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Cauchy-Schwarz Inequality for Shifted Quantum Integral Operator

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Abstract. We show that Cauchy-Schwarz inequality for shifted quantum integral operator does not hold in general and we prove the conditions under which it is valid. We apply it to Ostrowski type inequalities for shifted quantum integral operator.

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1. Introduction

Quantum calculus is a type of calculus where the main notions of derivative and integral are defined without using the limits. Instead, it is based on the concept of finite differences. These can be written in two distinct ways which relate to two principal branches of the quantum calculus. One of them is h-calculus, which is based on standard finite differences. It has numerous applications in physics and differential equations. The second branch is q-calculus based on finite difference form as follows

$$\frac{f(x_0) - f(qx_0)}{(1-q)\,x_0}, \quad q \in \langle 0, 1\rangle .$$

The q-calculus is applied in many scientific fields, as well. Its main applications are in number theory, numerical analysis, ordinary and partial difference equations, dynamic systems, algebra, communications and medical engineering (see for example [2], [4], [5], [8],[11] and references therein).

Within the q-calculus theory development, a systematic approach to investigation calls for obtaining the q-analogues of classic results (see [8], [5]). In this paper we are focused on obtaining Ostrowski type inequalities for shifted quantum integral operator by using the 2-seminorm. We will now introduce everything required to properly describe the problem we are dealing with.

The following **Ostrowski inequality** is well known [10]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\,\|f'\|_\infty . \tag{1}$$

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It holds for every $x \in [a,b]$, whenever $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) with derivative $f' : (a,b) \to \mathbb{R}$ bounded on (a,b) i.e.

$$\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < +\infty.$$

Ostrowski proved this inequality in 1938, and since then it has been generalized in a number of ways. This inequality can easily be proved by using the following **Montgomery identity** (see for instance [9])

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\,dt + \int_a^b P(x,t) f'(t)\,dt, \tag{2}$$

where $P(x,t)$ is the Peano kernel, defined by

$$P(x,t) = \begin{cases} \frac{1}{b-a}(t-a), & a \le t \le x, \\ \frac{1}{b-a}(t-b), & x < t \le b \end{cases}. \tag{3}$$

If we apply Hölder inequality to (2) we obtain

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| = \left| \int_a^b P(x,t) f'(t)\,dt \right| \le \|P(x,t)\|_1 \|f'\|_\infty,$$

where 1-norm is calculated with respect to variable t. Using

$$\begin{aligned} \|P(x,t)\|_1 = \int_a^b |P(x,t)|\,dt &= \frac{1}{b-a} \left(\int_a^x (t-a)\,dt + \int_x^b (b-t)\,dt \right) \\ &= \frac{(x-a)^2 + (b-x)^2}{2(b-a)} = \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a), \end{aligned}$$

we obtain (1). In a similar way, if f' is square-integrable function on the interval $[a,b]$ we can utilize the Cauchy-Schwarz inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| = \left| \int_a^b P(x,t) f'(t)\,dt \right| \le \|P(x,t)\|_2 \|f'\|_2$$

where

$$\|f'\|_2 = \left(\int_a^b |f'(t)|^2\,dt \right)^{\frac{1}{2}} < +\infty$$

and

$$\begin{aligned} \|P(x,t)\|_2 = \left(\int_a^b |P(x,t)|^2\,dt \right)^{\frac{1}{2}} &= \frac{1}{b-a} \left(\int_a^x (t-a)^2\,dt + \int_x^b (b-t)^2\,dt \right)^{\frac{1}{2}} \\ &= \frac{1}{b-a} \sqrt{\frac{(x-a)^3 + (b-x)^3}{3}} \end{aligned}$$

to obtain Ostrowski type inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \le \frac{1}{b-a} \sqrt{\frac{(x-a)^3 + (b-x)^3}{3}} \|f'\|_2.$$

In [2] Ostrowski inequality for shifted quantum integral operator is presented. The bound is given for every $x \in [a,b]$ and also tighter bound, but valid only on the q-lattice, that is for $x = a + q^m (b-a)$, $m \in \mathbb{N} \cup \{0\}$.

Theorem 1.1. *(**Ostrowski inequality for q-calculus**) Let $f : [a,b] \to \mathbb{R}$ be a q-integrable function over $[a,b]$ and continuous at a. Then the following inequalities hold*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, d_q^a t \right| \leq M(x) \left\| D_q^a f \right\|_\infty^{[a,b]}$$

where

$$M(x) = \begin{cases} (b-a)\left(\frac{1+2q^{m+1}}{1+q} - q^m\right), & \text{if } x = a + q^m (b-a) \text{ for } m \in \mathbb{N} \cup \{0\}, \\ (b-a)\left(\frac{x-a}{b-a} + \frac{1}{1+q}\right), & \text{otherwise.} \end{cases}$$

The aim of this paper is to obtain Ostrowski type inequalities for shifted quantum integral operator by using the 2-seminorm, that is for functions whose q-derivative is a q-square-integrable function on the interval $[a,b]$. The rest of the paper is organized as follows: in Section 2 we give preliminaries for shifted q-calculus, in Section 3 we show by counterexample that Cauchy-Schwarz inequality for quantum integral operator does not hold in general. We prove that it is valid on the q-lattice, that is, if the endpoints of the q-integral are $a + q^m (x-a)$ and x, for any $m \in \mathbb{N} \cup \{0\}$. In Section 4 we apply this result to Ostrowski type inequalities for shifted quantum integral operator.

2. Shifted q-derivative and q-integral

Taking $q \in \langle 0,1 \rangle$, the **shifted q-derivative** of an arbitrary function $f : [a,b] \to \mathbb{R}$ is defined by (see [11])

$$D_q^a f(x) = \frac{f(x) - f(a + q(x-a))}{(1-q)(x-a)}, \text{ if } x \in \langle a, b], $$
$$D_q^a f(a) = \lim_{x \to a} D_q^a f(x).$$

We say that f is q-differentiable for every $x \in \langle a,b]$, since the shifted q-derivative exists. Furthermore, if $\lim_{x \to a} D_q f(x)$ exists, we say that f is q-differentiable on $[a,b]$. The shifted q-derivative is generalization of the **Euler-Jackson q-difference operator** $D_q^0 f(x)$(see [6]). We can assume both to be discretizations of ordinary derivative. If f is differentiable function then it is valid

$$\lim_{q \to 1} D_q^a f(x) = f'(x).$$

As in [11], the shifted q-integral is defined by

$$\int_a^x f(t)\, d_q^a t = (1-q)(x-a) \sum_{k=0}^{\infty} q^k f\left(a + q^k (x-a)\right), \quad x \in [a,b]. \tag{4}$$

If the series on right hand-side converges we say that q-integral $\int_a^x f(t)\, d_q^a t$ exists. Furthermore, if f is continuous on $[a, b]$ as $q \to 1$, then the series

$$(1-q)(x-a)\sum_{k=0}^{\infty} q^k f\left(a + q^k (x-a)\right)$$

tends to the Riemann integral (see [3], [8]) and we have

$$\lim_{q\to 1}\int_a^x f(t)\, d_q^a t = \int_a^x f(t)\, dt.$$

Let us note that shifted q-integral generalizes the **Jacksons q-integral** $\int_0^x f(t)\, d_q^0 t$ (see [7]). For an arbitary $c \in \langle a, x\rangle$ shifted q-integral is defined by

$$\int_c^x f(t)\, d_q^a t = \int_a^x f(t)\, d_q^a t - \int_a^c f(t)\, d_q^a t. \tag{5}$$

In the following theorem important properties of shifted q-derivatives and q-integrals are given (see [11]).

Theorem 2.1. *For function $f : [a, b] \to \mathbb{R}$, $q \in \langle 0, 1\rangle$ and $x \in [a, b]$ the following identities hold*

(i)

$$D_q^a\left(\int_a^x f(t)\, d_q^a\, t\right) = f(x)$$

(ii)

$$\int_a^x D_q^a f(t)\, d_q^a\, t = f(x) - f(a)$$

(iii)

$$\int_a^x (f(t) + g(t))\, d_q^a\, t = \int_a^x f(t)\, d_q^a\, t + \int_a^x g(t)\, d_q^a\, t$$

(iv)

$$\int_a^x \alpha f(t)\, d_q^a\, t = \alpha\int_a^x f(t)\, d_q^a\, t, \quad \alpha \in \mathbb{R}$$

3. Cauchy-Schwarz inequality for shifted quantum integral operator

In this section we show that Cauchy-Schwarz inequality for shifted quantum integral operator does not hold in general. We prove the conditions under which it is valid. Here and hereafter we suppose that $f : [a, b] \to \mathbb{R}$ is q-differentiable on $[a, b]$, that is $\lim_{x\to a} D_q^a f(x)$ exists.

The symbol $L_q^2[a,b]$ denotes the space of q-square-integrable functions on the interval $[a,b]$ equipped with the seminorm $\|f\|_2^{[a,b],q}$ (or simply $\|f\|_2^{[a,b]}$) defined by

$$\|f\|_2^{[a,b]} = \left(\int_a^b |f(t)|^2 \, d_q^a t \right)^{\frac{1}{2}}.$$

Note that $\|\cdot\|_2^{[a,b]}$ is not actually a norm since it takes into account only values of f at $a + q^m (b-a)$, $m \in \mathbb{N} \cup \{0\}$. To make $L_q^2[a,b]$ fully fledged Banach space it is necessary so to take a quotient, considering functions equal if they coincide for $x = a + q^m (b-a)$, $m \in \mathbb{N} \cup \{0\}$.

We also define seminorm in the case when interval of integration is different from $[a,b]$. In the case when $a < x < b$ we define

$$\|f\|_2^{[a+q^m(x-a),x]} = \left(\int_{a+q^m(x-a)}^{x} |f(t)|^2 \, d_q^a t \right)^{\frac{1}{2}}.$$

In the next example we show that Cauchy-Schwarz inequality for quantum integral operator does not generally hold. We will also show that in some cases even 2-seminorm does not exist.

Example 3.1. Consider the functions $f, g : [a,b] \to \mathbb{R}$ defined by $f(x) = b - x$, $x \in [a,b]$ and $g(x) = \frac{1}{b-a}$, $x \in [a,b]$. Using the following result from [12]

$$\int_a^x (t-a)^n \, d_q^a \, t = \left(\frac{1-q}{1-q^{n+1}} \right) (x-a)^{n+1}$$

we calculate the left hand side of the Cauchy-Schwarz inequality

$$\int_{\frac{a+b}{2}}^{b} |f(t)| \, |g(t)| \, d_q^a \, t =$$

$$= \frac{1}{b-a} \int_{\frac{a+b}{2}}^{b} (b-t) \, d_q^a \, t = \frac{1}{b-a} \left(\int_{\frac{a+b}{2}}^{b} (b-a) \, d_q^a \, t - \int_{\frac{a+b}{2}}^{b} (t-a) \, d_q^a \, t \right)$$

$$= \frac{1}{b-a} \left(\int_a^b (b-a) \, d_q^a \, t - \int_a^{\frac{a+b}{2}} (b-a) \, d_q^a \, t - \int_a^b (t-a) \, d_q^a \, t + \int_a^{\frac{a+b}{2}} (t-a) \, d_q^a \, t \right)$$

$$= \frac{1}{b-a} \left(\frac{(b-a)^2}{2} - \frac{3(b-a)^2}{4(1+q)} \right) = (b-a) \frac{2q-1}{4(1+q)}.$$

For the right hand side of the Cauchy-Schwarz inequality we need

$$\|g\|_2^{\left[\frac{a+b}{2},b\right]} = \left(\int_{\frac{a+b}{2}}^{b} \left| \frac{1}{b-a} \right|^2 d_q^a t \right)^{\frac{1}{2}} = \frac{1}{b-a} \sqrt{b - \frac{b+a}{2}} = \sqrt{\frac{1}{2(b-a)}}.$$

and

$$\|f\|_2^{\left[\frac{a+b}{2},b\right]} = \left(\int\limits_{\frac{a+b}{2}}^{b} |b-t|^2 \, d_q^a t\right)^{\frac{1}{2}}$$

$$\int\limits_{\frac{a+b}{2}}^{b} |b-t|^2 \, d_q^a t = \int\limits_{\frac{a+b}{2}}^{b} ((t-a)-(b-a))^2 \, d_q^a t$$

$$= \int\limits_{\frac{a+b}{2}}^{b} \left((t-a)^2 - 2(t-a)(b-a) + (b-a)^2\right) d_q^a t$$

$$= \left(\frac{1}{1+q+q^2}\right) \frac{7}{8} (b-a)^3 - 2(b-a)\left(\frac{1}{1+q}\right) \frac{3}{4} (b-a)^2 + \frac{(b-a)^3}{2}$$

$$= (b-a)^3 \frac{4q^3-4q^2+3q-1}{4(1+q)(1+q+q^2)}.$$

Thus $\|f\|_2^{\left[\frac{a+b}{2},b\right]}$ exists only if $4q^3-4q^2+3q-1 \geq 0$, that is for $q \in \left[\frac{1}{2},1\right\rangle$. In this case Cauchy-Schwarz inequality reduces to inequality

$$\frac{2q-1}{4(1+q)} \leq \sqrt{\frac{4q^3-4q^2+3q-1}{8(1+q)(1+q+q^2)}}$$

which is equivalent to

$$4q^4 - 3q^2 + 7q - 3 \geq 0.$$

By factorizing the polynomial we get

$$4q^4 - 3q^2 + 7q - 3 = (2q-1)(2q+3)\left(q^2-q+1\right) \geq 0$$

Last inequality does not hold for $q \in \left\langle 0, \frac{1}{2}\right\rangle$, consequently Cauchy-Schwarz inequality in this example holds only for $q \in \left[\frac{1}{2},1\right\rangle$.

In the next theorem we show that Cauchy-Schwarz inequality is valid for every $q \in \langle 0,1\rangle$ if the endpoints of the q-integral are $a+q^m(x-a)$ and x, for any $m \in \mathbb{N} \cup \{0\}$.

Theorem 3.1. *Let $q \in \langle 0,1\rangle$, and let $f,g : [a,b] \to \mathbb{R}$ be functions such that f, $g \in L_q^2[a,b]$. Then for every $x \in \langle a,b]$ and $m \in \mathbb{N}$ the following identity holds*

$$\int\limits_{a+q^m(x-a)}^{x} |f(t)|\,|g(t)| \, d_q^a \, t \leq \|f\|_2^{[a+q^m(x-a),x]} \, \|g\|_2^{[a+q^m(x-a),x]} . \tag{6}$$

Proof. From the definitions (4) and (5) we have

$$\int\limits_{a+q^m(x-a)}^{x} |f(t)|\,|g(t)| \, d_q^a \, t = \int\limits_{a}^{x} |f(t)|\,|g(t)| \, d_q^a \, t - \int\limits_{a}^{a+q^m(x-a)} |f(t)|\,|g(t)| \, d_q^a \, t$$

$$= (1-q)(x-a)\sum_{k=0}^{\infty} q^k \left|f\left(a+q^k(x-a)\right)\right| \left|g\left(a+q^k(x-a)\right)\right|$$

$$- (1-q)(q^m(x-a)) \sum_{k=0}^{\infty} q^k \left| f\left(a+q^{k+m}(x-a)\right)\right| \left| g\left(a+q^{k+m}(x-a)\right)\right|$$
$$= (1-q)(x-a) \sum_{k=0}^{m-1} q^k \left| f\left(a+q^k(x-a)\right)\right| \left| g\left(a+q^k(x-a)\right)\right|.$$

By using the discrete Cauchy-Schwarz inequality for sequence spaces (the discrete Hölder's inequality for equal conjugate exponents) we obtain

$$(1-q)(x-a) \sum_{k=0}^{m-1} q^k \left| f\left(a+q^k(x-a)\right)\right| \left| g\left(a+q^k(x-a)\right)\right|$$
$$= (1-q)(x-a) \sum_{k=0}^{m-1} \left(q^{k/2} \left| f\left(a+q^k(x-a)\right)\right|\right) \left(q^{k/2} \left| g\left(a+q^k(x-a)\right)\right|\right)$$
$$\leq \left((1-q)(x-a) \sum_{k=0}^{m-1} q^k \left| f\left(a+q^k(x-a)\right)\right|^2\right)^{\frac{1}{2}}$$
$$\cdot \left((1-q)(x-a) \sum_{k=0}^{m-1} q^k \left| g\left(a+q^k(x-a)\right)\right|^2\right)^{\frac{1}{2}}$$
$$= \left(\int_a^x |f(t)|^2 \, d_q^a t - \int_a^{a+q^m(x-a)} |f(t)|^2 \, d_q^a t\right)^{\frac{1}{2}} \left(\int_a^x |g(t)|^2 \, d_q^a t - \int_a^{a+q^m(x-a)} |g(t)|^2 \, d_q^a t\right)^{\frac{1}{2}}$$
$$= \left(\int_{a+q^m(x-a)}^x |f(t)|^2 \, d_q^a t\right)^{\frac{1}{2}} \left(\int_{a+q^m(x-a)}^x |g(t)|^2 \, d_q^a t\right)^{\frac{1}{2}}$$
$$= \|f\|_2^{[a+q^m(x-a),x]} \, \|g\|_2^{[a+q^m(x-a),x]}.$$

□

Remark 3.1. If we take $m \to \infty$ in (6) we obtain result from [12]

$$\int_a^x |f(t)| \, |g(t)| \, d_q^a t \leq \|f\|_2^{[a,x]} \, \|g\|_2^{[a,x]}. \tag{7}$$

4. Ostrowski type inequalities for shifted q-integrals

In this section, we give two generalizations of Ostrowski inequality for shifted quantum integral operator. The first inequality is valid for every $x \in [a,b]$ and the second, with tighter bound then the first, that holds only on the q-lattice, that is for $x = a + q^m(b-a)$, $m \in \mathbb{N} \cup \{0\}$. Here we will use Montgomery identity and Cauchy-Schwarz inequality. We state the identity for q-integrals that is given in [1].

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be arbitrary function and $x\in[a,b]$. Then the following identity holds*

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,d_q^a\,t \tag{8}$$
$$=(b-a)\left(\int_0^{\frac{x-a}{b-a}}(qt)\,D_qf(tb+(1-t)a)\,d_q^0\,t+\int_{\frac{x-a}{b-a}}^1(qt-1)\,D_qf(tb+(1-t)a)\,d_q^0\,t\right).$$

Remark 4.1. In the case when $q=1$, identity (8) reduces to classic Montgomery identity for Riemann integral.

We will need the following result from [8]

$$\int_0^x t^n d_q^0\,t=\left(\frac{1-q}{1-q^{n+1}}\right)x^{n+1}. \tag{9}$$

In the next theorem we give a generalization of **Ostrowski inequality for q-integrals** that is valid for every $x\in[a,b]$.

Theorem 4.2. *Let $q\in\langle 0,1\rangle$, $f:[a,b]\to\mathbb{R}$ be a q-integrable function over $[a,b]$ such that $D_q^af\in L_q^2[a,b]$ and $x\in[a,b]$. Then the following inequalities hold*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,d_q^at\right|\le\sqrt{\frac{(b-a)(1+q^2)}{(1+q)(1+q+q^2)}}\left\|D_q^af\right\|_2^{[a,b]}+\sqrt{x-a}\left\|D_q^af\right\|_2^{[a,x]}.$$

Proof. Starting from (8) we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,d_q^a\,t\right|$$
$$=(b-a)\left|\int_0^{\frac{x-a}{b-a}}(qt)\,D_q^af(tb+(1-t)a)\,d_q^0\,t+\int_{\frac{x-a}{b-a}}^1(qt-1)\,D_q^af(tb+(1-t)a)\,d_q^0\,t\right|$$
$$=(b-a)\left|\int_0^1(qt-1)\,D_q^af(tb+(1-t)a)\,d_q^0\,t+\int_0^{\frac{x-a}{b-a}}D_q^af(tb+(1-t)a)\,d_q^0\,t\right|$$
$$\le(b-a)\left(\left|\int_0^1(qt-1)\,D_q^af(tb+(1-t)a)\,d_q^0\,t\right|+\left|\int_0^{\frac{x-a}{b-a}}D_q^af(tb+(1-t)a)\,d_q^0\,t\right|\right).$$

By using triangle inequality for q-integrals (see [8]) and inequality (7) if follows

$$
\begin{aligned}
&(b-a)\left(\left|\int_0^1 (qt-1)\, D_q^a f(tb+(1-t)a)\, d_q^0 t\right| + \left|\int_0^{\frac{x-a}{b-a}} D_q^a f(tb+(1-t)a)\, d_q^0 t\right|\right) \\
&\leq (b-a)\left(\int_0^1 |qt-1| \left|D_q^a f(tb+(1-t)a)\right| d_q^0 t + \int_0^{\frac{x-a}{b-a}} \left|D_q^a f(tb+(1-t)a)\right| d_q^0 t\right) \\
&\leq (b-a)\left\|D_q^a f(tb+(1-t)a)\right\|_2^{[0,1]} \|(1-qt)\|_2^{[0,1]} \\
&\quad + (b-a)\sqrt{\frac{x-a}{b-a}}\left\|D_q^a f(tb+(1-t)a)\right\|_2^{\left[0,\frac{x-a}{b-a}\right]}.
\end{aligned}
$$

Furthermore, for $x \in [a,b]$ we obtain

$$
\begin{aligned}
&\left\|D_q^a f(tb+(1-t)a)\right\|_2^{\left[0,\frac{x-a}{b-a}\right]} = \left\|\frac{f(a+t(b-a)) - f(a+qt(b-a))}{(1-q)\,t\,(b-a)}\right\|_2^{\left[0,\frac{x-a}{b-a}\right]} \\
&= \left((1-q)\frac{x-a}{b-a}\sum_{k=0}^{\infty} q^k \left(\frac{f\left(a+q^k(x-a)\right) - f\left(a+q^{k+1}(x-a)\right)}{(1-q)\,q^k(x-a)}\right)^2\right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{b-a}}\left((1-q)(x-a)\sum_{k=0}^{\infty} q^k \left(\frac{f\left(a+q^k(x-a)\right) - f\left(a+q^{k+1}(x-a)\right)}{(1-q)\,q^k(x-a)}\right)^2\right)^{\frac{1}{2}} \\
&= \frac{1}{\sqrt{b-a}}\left\|\frac{f(t) - f(a+q(t-a))}{(1-q)(t-a)}\right\|_2^{[a,x]} = \frac{1}{\sqrt{b-a}}\left\|D_q^a f(t)\right\|_2^{[a,x]}.
\end{aligned}
$$

In particular, we get $\left\|D_q^a f(tb+(1-t)a)\right\|_2^{[0,1]} = \frac{1}{\sqrt{b-a}}\left\|D_q^a f\right\|_2^{[a,b]}$.
Afterwards, we apply (9) to obtain

$$
\begin{aligned}
\|(1-qt)\|_2^{[0,1]} &= \left(\int_0^1 \left(1-2qt+q^2t^2\right) d_q^0 t\right)^{\frac{1}{2}} = \left(1 - \frac{2q}{1+q} + \frac{q^2}{1+q+q^2}\right)^{\frac{1}{2}} \\
&= \left(\frac{1+q^2}{(1+q)\left(1+q+q^2\right)}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
&(b-a)\left\|D_q^a f(tb+(1-t)a)\right\|_2^{[0,1]} \|(1-qt)\|_2^{[0,1]} \\
&+ (b-a)\sqrt{\frac{x-a}{b-a}}\left\|D_q^a f(tb+(1-t)a)\right\|_2^{\left[0,\frac{x-a}{b-a}\right]} \\
&= \sqrt{\frac{(b-a)\left(1+q^2\right)}{(1+q)\left(1+q+q^2\right)}}\left\|D_q^a f(t)\right\|_2^{[a,b]} + \sqrt{x-a}\left\|D_q^a f(t)\right\|_2^{[a,x]}.
\end{aligned}
$$

□

Theorem that follows gives a generalization of **Ostrowski inequality for q-integrals** that is valid only on the q-lattice, that is for $x = a+q^m(b-a)$, $m \in \mathbb{N}\cup\{0\}$.

Theorem 4.3. *Let $f:[a,b]\to\mathbb{R}$ be a q-integrable function over $[a,b]$ such that $D_q^a f\in L_q^2[a,b]$ and $m\in\mathbb{N}\cup\{0\}$. Then the following inequality holds*

$$\left|f\left(a+q^m(b-a)\right)-\frac{1}{b-a}\int_a^b f(t)\,d_q^a t\right|\le\sqrt{b-a}\left(\sqrt{\frac{q^{3m+2}}{1+q+q^2}}\left\|D_q^a f\right\|_2^{[a,a+q^m(b-a)]}\right.$$
$$\left.+\sqrt{1-q^m-\frac{2q\left(1-q^{2m}\right)}{1+q}+\frac{q^2\left(1-q^{3m}\right)}{1+q+q^2}}\left\|D_q^a f\right\|_2^{[a+q^m(b-a),b]}\right).$$

Proof. Starting from (8) we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,d_q^a\,t\right|$$
$$=(b-a)\left|\int_0^{\frac{x-a}{b-a}}(qt)\,D_q^a f\left(tb+(1-t)a\right)d_q^0\,t+\int_{\frac{x-a}{b-a}}^1(qt-1)\,D_q^a f\left(tb+(1-t)a\right)d_q^0\,t\right|$$
$$\le(b-a)\left(\left|\int_0^{\frac{x-a}{b-a}}(qt)\,D_q^a f\left(tb+(1-t)a\right)d_q^0\,t\right|+\left|\int_{\frac{x-a}{b-a}}^1(qt-1)\,D_q^a f\left(tb+(1-t)a\right)d_q^0\,t\right|\right).$$

In order to apply triangle inequality for q-integrals on the q-lattice ([8]) and inequality (6) we have to take $x=a+q^m(b-a)$. Therefore, we obtain

$$\left|\int_0^{\frac{x-a}{b-a}}(qt)\,D_q^a f\left(tb+(1-t)a\right)d_q^0\,t\right|\le\int_0^{\frac{x-a}{b-a}}\left|(qt)\,D_q^a f\left(tb+(1-t)a\right)\right|d_q^0\,t$$
$$\le\|qt\|_2^{\left[0,\frac{x-a}{b-a}\right]}\left\|D_q^a f\left(tb+(1-t)a\right)\right\|_2^{\left[0,\frac{x-a}{b-a}\right]}$$
$$=\frac{1}{\sqrt{b-a}}\sqrt{\frac{q^2}{1+q+q^2}\left(\frac{x-a}{b-a}\right)^3}\left\|D_q^a f\right\|_2^{[a,x]}.$$

Here we have used equalities: $\left\|D_q^a f\left(tb+(1-t)a\right)\right\|_2^{\left[0,\frac{x-a}{b-a}\right]}=\frac{1}{\sqrt{b-a}}\left\|D_q^a f\right\|_2^{[a,x]}$ and $\|qt\|_2^{\left[0,\frac{x-a}{b-a}\right]}=\left(\int_0^{\frac{x-a}{b-a}}q^2t^2d_q^0\,t\right)^{\frac{1}{2}}=\sqrt{\frac{q^2}{1+q+q^2}\left(\frac{x-a}{b-a}\right)^3}$.
Thus, for $x=a+q^m(b-a)$ we have

$$\left|\int_0^{\frac{x-a}{b-a}}(qt)\,D_q^a f\left(tb+(1-t)a\right)d_q^a\,t\right|\le\sqrt{\frac{q^{3m+2}}{(b-a)\left(1+q+q^2\right)}}\left\|D_q^a f\right\|_2^{[a,x]}.$$

Similarly, we obtain

$$\left|\int_{\frac{x-a}{b-a}}^{1}(qt-1)\,D_q^a f(tb+(1-t)a)\,d_q^a t\right| \le \int_{\frac{x-a}{b-a}}^{1}\left|(qt-1)\,D_q^a f(tb+(1-t)a)\right| d_q^a t$$
$$\le \left\|D_q^a f(tb+(1-t)a)\right\|_2^{\left[\frac{x-a}{b-a},1\right]}\|qt-1\|_2^{\left[\frac{x-a}{b-a},1\right]}.$$

Furthermore, for $x\in[a,b]$

$$\left\|D_q^a f(tb+(1-t)a)\right\|_2^{\left[\frac{x-a}{b-a},1\right]} = \left\|\frac{f(a+t(b-a))-f(a+qt(b-a))}{(1-q)\,t\,(b-a)}\right\|_2^{\left[\frac{x-a}{b-a},1\right]}$$
$$=\left((1-q)\sum_{k=0}^{\infty}q^k\left(\frac{f\left(a+q^k(b-a)\right)-f\left(a+q^{k+1}(b-a)\right)}{(1-q)\,q^k\,(b-a)}\right)^2\right.$$
$$\left.-(1-q)\,\frac{x-a}{b-a}\sum_{k=0}^{\infty}q^k\left(\frac{f\left(a+q^k(x-a)\right)-f\left(a+q^{k+1}(x-a)\right)}{(1-q)\,q^k\,(x-a)}\right)^2\right)^{\frac12}$$
$$=\frac{1}{\sqrt{b-a}}\left((1-q)(b-a)\sum_{k=0}^{\infty}q^k\left(\frac{f\left(a+q^k(b-a)\right)-f\left(a+q^{k+1}(b-a)\right)}{(1-q)\,q^k\,(b-a)}\right)^2\right.$$
$$\left.-(1-q)(x-a)\sum_{k=0}^{\infty}q^k\left(\frac{f\left(a+q^k(x-a)\right)-f\left(a+q^{k+1}(x-a)\right)}{(1-q)\,q^k\,(x-a)}\right)^2\right)^{\frac12}$$
$$=\frac{1}{\sqrt{b-a}}\left\|\frac{f(t)-f(a+q(t-a))}{(1-q)(t-a)}\right\|_2^{[x,b]}=\frac{1}{\sqrt{b-a}}\left\|D_q^a f(t)\right\|_2^{[x,b]}$$

and

$$\|qt-1\|_2^{\left[\frac{x-a}{b-a},1\right]}=\left(\int_0^1\left(1-2qt+q^2t^2\right)d_q^0 t-\left(\int_0^{\frac{x-a}{b-a}}\left(1-2qt+q^2t^2\right)d_q^0 t\right)\right)^{\frac12}$$
$$=\left(1-\frac{2q}{1+q}+\frac{q^2}{1+q+q^2}-\frac{x-a}{b-a}+\frac{2q}{1+q}\left(\frac{x-a}{b-a}\right)^2-\frac{q^2}{1+q+q^2}\left(\frac{x-a}{b-a}\right)^3\right)^{\frac12}.$$

For $x=a+q^m(b-a)$ finally we get

$$\|qt-1\|_2^{\left[\frac{x-a}{b-a},1\right]}=\left(1-q^m-\frac{2q}{1+q}\left(1-q^{2m}\right)+\frac{q^2}{1+q+q^2}\left(1-q^{3m}\right)\right)^{\frac12}$$

and the inequality is proved. □

Remark 4.2. The bound in the inequality form Theorem 4.3 is tighter than the one in the inequality from the Theorem 4.2 for $x=a+q^m(b-a)$. Indeed, it's not hard to see that for $m\in\mathbb{N}\cup\{0\}$

$$\left\|D_q^a f\right\|_2^{[a+q^m(b-a),b]}\le\left\|D_q^a f\right\|_2^{[a,b]},$$

and also for $q \in \langle 0,1\rangle$

$$\sqrt{\frac{x-a}{b-a}} = \sqrt{q^m} \geq \sqrt{\frac{q^{3m+2}}{1+q+q^2}},$$

$$\sqrt{1-q^m - \frac{2q\left(1-q^{2m}\right)}{1+q} + \frac{q^2\left(1-q^{3m}\right)}{1+q+q^2}}$$

$$= \sqrt{\frac{1+q^2-q^m\left(1-q^{m+3}\right)-\left(2q^{m+1}+2q^{m+2}+q^{m+3}\right)\left(1-q^m\right)-q^{3m+2}\left(1+q\right)}{\left(1+q\right)\left(1+q+q^2\right)}}$$

$$\leq \sqrt{\frac{1+q^2}{\left(1+q\right)\left(1+q+q^2\right)}}.$$

5. Conclusion

We have shown generalizations of Ostrowski inequalities for q-integrals, one valid for every $x \in [a,b]$, and the other one valid on q-lattice giving also the tighter bound on the q-lattice. We have shown that q-analogue of Cauchy-Schwarz inequality for quantum integral operator is only valid on the q-lattice, that is for $x = a + q^m\left(b-a\right)$, $m \in \mathbb{N} \cup \{0\}$, and it can not be obtained in general. Applications and further generalizations are possible for classes of functions that fulfill additional properties.

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