strongly almost lacunary convergence of order (β,γ) of sequences of fuzzy numbers.

2. Main results

In this section we give the main results of this paper.

Definition 2.1 Let $\theta=(k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0<\beta\leq\gamma\leq 1$. A sequence $X=(X_k)$ is said to be almost lacunary statistically convergent of order (β,γ) to X_0, with respect to the Orlicz function N, if for every $\varepsilon>0$

$$\lim_{r\to\infty}\frac{1}{h_r^{\beta}}\left|\left\{k\in I_r : \left[N\left(\frac{d\left(t_{kn}\left(X\right),X_0\right)}{s}\right)\right]\geq\varepsilon\right\}\right|^{\gamma}=0,\ \text{uniformly in } n \text{ and } s>0$$

where

$$t_{kn}\left(X\right)=\frac{X_n+X_{n+1}+...+X_{n+k}}{k+1}=\frac{1}{k+1}\sum_{i=0}^{k}X_{n+i}\ .$$

The set of all almost lacunary statistically convergent sequences of order (β,γ) with respect to the Orlicz function N will be denoted by $\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$. In this case we write $X_k \to X_0\left(\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)\right)$. In case of $\theta=(2^r)$, for all $r\in\mathbb{N}$ we shall write $\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N)$ instead of $\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$ and $N(x)=x$, we shall write $\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(\theta)$ instead of $\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$.

Definition 2.2 Let $\theta=(k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0<\beta\le\gamma\le 1$. We define the following sets

$$\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)=\left\{\begin{array}{c} X\in w^{\mathcal{F}}:\lim_{r\to\infty}\frac{1}{h_r^{\beta}}\left[\sum\limits_{k\in I_r}N\left(\dfrac{d\left(t_{kn}(X),\ X_0\right)}{s}\right)\right]^{\gamma}=0, \\ \text{uniformly in } n \text{ and } s>0\end{array}\right\},$$

$$\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_0=\left\{\begin{array}{c} X\in w^{\mathcal{F}}:\lim_{r\to\infty}\frac{1}{h_r^{\beta}}\left[\sum\limits_{k\in I_r}N\left(\dfrac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^{\gamma}=0, \\ \text{uniformly in } n \text{ and } s>0\end{array}\right\},$$

$$\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_\infty=\left\{X\in w^{\mathcal{F}}:\sup_{r,n}\frac{1}{h_r^{\beta}}\left[\sum_{k\in I_r}N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^{\gamma}<\infty,\qquad s>0\right\},$$

If $X\in\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$, we say that X is strongly almost lacunary convergent of order (β,γ) with respect to the Orlicz function N. In this case we write $X_k\to X_0\left(\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)\right)$.

i) If we take $N(x)=x$ then we get $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)=\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(\theta)$, $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_0=\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(\theta)_0$ and $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_\infty=\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(\theta)_\infty$,

ii) If we take $\beta=\gamma=1$ then we get $\hat{w}^{\mathcal{F}}(N,\theta)=\hat{w}^{\mathcal{F}}(\theta)$, $\hat{w}^{\mathcal{F}}(N,\theta)_0=\hat{w}^{\mathcal{F}}(\theta)_0$ and $\hat{w}^{\mathcal{F}}(N,\theta)_\infty=\hat{w}^{\mathcal{F}}(\theta)_\infty$,

i) If we take $\theta=(2^r)$then we get $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)=\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N)$, $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_0=\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N)_0$ and $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_\infty=\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N)_\infty$.

The proof of each of the following results are straightforward, so we state these.

Theorem 2.3 Let $\theta=(k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0<\beta\le\gamma\le 1$, then $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_0\subset\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)\subset\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_\infty$.

Theorem 2.4 Let $\theta=(k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0<\beta\le\gamma\le 1$, then $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_0$, $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$, $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_\infty$ and $\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$ are closed under the operations of addition and scalar multiplication.

Theorem 2.5 Let $\theta=(k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0<\beta\le\gamma\le 1$, then

i) $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_1,\theta)_0\cap\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_2,\theta)_0\subset\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_1+N_2,\theta)_0$,

ii) $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_1,\theta)\cap\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_2,\theta)\subset\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_1+N_2,\theta)$,

iii) $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_1,\theta)_\infty\cap\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_2,\theta)_\infty\subset\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N_1+N_2,\theta)_\infty$.

Theorem 2.6 Let $\theta = (k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0 < \beta \leq \gamma \leq 1$. If $\liminf_r \frac{h_r^\beta}{r^\beta} > 0$, then $\hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N) \subset \hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$.

Proof. For given $\varepsilon > 0$ we have

$$\left\{k \leq k_r : \left[N\left(\frac{d\left(t_{kn}(X), X_0\right)}{s}\right)\right]^\gamma \geq \varepsilon\right\} \supset \left\{k \in I_r : \left[N\left(\frac{d\left(t_{kn}(X), X_0\right)}{s}\right)\right]^\gamma \geq \varepsilon\right\}.$$

and so

$$\begin{aligned} &\frac{1}{r^\beta}\left|\left\{k \leq k_r : \left[N\left(\frac{d\left(t_{kn}(X), X_0\right)}{s}\right)\right]^\gamma \geq \varepsilon\right\}\right| \\ &\quad \geq \frac{h_r^\beta}{r^\beta}\frac{1}{h_r^\beta}\left|\left\{k \leq k_r : \left[N\left(\frac{d\left(t_{kn}(X), X_0\right)}{s}\right)\right]^\gamma \geq \varepsilon\right\}\right|. \end{aligned}$$

Hence we get $x \in \hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$.

Theorem 2.7 Let $\theta = (k_r)$ be a lacunary sequence, N be an Orlicz function and β_1, β_1, γ_1 and γ_2 are fixed real numbers such that $0 < \beta_1 \leq \beta_2 \leq \gamma_1 \leq \gamma_2 \leq 1$, then

i) $\hat{S}^{\mathcal{F}}_{(\beta_1,\gamma_2)}(N,\theta) \subset \hat{S}^{\mathcal{F}}_{(\beta_2,\gamma_1)}(N,\theta)$,
ii) $\hat{w}^{\mathcal{F}}_{(\beta_1,\gamma_2)}(N,\theta) \subseteq \hat{w}^{\mathcal{F}}_{(\beta_2,\gamma_1)}(N,\theta)$,
iii) $\hat{w}^{\mathcal{F}}_{(\beta_1,\gamma_2)}(N,\theta)_0 \subseteq \hat{w}^{\mathcal{F}}_{(\beta_2,\gamma_1)}(N,\theta)_0$,
iv) $\hat{w}^{\mathcal{F}}_{(\beta_1,\gamma_2)}(N,\theta)_\infty \subseteq \hat{w}^{\mathcal{F}}_{(\beta_2,\gamma_1)}(N,\theta)_\infty$,
v) $\hat{w}^{\mathcal{F}}_{(\beta_1,\gamma_2)}(N,\theta) \subset \hat{S}^{\mathcal{F}}_{(\beta_2,\gamma_1)}(N,\theta)$.

Proof. Omitted.

Theorem 2.8 Let $\theta = (k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0 < \beta \leq \gamma \leq 1$. If $\liminf_r q_r > 1$, then $\hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N)_0 \subseteq \hat{w}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)_0$.

Proof. Proof follows from the following inequality

$$\begin{aligned} &\frac{1}{h_r^\beta}\left[\sum_{k\in I_r} N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^\gamma \\ &= \frac{1}{h_r^\beta}\left[\sum_{i=1}^{k_r} N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right) - \sum_{i=1}^{k_{r-1}} N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^\gamma \\ &\leq \frac{1}{h_r^\beta}\left[\sum_{i=1}^{k_r} N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^\gamma + \frac{1}{h_r^\beta}\left[\sum_{i=1}^{k_{r-1}} N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^\gamma \\ &= \frac{k_r^\beta}{h_r^\beta}\frac{1}{k_r^\beta}\left[\sum_{i=1}^{k_r} N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^\gamma + \frac{k_{r-1}^\beta}{h_r^\beta}\frac{1}{k_{r-1}^\beta}\left[\sum_{i=1}^{k_{r-1}} N\left(\frac{d\left(t_{kn}(X),\ \bar{0}\right)}{s}\right)\right]^\gamma. \end{aligned}$$

The proof of the following theorem is similar to that of Theorem 2.6, therefore we choose to give it without proof.

Theorem 2.9 Let $\theta = (k_r)$ be a lacunary sequence, N be an Orlicz function and β and γ are two fixed real numbers such that $0 < \beta \leq \gamma \leq 1$. If $\liminf_r \frac{h_r^\beta}{k_r} > 0$, then $\hat{S}^{\mathcal{F}}(N) \subset \hat{S}^{\mathcal{F}}_{(\beta,\gamma)}(N,\theta)$.

Definition 2.10 Let $\theta = (k_r)$ be a lacunary sequence, N be an Orlicz function, $r = (r_k)$ be any sequence of strictly positive real numbers and $0 < \beta \leq 1$ be given. We define the following sets

$$\hat{w}^{\mathcal{F}}_\beta(N,r,\theta) = \left\{ \begin{array}{c} X \in w^{\mathcal{F}} : \lim_{r\to\infty} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(t_{kn}(X),\, X_0)}{s}\right)\right]^{r_k} = 0, \\ \text{uniformly in } n \text{ and } s > 0 \end{array} \right\},$$

$$\hat{w}^{\mathcal{F}}_\beta(N,r,\theta)_0 = \left\{ \begin{array}{c} X \in w^{\mathcal{F}} : \lim_{r\to\infty} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(t_{kn}(X),\, \bar{0})}{s}\right)\right]^{r_k} = 0, \\ \text{uniformly in } n \text{ and } s > 0 \end{array} \right\},$$

$$\hat{w}^{\mathcal{F}}_\beta(N,r,\theta)_\infty = \left\{ X \in w^{\mathcal{F}} : \sup_{r,n} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(t_{kn}(X),\, \bar{0})}{s}\right)\right]^{r_k} < \infty, \qquad s > 0 \right\},$$

If $X \in \hat{w}^{\mathcal{F}}_\beta(N,r,\theta)$, we say that X is strongly almost lacunary convergent of order β with respect to the Orlicz function N. In this case we write $X_k \to X_0\left(\hat{w}^{\mathcal{F}}_\beta(N,r,\theta)\right)$.

Theorem 2.11 If $\lim r_k > 0$ and X is strongly almost lacunary convergent of order β to X_0, with respect to the Orlicz function N, then X_0 is unique.

Proof. Suppose that $X_k \to X_0\left(\hat{w}^{\mathcal{F}}_\beta(N,r,\theta)\right)$, $X_k \to Y_0\left(\hat{w}^{\mathcal{F}}_\beta(N,r,\theta)\right)$ and $\lim r_k = \ell > 0$, then there exist s_1 and s_2 such that

$$\lim_{r\to\infty} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(t_{kn}(X), X_0)}{s_1}\right)\right]^{r_k} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(t_{kn}(X), Y_0)}{s_2}\right)\right]^{p_k} = 0, \text{ uniformly in } n.$$

Let $s = \max(2s_1, 2s_2)$. Then we have

$$\begin{aligned} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(X_0,Y_0)}{s}\right)\right]^{r_k} &\leq \frac{G}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(t_{kn}(X), X_0)}{s_1}\right)\right]^{r_k} \\ &+ \frac{G}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(t_{kn}(X), Y_0)}{s_2}\right)\right]^{r_k} \to 0, \qquad (r\to\infty) \end{aligned}$$

where $\sup_k r_k = H$ and $G = \max\left(1, 2^{H-1}\right)$. Thus

$$\lim_{r\to\infty} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[N\left(\frac{d(X_0,Y_0)}{s}\right)\right]^{r_k} = 0.$$

Also, since clearly

$$\lim_k \left[N\left(\frac{d(X_0,Y_0)}{s}\right)\right]^{r_k} = \left[N\left(\frac{d(X_0,Y_0)}{s}\right)\right]^{\ell},$$

and so $\left[N\left(\frac{d(X_0,Y_0)}{s}\right)\right]^{\ell}=0$. Hence $X_0=Y_0$.

In Theorem 2.12 and Theorem 2.13, we shall assume that the sequence $r=(r_k)$ is bounded and $0<h=\inf_k r_k\le r_k\le \sup_k p_k=H<\infty$. From now on, $\sum_1$ will be denote the sum over $k\in I_r$ with $d(t_{kn}(X),X_0)\ge\varepsilon$ and $\sum_2$ the sum over $k\in I_r$ with $d(t_{kn}(X),X_0)<\varepsilon$.

Theorem 2.12 Let $\theta=(k_r)$ be a lacunary sequence, N be an Orlicz function, β and γ are two fixed real numbers such that $0<\beta\le\gamma\le1$, then $\hat{w}_{\beta}^{\mathcal{F}}(N,r,\theta)\subset\hat{S}_{\gamma}^{\mathcal{F}}(\theta)$.

Proof. $X\in\hat{w}_{\beta}^{\mathcal{F}}(N,r,\theta)$, then we have

$$\begin{aligned}\frac{1}{h_r^{\beta}}\sum_{k\in I_r}\left[N\left(\frac{d(t_{kn}(X),X_0)}{s}\right)\right]^{r_k} &\ge \frac{1}{h_r^{\gamma}}\sum_1\left[N\left(\frac{d(t_{kn}(X),X_0)}{s}\right)\right]^{r_k}\\ &\ge \frac{1}{h_r^{\gamma}}\sum_1\min\left([N(\sigma)]^h,[N(\sigma)]^H\right),\qquad \sigma=\frac{\varepsilon}{s}\\ &\ge \frac{1}{h_r^{\gamma}}\left|\{k\in I_r: d(t_{kn}(X),X_0)\ge\varepsilon\}\right|.\min\left([N(\sigma)]^h,[N(\sigma)]^H\right).\end{aligned}$$

Hence $X\in\hat{S}_{\gamma}^{\mathcal{F}}(\theta)$.

Theorem 2.13 Let $\theta=(k_r)$ be a lacunary sequence, N be an Orlicz function, β and γ are two fixed real numbers such that $0<\beta\le\gamma\le1$, $X\in\ell_{\infty}^{\mathcal{F}}$ and $\lim_{r\to\infty}\frac{h_r}{h_r^{\beta}}=1$, then $\hat{S}_{\beta}^{\mathcal{F}}(\theta)\subset\hat{w}_{\gamma}^{\mathcal{F}}(N,r,\theta)$.

Proof. Suppose that $X\in\ell_{\infty}^{\mathcal{F}}$ and $X_k\to X_0\left(\hat{S}_{\gamma}^{\mathcal{F}}(\theta)\right)$. Since X is bounded, there exists a constant $B>0$ such that $d(t_{kn}(X),X_0)\le B$. Let $\varepsilon>0$, then we have

$$\begin{aligned}&\frac{1}{h_r^{\gamma}}\sum_{k\in I_r}\left[N\left(\frac{d(t_{kn}(X),X_0)}{s}\right)\right]^{r_k}\\ &\le \frac{1}{h_r^{\beta}}\sum_1\left[N\left(\frac{d(t_{kn}(X),X_0)}{s}\right)\right]^{r_k}+\frac{1}{h_r^{\beta}}\sum_2\left[N\left(\frac{d(t_{kn}(X),X_0)}{s}\right)\right]^{r_k}\\ &\le \frac{1}{h_r^{\beta}}\sum_1\max\left\{\left[N\left(\frac{B}{s}\right)\right]^h,\left[N\left(\frac{B}{s}\right)\right]^H\right\}+\frac{1}{h_r^{\beta}}\sum_2\left[N\left(\frac{\varepsilon}{s}\right)\right]^{r_k},\\ &\le \max\left\{[N(C)]^h,[N(C)]^H\right\}\frac{1}{h_r^{\beta}}\left|\{k\in I_r: d(t_{kn}(X),X_0)\ge\varepsilon\}\right|\\ &\qquad+\frac{h_r}{h_r^{\beta}}\max\left\{[N(\delta)]^h,[N(\delta)]^H\right\},\quad \frac{B}{s}=C,\ \frac{\varepsilon}{s}=\delta.\end{aligned}$$

Hence $X\in\hat{w}_{\gamma}^{\mathcal{F}}(N,r,\theta)$.

3. Conclusions

Fuzzy set theory and Fuzzy logic theory have many applications in everyday life. In this study, the concepts of almost lacunary statistical convergence and summability of fuzzy sequences are studied. The concepts in the study are related to the concepts of convergence in probability theory in statistics.

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