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Approximation Behaviour of Generalized Baskakov-Durrmeyer-Schurer Operators

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Abstract. The goal of this manuscript is to introduce a new sequence of generalized-Baskakov-Durrmeyer-Schurer Operators. Further, basic estimates are calculated. In the subsection sequence, rapidity of convergence and order of approximation are studied in terms of first and second order modulus of continuity. We prove a Korovkin-type approximation theorem and obtain the rate of convergence of these operators. Moreover, local and global approximation properties are discussed in different functional spaces. Lastly, A-statistical approximation results are presented.

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1. Introduction

In 1998, Mihesan [10] presented a generalized Baskakov type operators as:

$$L_{n,a}(h;x) \;=\; \sum_{k=1}^{\infty} W_{n,k}^{a}(x) h\left(\frac{k}{n}\right), \tag{1}$$

where $h \in C[0,\infty)$ and $W_{n,k}^{a}(x) = e^{\frac{ax}{1+x}} \frac{p_k(n,a)}{k!} \frac{x^k}{(1+x)^(k+n)}$ with $W_{n,k}^{a}(x) = 1$. He proved uniform convergence for these sequence of operators on $[0,b]$ for functions that have exponential growth and also discussed a pointwise estimate. In [3], Wafi and Khatoon studied the rate of convergence of these operators in terms of the modulus of continuity and obtained the Voronovskaja type theorem and a direct estimate of these operators in terms of the Ditzian-Totik modulus of smoothness. Subsequently, Wafi et al. ([2], [3]) and Rao et al. ([34, 35, 36]) studied the simultaneous approximation properties of these operators for functions of one and two variables in exponential and polynomial weighted spaces.

In [4], Erencin and Bascanbaz-Tunca studied the weighted approximation properties and they estimated the order of approximation in terms of the usual modulus of continuity of these operators. For $f \in C_B[0,\infty)$, the space of all bounded and continuous function on $[0,\infty)$, Erencin [5] introduced the Durrmeyer type modification of the operators defined and established some local results for these operators. Agrawal et al. [9] extended her study and discussed some direct results in simultaneous approximation by these operators, e.g., pointwise convergence theorem, Voronovskaja type

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theorem and error in the estimate of the modulus of continuity. Also, they obtained the error in the approximation of functions having derivatives of bounded variation.

To approximate in a bigger class, i.e., class of Lebesgue measurable function, Durrmeyer type modification of Baskakov operators was introduced by [5]. In the last few decades, the integral modifications of several operators were constructed and their approximation behavior studied, we mention some of the work in this direction, e.g., Raiz et al. ([6], [7]), Rao et al.([27], [34], [35], [36], [42]), constructed a new sequence of the operators and Karsli[13], Mohiuddine et al. [37], Mursaleen et al. [38], Ansari [40], Nasiruzzaman et al. [39], Devdhara et al. [41] etc. On the other hand, Stancu [14] introduced and investigated a new parameter-dependent linear positive operators of Bernstein type associated to a function $h \in C[0,1]$. The new construction of his operators shows that the new sequence of Bernstein polynomials present a better approach with the suitable selection of the parameters. We also refer to reader for a deep historical background [15, 16, 17, 19, 20, 21, 22, 23, 41, 42]. Kumar et al. [24] proposed Stancu type modification of generalized Baskakov Durrmeyer operators. They defined the Stancu type generalization of Baskakov Durrmeyer type of the operators as:

$$L_{n,a}^{\alpha,\beta}(h;x) = \sum_{k=1}^{\infty} \frac{W_{n,k}^{a}(x)}{B(k,n)} \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k}} f\left(\frac{nt+\alpha}{n+\beta}\right) dt + W_{n,0}^{a}(x) h\left(\frac{\alpha}{n+\beta}\right). \tag{2}$$

Motivated by the above development, we introduce Schurer type modification of Generalized Baskakov Durrmeyer operators. For $h \in \mathbf{D}, s+d \in \mathbf{N}$ and $0 \leq \sigma \leq \tau$, we defined the Schurer type generalization of the operators defined in (2) as:

$$D_{s+d,c}^{\sigma,\tau}(h;u) = \sum_{i=1}^{\infty} \frac{P_{s+d,i}^{c}(u)}{B(i,s+d)} \int_0^{\infty} \frac{t^{i-1}}{(1+t)^{s+d+i}} h\left(\frac{(s+d)t+\sigma}{s+d+\tau}\right) dt + P_{s+d,0}^{c}(u) h\left(\frac{\sigma}{s+d+\tau}\right), \tag{3}$$

where

$$P_{s+d,i}^{c}(u) = e^{-\frac{cu}{1+u}} \frac{Q_i(s+d,c)}{i!} \frac{u^i}{(1+u)^{s+d+i}}, \quad \sum_{i=0}^{\infty} P_{s+d,i}^{c}(u) = 1,$$

$$Q_i(s+d,c) = \sum_{j=0}^{i} \binom{i}{j,} (s+d)_j c^{i-j},$$

where $(s+d)_j$ denote the Pochhammer symbol given by $(s+d)_0 = 1, (s+d)_j = (s+d)(s+d+1)\dots(s+d+j-1)$, for $j \geq 1$ and $\mathbf{D}$ denotes the class of all Lebesgue measurable functions h on $[0,\infty)$ as:

$$D = \left\{ h : \int_0^{\infty} \frac{|h(t)|}{(1+t)^{s+d}} dt < \infty \text{ for some positive integer } s+d \right\}.$$

We observe that $C_B[0,\infty) \subset \boldsymbol{D}$. The operators (3) are linear, positive and for $\sigma = \tau = 0$ and $s+d=n$ they reduce to generalized Baskakov Durremeyer operators.

The purpose of this paper is to investigate and study the convergence properties of the operators defined in (3). First, we derive the recurrence relation and central moments of these operators and then we study the local approximation, weighted approximation and A-statistical convergence of the operators.

2. Basic results

In the sequel, we shall need the following auxiliary results which will be necessary to prove our main results.

Lemma 2.1. [9] *For every $u \in (0,\infty)$, we have*

$$u(1+u)^2\left\{\frac{d}{du}P^c_{s+d,i}(u)\right\} = \{(i-s+du)(1+u)-cu\}P^c_{s+d,i}(u).$$

Lemma 2.2. *We have the following recurrence relation:*

$$\begin{aligned}u(1+u)^2\left(D^{\sigma,\tau}_{s+d,c}(t^m;u)\right)' = {} & D^{\sigma,\tau}_{n+p,c}(t^m;u)\left[(1+u)\left(\frac{2\sigma(m+1)}{s+d}-m\right.\right.\\ & \left.\left.-\frac{\sigma}{s+d+\tau}-s+du\right)-cu\right]\\ & +D^{\sigma,\tau}_{s+d,c}(t^{m+1};u)\frac{(s+d)(1+u)}{s+d+\tau}(s+d-m-1)\\ & +D^{\sigma,\tau}_{s+d,c}(t^{m-1};u)(1+u)\frac{m\sigma}{s+d+\tau}\left(1-\frac{\sigma}{s+d}\right).\end{aligned}$$

Proof. For $u=0$, the above relation is easily verified. For $u \in (0,\infty)$, we proceed as follows:
From Lemma 2.1, we can write

$$\begin{aligned}\left(D^{\sigma,\tau}_{s+d,c}(t^m;u)\right)' = & \sum_{i=0}^{\infty}\frac{\frac{d}{du}\left(P^c_{s+d,i}(u)\right)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{s+dt+\sigma}{s+d+\tau}\right)^m dt\\ & +\frac{d}{du}\left(P^c_{s+d,0}(u)\right)\left(\frac{\sigma}{s+d+\tau}\right)^m\\ = & J_1\frac{1}{u(1+u)}-\frac{c}{(1+u)^2}D^{\sigma,\tau}_{s+d,c}(t^m;u)\\ & -\frac{s+d}{(1+u)}D^{\sigma,\tau}_{s+d,c}(t^m;u).\end{aligned} \tag{4}$$

We may write J_1 as

$$\begin{aligned}J_1 = & \sum_{i=0}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}(i-1-(s+d+1)t)}{(1+t)^{s+d+i}}\left(\frac{s+dt+\sigma}{s+d+\tau}\right)^m dt\\ & +\sum_{i=0}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,n+p)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{s+dt+\sigma}{s+d+\tau}\right)^m dt\\ & +(s+d+1)\sum_{i=0}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}t\left(\frac{(s+d)t+\sigma}{s+d+\tau}\right)^m dt\\ = & J_2+J_3.\end{aligned} \tag{5}$$

Using $t=\frac{s+d+\tau}{s+d}\left[\left(\frac{s+dt+\sigma}{s+d+\tau}\right)-\left(\frac{\sigma-\tau}{s+d+\tau}\right)\right]$ and integrating by parts, we obtain

$$\begin{aligned}J_2=&-mD^{\sigma,\tau}_{s+d,c}\left(t^m;u\right)+\frac{m\sigma}{s+d+\tau}D^{\sigma,\tau}_{s+d,c}\left(t^{m-1};u\right)\\&-\frac{(m+2)(n+p+\tau)}{s+d}D^{\sigma,\tau}_{s+d,c}\left(t^{m+1};u\right)-\frac{m\sigma^2}{(s+d)(n+\tau)}D^{\sigma,\tau}_{s+d,c}\left(t^{m-1};u\right)\\&+\frac{2\sigma(m+1)}{s+d}D^{\sigma,\tau}_{s+d,c}\left(t^m;u\right).\end{aligned}\tag{6}$$

Again using $t=\frac{n+p+\tau}{s+d}\left[\left(\frac{(s+d)t+\sigma}{s+d+\tau}\right)-\left(\frac{\sigma-\tau}{s+d+\tau}\right)\right]$, we get

$$J_3=\frac{(s+d+1)(s+d+\tau)}{s+d}D^{\sigma,\tau}_{s+d,c}\left(t^{m+1};u\right)-\frac{\sigma}{s+d+\tau}D^{\sigma,\tau}_{s+d,c}\left(t^m;u\right).\tag{7}$$

Combining the equalities (4)-(7), we get the desired result. □

Corollary 2.3. *For the function $D^{\sigma,\tau}_{s+d,c}\left(t^m;u\right)$, we have*

(i) $D^{\sigma,\tau}_{s+d,c}(t;u)=\frac{(s+d)^2}{(n+p-1)(s+d+\tau)}\left(u+\frac{cu}{(s+d)(1+u)}\right)+\frac{\sigma}{s+d+\tau}$, $(s+d)>1$,

(ii) $$D^{\sigma,\tau}_{s+d,c}\left(t^2;u\right)=\frac{(s+d)^3}{(s+d-1)(s+d-2)(s+d+\tau)^2}\left(u+\frac{cu}{(s+d)(1+u)}\right)$$

$$\begin{aligned}&+\frac{(s+d)^4}{(s+d-1)(s+d-2)(s+d+\tau)^2}\left(\frac{u^2}{(s+d)^2}+\frac{u}{s+d}+u^2+\frac{c^2u^2}{(s+d)^2(1+u)^2}\right.\\&\left.+\frac{2cu^2}{(s+d)(1+u)}+\frac{cu}{(s+d)^2(1+u)}\right)+\left(\frac{2(s+d)^2\sigma}{(s+d-1)(s+d+\tau)}\right)\\&\times\left(u+\frac{cu}{(s+d)(1+u)}\right)+\left(\frac{\sigma}{s+d+\tau}\right)^2,s+d>2.\end{aligned}$$

(iii) *For each $u\in(0,\infty)$ and $m\in\mathbf{N}$,*

$$D^{\sigma,\tau}_{s+d,c}\left(t^m;u\right)=c_m(s+d,\tau)u^m+(s+d)^{-1}\left(r_m(u,c,\sigma,\tau)+o(1)\right),$$

where $c_m(s+d,\tau)=\left(\frac{s+d}{s+d+\tau}\right)^m\prod_{l=0}^{m-1}\frac{(s+d+l)}{(s+d-l-1)}$ and $r_m(u,c,\sigma,\tau)$ is a rational function of u depending on the parameters c,σ and τ.

For $m\in\mathbf{N}^0=\mathbf{N}\cup\{0\}$, the $m-$th order central moment for the operators (3) is defined as:

$$\begin{aligned}\theta^{c,\sigma,\tau}_{s+d,m}(x):=&D^{\sigma,\tau}_{s+d,c}\left((t-u)^m;u\right)\\=&\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^m dt\\&+P^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^m,\quad s+d>m.\end{aligned}$$

Lemma 2.4. *For the function* $\theta^{c,\sigma,\tau}_{s+d,m}(u)$*, we have* $\theta^{c,\sigma,\tau}_{s+d,0}(u)=1$ *and the following recurrence relation holds:*

$$\begin{aligned}&(s+d+\tau)(s+d-m-1)\theta^{c,\sigma,\tau}_{s+d,m+1}(u)\\ &=(s+d)u(1+u)\left(\left(\theta^{c,\sigma,\tau}_{s+d,m}(u)\right)'+m\theta^{c,\sigma,\tau}_{s+d,m-1}(x)\right)\\ &\quad+\theta^{c,\sigma,\tau}_{s+d,m}(u)\Bigg(ms+d-(s+d+\tau)\left(\frac{\sigma}{s+d+\tau}-u\right)(2m-(s+d)+1)\\ &\quad+(s+d)^2u+\frac{s+dcu}{(1+u)}\Bigg)+\theta^{c,\sigma,\tau}_{s+d,m-1}(u)\Bigg(m(s+d+\tau)\left(\frac{\sigma}{s+d+\tau}-u\right)^2\\ &\quad-ms+d\left(\frac{\sigma}{s+d+\tau}-u\right)\Bigg).\end{aligned}\tag{8}$$

Proof. For $u=0$, the relation (8) holds. For $u\in(0,\infty)$, we proceed as follows from Lemma 2.1, we can write

$$\begin{aligned}\left(\theta^{c,\sigma,\tau}_{s+d,m}(u)\right)'=&\sum_{i=1}^{\infty}\frac{\frac{d}{du}\left(P^c_{s+d,i}(u)\right)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{s+dt+\sigma}{s+d+\tau}-u\right)^m dt\\ &-m\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m-1}dt\\ &+\frac{d}{du}\left(P^c_{s+d,0}(u)\right)\left(\frac{\sigma}{s+d+\tau}-u\right)^m-mP^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^{m-1}\\ =&J_1-m\theta^{c,\sigma,\tau}_{s+d,m-1}(u),\ \text{(say)}\ .\end{aligned}\tag{9}$$

On making use of Lemma 2.1, we obtain

$$\begin{aligned}u(1+u)^2I_1=&\sum_{i=1}^{\infty}u(1+u)^2\frac{\frac{d}{du}\left(P^c_{s+d,i}(u)\right)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^m dt,\\ &+u(1+u)^2\frac{d}{du}\left(P^c_{s+d,0}(u)\right)\left(\frac{\sigma}{s+d+\tau}-u\right)^m\\ =&\sum_{i=1}^{\infty}\frac{((i-(s+d)u)(1+u)-cu)P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+k}}\\ &\times\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^m dt-((s+d)u(1+u)+cu)P^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^m\\ =&\ (1+u)I_2-(s+d)u(1+u)\theta^{c,\sigma,\tau}_{s+d,m}(u)-cu\theta^{c,\sigma,\tau}_{s+d,m}(u).\end{aligned}\tag{10}$$

We can write J_2 as

$$\begin{aligned}J_2=&\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}((i-1)-(s+d+1)t)\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^m dt\\ &+\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(k,s+d)}\int_0^\infty\frac{t^{i-1}}{(1+t)^{s+d+i}}((s+d+1)t+1)\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^m dt\\ =&\ J_3+J_4,\ \text{say}.\end{aligned}\tag{11}$$

Now, using the identity $t=\frac{s+d+\tau}{s+d}\left[\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)-\left(\frac{\sigma-\tau u}{s+d+\tau}\right)\right]$, we have

$$\begin{aligned}J_4 =&\frac{(s+d+1)(s+d+\tau)}{s+d}\left[\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+k}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m+1}dt\right.\\&\left.+P^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^{m+1}\right]-\left(\frac{\sigma}{s+d+\tau}-u\right)\frac{(s+d+1)(s+d+\tau)}{s+d}\\&\times\left[\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m}dt\right.\\&\left.+P^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^{m}\right]\\=&\frac{(s+d+1)(s+d+\tau)}{s+d}\theta^{c,\sigma,\tau}_{s+d,m+}(u)-\left(\frac{\sigma}{s+p+\beta}-u\right)\\&\times\frac{(ns+d+1)(s+d+\tau)}{s+d}\theta^{c,\sigma,\tau}_{s+d,m}(u). \qquad (12)\end{aligned}$$

Making use of the identity $t(1+t)\frac{d}{dt}\left(\frac{t^i}{(1+t)^{s+d+i+1}}\right)=(i-(s+d+1)t)\frac{t^i}{(1+t)^{s+d+i+1}}$ and then integrating by parts, we obtain

$$\begin{aligned}J_3 =&\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m}dt\\&-\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\frac{d}{dt}t(1+t)\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m}dt. \qquad (13)\end{aligned}$$

Again using $t=\frac{s+d+\tau}{s+d}\left[\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)-\left(\frac{\sigma-\tau u}{s+d+\tau}\right)\right]$, we have

$$\begin{aligned}J_3 =&\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m}dt\\&-(m+1)\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m}dt\\&+m\left(\frac{\sigma}{s+d+\tau}-u\right)\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m-1}dt\\&-\frac{(m+2)(s+d+\tau)}{s+d}\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m+1}dt\\&-\frac{m(s+d+\tau)}{s+d}\left(\frac{\sigma}{s+d+\tau}-u\right)^2\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\\&\times\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m-1}dt\\=&J_1+J_2,\ \text{say}. \qquad (14)\end{aligned}$$

Now, we estimate J_1

$$\begin{aligned} J_1 &= -(m+1)\left[\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^m dt \right.\\ &\left. +P^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^m\right]+m\left(\frac{\sigma}{s+d+\tau}-u\right)\\ &\left[\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^{m-1} dt\right.\\ &\left.+P^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^{m-1}\right]\\ &+\left[\sum_{i=1}^{\infty}\frac{P^c_{s+d,i}(u)}{B(i,s+d)}\int_0^{\infty}\frac{t^{i-1}}{(1+t)^{s+d+i}}\left(\frac{(s+d)t+\sigma}{s+d+\tau}-u\right)^m dt\right.\\ &\left.+P^c_{s+d,0}(u)\left(\frac{\sigma}{s+d+\tau}-u\right)^m\right]\\ &= -m\theta^{c,\sigma,\tau}_{s+d,m}(u)+m\left(\frac{\sigma}{s+d+\tau}-u\right)\theta^{c,\sigma,\tau}_{s+d,m-1}(u). \end{aligned} \tag{15}$$

Proceeding in a similar manner, we obtain the estimate of J_2 as

$$\begin{aligned} J_2 = &-\frac{(m+2)(s+d+\tau)}{s+d}\theta^{c,\sigma,\tau}_{s+d,m+1}(u)+\frac{2(m+1)(s+d+\tau)}{s+d}\theta^{c,\sigma,\tau}_{s+d,m}(u)\\ &-\frac{m(s+d+\tau)}{s+d}\left(\frac{\sigma}{s+d+\tau}-u\right)^2\theta^{c,\sigma,\tau}_{s+d,m-1}(u). \end{aligned} \tag{16}$$

Combining (9)-(16), we get the desired recurrence relation. □

Corollary 2.5. *For the function $\theta^{c,\sigma,\tau}_{s+d,m}(u)$, the following hold (i) $\theta^{c,\sigma,\tau}_{s+d,m}(u)$ is a rational function of u and $s+d$. (ii) For every $u\in(0,\infty)$, $\theta^{c,\sigma,\tau}_{s+d,m}(u)=O\left(\frac{1}{s+d^{\left[\frac{m+1}{2}\right]}}\right)$, where $[\eta]$ denotes the integer part of η.*

Proof. The assertions (i) and (ii) easily follow from the recurrence relation (8) by using the mathematical induction on m. □

3. Uniform convergence of the operators $D^{\sigma,\tau}_{s+d,c}(.;.)$

Definition 3.1. Let $k\in\mathcal{C}[0,\infty)$, then, modulus of continuous for a uniformly continuous function h is presented as:

$$\omega(h;\eta)=sup_{|s_1-s_2|\le\eta}|h(s_1)-h(s_2)|,\, s_1,s_2\in[0,\infty).$$

For a uniformly continuous function h in $\mathcal{C}[0,\infty)$ and $\eta>0$, one has

$$|h(s_1)-h(s_2)|\le\left(1+\frac{(1-s)^2}{\eta^2}\right)\omega(h;\eta). \tag{17}$$

Theorem 3.1. *For $D^{\sigma,\tau}_{s+d,c}(.;.)$ the operators introduced by (3), with $D^{\sigma,\tau}_{s+d,c}(.;.)$ converges to g uniformly on each bounded subset of $[0,\infty]$ where $g \in \mathcal{C}[0,\infty] \cap \{g : v \geq 0, \dfrac{g(v)}{1+v^2}$ converges as $v \to \infty\}$.*

Proof. In the light of Korovkin-type property (iv) of Theorem 4.1.4 in [11], it is sufficient to show that $D^{\sigma,\tau}_{s+d,c}(.;.) \to t^m(u)$, for $m = 0,1,2$. By Corollary 2.3, it is obvious $D^{\sigma,\tau}_{s+d,c}(.;.) \to t(u)$ as $(s+d) \to \infty$ for $m = 0,1,2...$, which completes the proof of Theorem 3.1. □

Theorem 3.2. [12] *Suppose $\mathcal{L} : \mathcal{C}[p,q] \to \mathcal{B}[c,d]$ be a linear and positive operator and suppose β_u be the function defined by*

$$\beta_v(u) = |u - v|, (u,v) \in [c,d] \times [p,q].$$

If $g \in \mathcal{C}_B([p,q])$ for any $u \in [p,q]$ and $\eta > 0$, the operator L verifies:

$$|(Lg)(u) - g(u)| \leq |g(u)||(Lt)(u) - 1|\{(Lt)(u) + \eta^{-1}\sqrt{(Lt)(u)(L\beta_u^2(u))}\}\omega_g(\eta).$$

Theorem 3.3. *Let $g \in \mathcal{C}_B[0,\infty)$. Then, for the operator $D^{\sigma,\tau}_{s+d,c}(.;.)$ presented by (3), we get*

$$|D^{\sigma,\tau}_{s+d,c}(.;.)(g,u) - g(u)| \leq 2\omega(g;\eta), \ \textit{where}\ \eta = \sqrt{D^{\sigma,\tau}_{s+d,c}(\phi_u^2;u)}.$$

Proof. In term of Lemma 2.2, Corollary 2.3, and Theorem 3.2, one has

$$\left|D^{\sigma,\tau}_{s+d,c}(g,u) - g(u)\right| \leq \left\{1 + \delta^{-1}\sqrt{D^{\sigma,\tau}_{s+d,c}(\phi_u^2);u}\right\}\omega(g;\eta),$$

which completes the proof. □

4. Local approximation

Let $C_B[0,\infty)$ be the space of all real valued continuous and bounded functions h on the interval $[0,\infty)$, with the norm

$$\|h\|_{C_B} := \sup_{u\in[0,\infty)} |h(u)|.$$

Let $\delta > 0$ and $C_B^2[0,\infty) = \{g \in C_B[0,\infty); g', g'' \in C_B[0,\infty)\}$. For $h \in C_B[0,\infty)$, the Peetre's K-functional is given by

$$K_2(h,\delta) = \inf\left\{\|h - g\| + \delta\left\|g''\right\|; g \in C_B^2[0,\infty)\right\}. \tag{18}$$

By DeVore and Lorentz ([8] , p. 177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$K_2(h,\delta) \leq C\omega_2(h,\sqrt{\delta}), \tag{19}$$

where the second order modulus of continuity is defined as:

$$\omega_2(h,\sqrt{\delta}) = \sup_{0<g\leq\sqrt{\delta}} \sup_{u\in[0,\infty)} |h(u+2g) - 2h(u+g) + h(u)|. \tag{20}$$

For $h \in C_B[0,\infty)$, the usual modulus of continuity is defined as:

$$\omega(h,\delta) = \sup_{0<g\leq\sqrt{\delta} 0\leq u<\infty} \sup_{0} |h(u+g) - h(u)|.$$

Theorem 4.1. *Let $h \in C_B[0,\infty)$. Then, there exists a constant $K>0$ such that*

$$\left|D^{\sigma,\tau}_{s+d,c}(h;u)-h(u)\right| \leq K\omega_2\left(h;\sqrt{\eta^{(\sigma,\tau)}_{s+d,c}(u)}\right)$$
$$+\omega\left(h;\frac{\frac{(s+d)cu}{(1+u)}+\sigma(s+d-1)-u((s+d)\tau-(s+d)-\tau)}{(s+d+\tau)(s+d-1)}\right)$$

where

$$\eta^{(\sigma,\tau)}_{s+d,c}(u) = D^{\sigma,\tau}_{s+d,c}\left((t-u)^2;u\right) + \left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)}\left(u+\frac{cu}{(s+d)(1+u)}\right)+\frac{\sigma}{s+d+\tau}-u\right)^2.$$

Proof. First, let us consider the auxiliary operators

$$\bar{D}^{\sigma,\tau}_{s+d,c}(h;u) = D^{\sigma,\tau}_{s+d,c}(h;u)+h(u) -h\left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)}\left(u+\frac{cu}{(s+d)(1+u)}\right)+\frac{\sigma}{s+d+\tau}\right). \tag{21}$$

According to definition of the auxiliary operators, $\bar{D}^{\sigma,\tau}_{s+d,c}(1,u)=1$ and $\bar{D}^{\sigma,\tau}_{s+d,c}(t;u)=u$ Let $h \in C^2_B[0,\infty)$. Using Taylor's expansion of g, we write

$$h(t)=h(u)+(t-u)g'(u)+\int_u^t (t-v)h''(v)dv.$$

Applying the operators $\bar{D}^{\sigma,\tau}_{s+d,c}(.;.)$, to both sides of above equality, we deduce

$$\begin{aligned}\left|\bar{D}^{\sigma,\tau}_{s+d,c}(h;u)-h(u)\right| &\leq D^{\sigma,\tau}_{s+d,c}\left(\left|\int_u^t (t-v)h''(v)dv\right|;u\right)\\ &+\left|\int_u^{\Big\{} \frac{(s+d)^2}{(s+d+\tau)(s+d-1)}\left(u+\frac{cu}{(s+d)(1+u)}\right)\right.\\ &+\frac{\sigma}{s+d+\tau}\Big\}\Bigg|\left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)}\left(u+\frac{cu}{(s+d)(1+u)}\right)\right.\\ &\left.+\frac{\sigma}{s+d+\tau}-v\right)\mid |g''(v)|\,dv \mid\leq D^{\sigma,\tau}_{s+d,c}\left((t-u)^2;u\right)\|g''\|\\ &+\left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)}\left(u+\frac{cu}{(s+d)(1+u)}\right)\right.\\ &\left.+\frac{\sigma}{s+d+\tau}-u\right)^2\|g''\|\\ &=\eta^{(\sigma,\tau)}_{s+d,c}(u)\,\|g''\|\,.\end{aligned} \tag{22}$$

From (21), we have

$$\left|\bar{D}^{\sigma,\tau}_{s+d,c}(h;u)\right| \leq 3\|h\|. \tag{23}$$

Since $g \in C_B^2[0,\infty)$ and using (22),(23) in (21), we obtain

$$\begin{aligned}
\left|D_{s+d,c}^{\sigma,\tau}(h;u)-h(u)\right| &\leq \left|\bar{D}_{s+d,c}^{\sigma,\tau}(h-g;u)\right| + \left|\bar{D}_{s+d,c}^{\sigma,\tau}(g;u)-g(u)\right| + |g(u)-h(u)| \\
&\quad + \left|h\left(\frac{s+d}{(s+d+\tau)(s+d-1)}\left((s+d)u+\frac{cu}{1+u}+1\right)+\frac{\sigma}{s+d+\tau}\right)-h(u)\right| \\
&\leq 4\|h-g\| + \left|\bar{D}_{s+d,c}^{\sigma,\tau}(g;u)-g(u)\right| \\
&\quad + \left|h\left(\frac{s+d}{(s+d+\tau)(s+d-1)}\left((s+d)u+\frac{cu}{1+u}+1\right)+\frac{\sigma}{s+d+\tau}\right)-h(u)\right| \\
&\leq 4\|h-g\| + \eta_{s+d,c}^{(\sigma,\tau)}(u)\,\|g''\| \\
&\quad + \left|h\left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)}\left(u+\frac{cu}{(s+d)(1+u)}\right)+\frac{\sigma}{s+d+\tau}\right)-h(u)\right| \\
&\leq 4\|h-g\| + \eta_{s+d,c}^{(\sigma,\tau)}(u)\,\|g''\| \\
&\quad + \omega\left(h;\frac{\frac{(s+d)cu}{(1+u)}+\sigma(s+d-1)-u((s+d)\tau-(s+d)-\tau)}{(s+d+\tau)(s+d-1)}\right).
\end{aligned}$$

Taking the infimum on the right hand side over all $g \in C_B^2[0,\infty)$, we get

$$\begin{aligned}
\left|D_{s+d,c}^{\sigma,\tau}(h;u)-h(u)\right| \leq & 4K_2\left(h;\eta_{s+d,c}^{(\sigma,\tau)}(u)\right) \\
& + \omega\left(h;\frac{\frac{(s+d)cu}{(1+u)}+\sigma(s+d-1)-u((s+d)\tau-(s+d)-\tau)}{(s+d+\tau)(s+d-1)}\right).
\end{aligned}$$

Thus by (19), we get

$$\begin{aligned}
\left|D_{s+d,c}^{\sigma,\tau}(h;u)-h(u)\right| \leq & C\omega_2\left(h;\sqrt{\eta_{s+d,c}^{(\sigma,\tau)}(u)}\right) \\
& + \omega\left(h;\frac{\frac{(s+d)cu}{(1+u)}+\sigma(s+d-1)-u((s+d)\tau-(s+d)-\tau)}{(s+d+\tau)(s+d-1)}\right),
\end{aligned}$$

which completes the proof. □

5. Weighted approximation

Let $B_\rho[0,\infty)$ be the space of all real valued functions on $[0,\infty)$ satisfying the condition $|h(u)| \leq M_h\rho(u)$, where M_h is a constant depending only on h and $\rho(u)$ is a weight function. Let $C_\rho[0,\infty)$ be the space of all continuous functions in $B_\rho[0,\infty)$ with the norm $\|h\|_\rho = \sup_{u\in[0,\infty)} \frac{|h(u)|}{\rho(u)}$ and $C_\rho^0 = \left\{h \in C_\rho[0,\infty) : \lim_{u\to\infty} \frac{|h(u)|}{\rho(u)} < \infty\right\}$. In what follows, we assume the weight function as $\rho(u) = 1+u^2$.

Theorem 5.1. *If $h \in C_\rho[0,\infty)$, then for each $u \in [0,b]$ and $s+d > 2$,then we have*

$$\left|D_{s+d,c}^{\sigma,\tau}(h,u)-h(u)\right| \leq 6M_h\left(1+b^2\right)\xi_{s+d,c}^{(\sigma,\tau)}(u) + 2\omega_{b+1}\left(h,\sqrt{\xi_{s+d,c}(u)}\right),$$

where $\xi^{(\sigma,\tau)}_{s+d,c}(u) = D^{(\sigma,\tau)}_{s+d,c}\left((t-u)^2;u\right)$ *and* $\omega_b(h,\delta)$ *is the modulus of continuity of* h *on* $[0,b]$.

Proof. Let $u \in [0,b]$ and $t > b+1$. Then, in view of $t-u>1$ we have

$$\begin{aligned} |h(t)-h(u)| &\leq M_h\left(2+t^2+u^2\right) \\ &\leq M_h(t-u)^2\left(3+2u+2u^2\right) \\ &\leq 6M_h(t-u)^2\left(1+u^2\right) \\ &\leq 6M_h(t-u)^2\left(1+b^2\right). \end{aligned} \tag{24}$$

For $u \in [0,b]$ and $t \leq b+1$, we have

$$|h(t)-h(u)| \leq \omega_{b+1}(h,|t-u|) \leq \omega_{b+1}(h,\delta)\left(1+\frac{|t-u|}{\delta}\right), \delta>0. \tag{25}$$

Combining (24) and (25), we obtain

$$|h(t)-f(u)| \leq 6M_h\left(1+b^2\right)(t-u)^2 + \omega_{b+1}(h,\delta)\left(1+\frac{|t-u|}{\delta}\right).$$

Applying the operators $D^{\sigma,\tau}_{s+d,c}(.;.)$ given by (3) to the above inequality and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \left|D^{\sigma,\tau}_{s+d,c}(h,u)-h(u)\right| \leq & 6M_h\left(1+b^2\right)D^{\sigma,\tau}_{s+d,c}\left((t-u)^2,u\right) \\ & + \omega_{b+1}(h,\delta)\left(1+\frac{1}{\delta}\sqrt{D^{\sigma,\tau}_{s+d,c}\left((t-u)^2,u\right)}\right). \end{aligned}$$

Choosing $\delta = \sqrt{\xi^{(\sigma,\tau)}_{s+d,c}(u)}$, we get the required result. □

Theorem 5.2. *Let* $f \in C^0_\rho$. *Then we have*

$$\lim_{s+d\to\infty}\left\|D^{\sigma,\tau}_{s+d,c}(h)-h\right\|_\rho = 0.$$

Proof. With elementary calculations, it follows easily that

$$\lim_{(s+d)\to\infty}\|D^{\sigma,\tau}_{(s+d),c}(e_i,.)-e_i\|_\rho = 0,\ \textit{where}\ e_i(u)=u^i, i=0,1,2.$$

By weighted Korovkin theorem given in [25], we get the required result. □

Next we give the following theorem to approximate all functions in C^0_ρ. This type of result is discussed in [26] for locally integrable functions.

Theorem 5.3. *For each* $h \in C^0_\rho$ *and* $\lambda>0$, *we have*

$$\lim_{s+d\to\infty}\sup_{u\in[0,\infty)}\frac{\left|D^{\sigma,\tau}_{s+d,c}(h,u)-h(u)\right|}{\left(1+u^2\right)^{1+\lambda}} = 0.$$

Proof. For any fixed $x_0>0$,

$$\sup_{u\in[0,\infty)}\frac{\left|D^{\sigma,\tau}_{s+d,c}(h,u)-h(u)\right|}{\left(1+u^2\right)^{1+\lambda}} \leq \sup_{u\leq u_0}\frac{\left|D^{\sigma,\tau}_{s+d,c}(h,u)-h(u)\right|}{\left(1+u^2\right)^{1+\lambda}} + \sup_{u\geq u_0}\frac{\left|D^{\sigma,\tau}_{s+d,c}(h,u)-h(u)\right|}{\left(1+u^2\right)^{1+\lambda}}$$

$$\leq \left\| D^{\sigma,\tau}_{s+d,c}(h) - h \right\|_{C[0,u_0]} + \|h\|_\rho \sup_{u\geq x_0} \frac{\left| D^{\sigma,\tau}_{s+d,c}\left(1+t^2,u\right)\right|}{\left(1+u^2\right)^{1+\lambda}} + \sup_{u\geq u_0} \frac{|h(x)|}{\left(1+u_0^2\right)^{1+\lambda}}$$

$$= I_1 + I_2 + I_3, \text{ say.} \tag{26}$$

Since $|h(u)| \leq \|h\|_\rho \left(1+u^2\right)$, we have

$$I_3 = \sup_{u\geq u_0} \frac{|h(u)|}{\left(1+u^2\right)^{1+\lambda}} \leq \sup_{u\geq u_0} \frac{\|h\|_\rho}{\left(1+u^2\right)^{\lambda}} \leq \frac{\|h\|_\rho}{\left(1+u_0^2\right)^{\lambda}}.$$

Let $\epsilon > 0$ be arbitrary. In view of Theorem 3.1, there exists $n_1 \in \mathbf{N}$ such that

$$\|h\|_\rho \frac{\left| D^{\sigma,\tau}_{s+d,c}\left(1+t^2,u\right)\right|}{\left(1+u^2\right)^{1+\lambda}} < \frac{1}{\left(1+u^2\right)^{1+\lambda}} \|h\|_\rho \left(\left(1+u^2\right) + \frac{\epsilon}{3\|h\|_\rho} \right), \forall (s+d) \geq n_1$$

$$< \frac{\|h\|_\rho}{\left(1+u^2\right)^{\lambda}} + \frac{\epsilon}{3}, \forall (s+d) \geq n_1. \tag{27}$$

Hence, $\|h\|_\rho \sup_{u\geq u_0} \frac{\left| D^{\sigma,\tau}_{s+d,c}\left(1+t^2,u\right)\right|}{\left(1+u^2\right)^{1+\lambda}} < \frac{\|h\|_\rho}{\left(1+u_0^2\right)^{\lambda}} + \frac{\epsilon}{3}, \forall (s+d) \geq n_1$. Thus, $I_2 + I_3 < \frac{2\|h\|_\rho}{\left(1+u_0^2\right)^{\lambda}} + \frac{\epsilon}{3}, \quad \forall (s+d) \geq n_1$. Now, let us choose u_0 to be so large that $\frac{\|h\|_\rho}{\left(1+u^2\right)^{\lambda}} < \frac{\epsilon}{6}$. Then,

$$I_2 + I_3 < \frac{2\epsilon}{3} \forall (s+d) \geq n_1. \tag{28}$$

By

$$I_1 = \left\| D^{\sigma,\tau}_{s+d,c}(h) - h \right\|_{C[0,u_0]} < \frac{\epsilon}{3}, \quad \forall (s+d) \geq n_2. \tag{29}$$

Let $n_0 = \max(n_1, n_2)$. Then, combining (27)-(29)

$$\sup_{u\in[0,\infty)} \frac{\left| D^{\sigma,\tau}_{s+d,c}(h,u) - h(u)\right|}{\left(1+u^2\right)^{1+\lambda}} < \epsilon, \quad \forall (s+d) \geq n_0.$$

This completes the proof. □

6. Statistical convergence

Let $A = \left(c_{(s+d)i}\right)$ be a non-negative infinite summability matrix. For a given sequence $u := (u_i)$, the A-transform of u denoted by $Au : ((Au)_{s+d})$ is defined as

$$(Au)_{s+d} = \sum_{i=1}^{\infty} c_{(s+d)i} u_i.$$

Provided the series converges for each $s+d$. A is said to be regular if $\lim_{s+d}(Au)_{s+d} = D$ whenever $\lim u = D$. Then $u = (u_{s+d})$ is said to be a A-statistically convergent to D i.e.$st_A - \lim u = D$ if for every $\epsilon > 0, \lim_{s+d} \sum_{i:|u_i - D|\geq\epsilon} c_{(s+d)i} = 0$. Recently, the statistical convergence properties have been investigated for several operators by following the work of Gadjiev and Orhan [32], for instance, Agratini [29], Özarslan [28], Erkus et al. [31], Doğru and Örkcü [30] etc.

Theorem 6.1. *Let $u \in [0,\infty)$ and $A = \left(c_{(s+d)i}\right)$ be a non-negative regular summability matrix. Then, for all $h \in C^0_\rho$ we have*

$$st_A - \lim_{s+d} \left\| D^{\sigma,\tau}_{s+d,c}(h,.) - h \right\|_{\rho_\sigma} = 0,$$

where $\rho_\sigma(u) = 1 + u^{2+\sigma}, \sigma > 0$.

Proof. In [33], it is enough to prove that

$$st_A - \lim_{s+d} \|D^{\sigma,\tau}_{s+d,c}(e_i,.) - e_i\|_{\rho_0} = 0,$$

where $e_i(u) = u^i, i = 0,1,2$

$$st_A - \lim_{s+d} \left\| D^{\sigma,\tau}_{s+d,c}(e_0,.) - e_0 \right\|_{\rho_0} = 0.$$

Using Corollary 2.3, we have

$$\begin{aligned}
\left| D^{\sigma,\tau}_{s+d,c}(t;u) - u \right\|_\rho &= \sup_{u\in[0,\infty)} \| \left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)} - 1 \right) u \\
&+ \frac{c(s+d)}{(s+d+\tau)(s+d-1)} \frac{u}{(1+u)} + \frac{\sigma}{(s+d+\tau)} \Big| \frac{1}{1+u^2} \\
&\leq \left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)} - 1 \right) + \frac{(s+d)c}{(s+d+\tau)(s+d-1)} + \frac{\sigma}{s+d+\tau}.
\end{aligned}$$

For each $\epsilon > 0$, we define the following sets

$$\begin{aligned}
C_1 : &= \left\{ s+d : \left\| D^{(\sigma,\tau)}_{s+d,c}(e_1,.) - e_1 \right\| \geq \epsilon \right\} \\
C_2 : &= \left\{ s+d : \left(\frac{(s+d)^2}{(s+d+\tau)(s+d-1)} - 1 \right) \geq \frac{\epsilon}{3} \right\} \\
C_3 : &= \left\{ s+d : \frac{(s+d)c}{(s+d+\tau)(s+d-1)} \geq \frac{\epsilon}{3} \right\} \\
C_4 : &= \left\{ s+d : \frac{\sigma}{s+d+\tau} \geq \frac{\epsilon}{3} \right\}.
\end{aligned}$$

It is clear that $C_1 \subseteq C_2 \cup C_3 \cup C_4$, which implies that $\sum_{i\in C_1} c_{(s+d)i} \leq \sum_{i\in C_2} c_{(s+d)i} + \sum_{i\in C_3} c_{(s+d)i} + \sum_{i\in C_4} c_{(s+d)i}$. Hence, we have

$$st_A - \lim_{s+d} \left\| D^{\sigma,\tau}_{s+d,c}(e_1,.) - e_1 \right\|_{\rho_0} = 0.$$

Again using Corollary 2.3, we obtain

$$\begin{aligned}
\left\| D^{\sigma,\tau}_{s+d,c}(t^2;x) - x^2 \right\|_\rho &= \sup_{u\in[0,\infty)} | u \left(\frac{2(s+d)^3}{(s+d+\tau)^2(s+d-1)(s+d-2)} \right. \\
&+ \left. \frac{2(s+d)^2\sigma}{(s+d+\tau)(s+d-1)} \right) + u^2 \left(\frac{(s+d)^2}{(s+d+\tau)^2(s+d-1)(s+d-2)} \right. \\
&+ \left. \frac{(s+d)^4}{(s+d+\tau)^2(s+d-1)(s+d-2)} - 1 \right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{u}{(1+u)}\left(\frac{2(s+d)^2c}{(s+d+\tau)^2(s+d-1)(s+d-2)}+\frac{2(s+d)c\sigma}{(s+d+\tau)(s+d-1)}\right)\\
&+\frac{(s+d)^2}{(s+d+\tau)^2(s+d-1)(s+d-2)}\frac{c^2u^2}{(1+u)^2}\\
&+\frac{(s+d)^3}{(s+d+\tau)^2(s+d-1)(s+d-2)}\frac{2cu^2}{(1+u)}+\frac{\sigma^2}{(s+d+\tau)^2}\mid\frac{1}{1+u^2}\\
\leq\ &\left(\frac{2(s+d)^3}{((s+d)+\tau)^2(s+d-1)(s+d-2)}+\frac{2(s+d)^2\sigma}{(s+d+\tau)(s+d-1)}\right)\\
&+\left(\frac{(s+d)^2}{(s+d+\tau)^2(s+d-1)(s+d-2)}+\frac{(s+d)^4}{(s+d+\tau)^2(s+d-1)(s+d-2)}-1\right)\\
&+\left(\frac{2(s+d)^2c}{(s+d+\tau)^2(s+d-1)(s+d-2)}+\frac{2(s+d)c\sigma}{(s+d+\tau)(s+d-1)}\right)\\
&+\frac{c^2(c+d)^2}{(s+d+\tau)^2(s+d-1)(s+d-2)}+\frac{2c(s+d)^3}{(s+d+\tau)^2(s+d-1)(s+d-2)}\\
&+\frac{\sigma^2}{(s+d+\tau)^2}.
\end{aligned}
$$

For a given $\epsilon>0$, we have the following sets

$$D_1:=\left\{s+d:\left\|D^{\sigma,\tau}_{s+d,c}(e_2,.)-e_2\right\|\geq\epsilon\right\}$$

$$D_2:=\left\{s+d:\frac{2(s+d)^3}{(s+d+\tau)^2(s+d-1)(s+d-2)}+\frac{2(s+d)^2\sigma}{(s+d+\tau)(s+d-1)}\geq\frac{\epsilon}{6}\right\}$$

$$
\begin{aligned}
D_3:=&\left\{s+d:\left(\frac{(s+d)^2}{(s+d+\tau)^2(s+d-1)(s+d-2)}\right.\right.\\
&\left.\left.\qquad+\frac{(s+d)^4}{(s+d+\tau)^2(s+d-1)(s+d-2)}-1\right)\geq\frac{\epsilon}{6}\right\}
\end{aligned}
$$

$$D_4:=\left\{s+d:\left(\frac{2(s+d)^2c}{(s+d+\tau)^2(s+d-1)(s=d-2)}+\frac{2(s+d)c\sigma}{(s+d+\tau)(s=d-1)}\right)\geq\frac{\epsilon}{6}\right\}$$

$$D_5:=\left\{s+d:\frac{c^2(s+d)^2}{(s+d+\tau)^2(s+d-1)(s+d-2)}\geq\frac{\epsilon}{6}\right\}$$

$$D_6:=\left\{s+d:\frac{2c(s+d)^3}{(s+d+\tau)^2(s+d-1)(s+d-2)}\geq\frac{\epsilon}{6}\right\}.$$

$$D_7:=\left\{s+d:\frac{\sigma^2}{(s+d+\tau)^2}\geq\frac{\epsilon}{6}\right\}.$$

Then it is clear that $D_1\subseteq D_2\cup D_3\cup D_4\cup D_5\cup D_6\cup D_7$. Hence, we observe that

$$
\begin{aligned}
\sum_{i\in D_1}c_{(c+d)i}\ \leq\ &\sum_{i\in D_2}c_{(s+d)i}+\sum_{i\in D_3}c_{(s+d)i}+\sum_{i\in D_4}c_{(s+d)i}\\
+\ &\sum_{i\in D_5}c_{(s+d)i}+\sum_{i\in D_6}c_{(s+d)i}+\sum_{i\in D_7}c_{(s+d)i}.
\end{aligned}
$$

Taking limit as $(s+d)\to\infty$, we have

$$st_A-\lim_{(s+d)}\left\|D^{\sigma,\tau}_{s+d,c}(e_2,.)-e_2\right\|_{\rho_0}=0.$$

Hence, the proof is completed. □

7. Conclusion

In this study, we provides a comprehensive analysis of the approximation properties of Baskakov-Durrmeyer-Schurer operators. We investigate their convergence behavior, we prove a Korovkin-type approximation theorem and obtain the rate of convergence of these operators. Moreover, local and global approximation properties are discussed in different functional spaces. Lastly, A-statistical approximation results are presented.

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