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An elementary argument regarding the long-time behaviour of the solution to a stochastic differential equation

Benjamin Jourdain, Claude Le Bris and Tony Lelièvre

Abstract. We describe the long-time behaviour of the solution to a stochastic differential equation without resorting to arguments based upon the study of the associated Fokker Planck equation. This work is motivated by our interest in the modelling of polymeric fluid flows [5, 6, 4].

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1. Introduction

1.1. General mathematical setting. In this short article, we are interested in the long-time behaviour of a stochastic process $X_t \in \mathbb{R}^N$ solution to a stochastic differential equation (SDE) of the following form:

$$dX_t = -\nabla\Pi(X_t)\,dt + b(t, X_t)\,dt + dW_t, \tag{1}$$

where the stochastic processes X_t and W_t are defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, and W_t is a standard N-dimensional $(\mathcal{F}_t)$-adapted Brownian motion.

We suppose that the function Π (called the potential, see Section 2 for the physical background) and the function b are such that there exists a unique strong global-in-time solution to (1) (see [7] for possible conditions).

Regarding the long-time behaviour of b, we assume that $\exists C < \infty$, s.t. for a.e. t, X,

$$|b(t, X)| \le Ce^{-\beta t}\left(1 + |X|\right), \tag{2}$$

where $|.|$ denotes the Euclidean norm on $\mathbb{R}^N$. It is thus expected that, at least in a vague sense that will be made precise below, X_t behaves in the long-time limit as a process X_t^∞ solution to:

$$dX_t^\infty = -\nabla\Pi(X_t^\infty)\,dt + dW_t.$$

For our analysis, we need to impose that Π is an α-convex function, i.e. a function satisfying, for all X and Y in $\mathbb{R}^N$, and for all $\lambda \in (0, 1)$:

$$\Pi(\lambda X + (1-\lambda)Y) \le \lambda\Pi(X) + (1-\lambda)\Pi(Y) - \frac{\alpha\lambda(1-\lambda)}{2}|X - Y|^2. \tag{3}$$

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1.2. The case of a radial potential. A case of particular interest in the applications we have in mind (see Section 2) is the case of a radial potential:

$$\Pi(X) = \pi(|X|) \tag{4}$$

where π is a $\mathcal{C}^2$ function on $\mathbb{R}_+$, which we can assume to be zero at zero.

In this particular case, the α-convexity of Π (3) is equivalent to:

$$\pi \text{ is an } \alpha\text{-convex function such that } \pi'(0) \geq 0. \tag{5}$$

1.3. Long-time behaviour. The stochastic differential equation (1) is standardly associated to the Fokker-Planck equation:

$$\partial_t \psi(t,X) = -\mathrm{div}_X \left((-\nabla\Pi(X) + b(t,X))\psi(t,X) - \frac{1}{2}\nabla_X \psi(t,X) \right) \tag{6}$$

where $\psi(t,X)$ is the density with respect to the Lebesgue measure of the random variable X_t.

An information on the long-time behaviour of X_t is provided by the study of (6). Indeed, entropy methods (see [2, 4]) allow one to deduce the convergence as t goes to infinity of $\psi(t,X)$ to $\psi_\infty(X)$ solution to:

$$0 = \mathrm{div}_X \left((\nabla\Pi(X))\psi_\infty(X) + \frac{1}{2}\nabla_X \psi_\infty(X) \right). \tag{7}$$

Explicitely, $\psi_\infty(X) \propto \exp(-2\Pi(X))$. The convergence typically holds in L^1-norm. Following this approach, we obtain that the law of the solution X_t to (1) converges in variation to the measure μ_∞ defined by $d\mu_\infty(X) = \psi_\infty(X)\,dX$ (in the case of continuous measures, the convergence in variation is equivalent to the convergence of the densities in L^1-norm). The method provides the additional information that the rate of convergence is exponential.

For this approach to work, it is well known that some conditions on the potential Π are required, which allow for some functional inequalities involving the measure μ_∞ (Poincaré or log-Sobolev inequalities). A usual assumption which ensures a log-Sobolev inequality with respect to μ_∞ is the α-convexity of Π. This can be somewhat weakened (for example, a log-Sobolev inequality also holds if Π is a bounded perturbation of an α-convex function, see [1]) but in any case, the strong convexity of Π is the natural assumption to obtain a log-Sobolev inequality. This is linked to some contractivity property of the semigroup associated with the operator $\mathrm{div}\left(\nabla\Pi\,. + \frac{1}{2}\nabla.\right)$ (see again [1]).

Our approach here is different. We wish to identify the long-time behaviour of X_t by manipulating X_t itself and not some non-linear functionals of the density of X_t, as is the case in entropy methods. A major reason that motivates this alternative approach is that non-linear functionals of the density of X_t cannot in general be easily expressed in terms of X_t. Consequently, the approaches that make use of them are not likely to be easily transposed at the discrete level and provide useful information on the discretized process, which is an important purpose for the numerical simulations. We therefore believe that a proof at the continuous level using only the process X_t is a first step towards the comprehension of the long-time behaviour of the discretized problem. We have here in mind the long-time behaviour of some observables $\mathbb{E}(\phi(X_t))$, the expectation value being discretized by an empirical mean (Monte Carlo method), and the process X_t being approximated by an Euler scheme on (1).

Actually, our alternative approach reveals also interesting for the following three reasons:

- since we use elementary arguments, we see in a very clear way why and how the α-convexity of Π plays a role,
- we can handle irregular initial conditions for the law of X_0 (like Dirac functions),
- we obtain a somehow stronger information on the behaviour of X_t since we prove the exponential convergence of the process $(X_t - X_t^\infty)$ to 0 (see Proposition 3.1).

On the other hand, as opposed to methods based on the Fokker-Planck equation that have a broad range of applications, we are restricted to consider very simple cases.

In Section 2, we motivate the study of (1) by explaining how such a SDE arises in the modelling of polymeric fluid flows. We next give in Section 3 the mathematical analysis of the long-time behaviour of the solution to (1).

2. Physical background

2.1. Multiscale models for polymeric fluids.

One motivation for this mathematical study stems from our interest in multiscale models for polymeric fluids. In such models, the macroscopic quantities (such as the velocity or the pressure of the fluid) evolve following a partial differential equation translating some macroscopic conservation laws (typically Navier-Stokes type equations with an additional non-newtonian contribution in the stress tensor). The microscopic variables which describe the conformation of the polymer chains in the fluid follow a stochastic differential equation. The probabilistic nature of the equations at the microscopic level arises through a Langevin description of the interaction between the (small) fluid molecules and the (larger) polymeric chains: the fluid influences the dynamics of the polymer chains through a drag force plus a Brownian term. The problem is fully coupled since the conformation of the polymer chains also influences the flow field through the perturbation of the stress tensor in the momentum equations.

2.2. The dumbbell model.

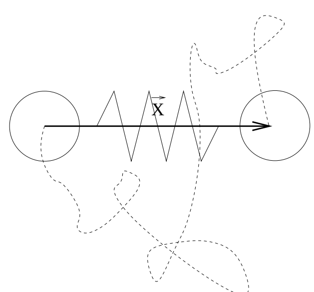

FIGURE 1. In the dumbbell model, the polymer (in dashed line) is modelled by two beads linked by a spring. The length and the orientation of the polymer is given by the so-called end-to-end vector X.

In the sequel, we focus on the dumbbell model (see Figure 1): the conformation of a polymer chain is represented by a vector X (of the dimension of the ambient space) which gives the orientation and the length of the molecule. To derive the dynamics of this vector in a given velocity field $u(t,x)$ (where t denotes time and x denotes position in space), a Langevin equation is written for each end of the vector X. One then obtains the following stochastic partial differential equation (see [3, 9]) on the vector $X_t(x)$ (which represents the conformation of the representative polymer chain at time t at a given point x in space):

$$dX_t(x) + u(t,x).\nabla X_t(x)\,dt = \left(\nabla u(t,x) X_t(x) - \nabla \Pi(X_t(x))\right) dt + dW_t. \tag{8}$$

In equation (8), $\Pi(X)$ models the potential of the entropic force between the two ends of the polymeric chain. Two commonly used models are the Hookean model for which $\Pi(X) = \frac{|X|^2}{4}$ and the FENE model (which takes into account the finite extensibility of the polymer chain) for which $\Pi(X) = -\frac{b}{4}\ln\left(1 - \frac{|X|^2}{b}\right)$, $\forall |X| < b$ and $\Pi(X) = +\infty$, $\forall |X| \geq b$ (this is the case of an "explosive potential"). For mathematical purposes, the case of a polynomial force $\Pi(X) = \frac{|X|^2}{4} + \frac{|X|^n}{2n}$ may also be considered. It is easy to check that in all these cases, Π is a radially symmetric (1/2)-convex potential.

The contribution of the polymer chains to the stress tensor is given by the Kramers formula (see [3, 9]):

$$\tau(t,x) = \left(\mathbb{E}(X_t(x) \otimes \nabla \Pi(X_t(x))) - \mathrm{Id}\right). \tag{9}$$

We henceforth assume that the velocity field is given and regular enough (say $\mathcal{C}^1$) so that one can use the characteristic method (by integrating the vector field u) to rewrite Equation (8) in the following form:

$$dX_t = \left(G(t)X_t - \nabla\Pi(X_t)\right) dt + dW_t \tag{10}$$

where $G(t)$ is $\nabla u(t, x(t))$, with $x(t)$ the position at time t of the fluid particle which was at $x(0)$ at time 0. We indeed recover a SDE of the form (1), with $b(t,X) = G(t)X$. We suppose that the velocity field goes exponentially fast to 0, in such a way that (2) is satisfied (for example, $||\nabla u||_{L^\infty_x} \leq Ce^{-\beta t}$).

Then, two questions of practical interest are for example:
- What is the long-time behaviour of the root mean square length $\sqrt{\mathbb{E}(|X_t|^2)}$?
- What is the long-time behaviour of the stress tensor τ defined by (9) ?

Remark 2.1. *A more accurate description of the polymer chains configuration leads to a SDE of the form (1) in a space of larger dimension (for example $N = 2n$ or $N = 3n$ in the case of a bead-spring model, with n springs).*

Remark 2.2. *Our approach is here completely decoupled, since we suppose that the velocity field is given* a priori. *In general, the velocity is not known in advance, and the velocity field is influenced by the conformation of the polymers modelled by X_t (see* [4] *for the long-time behaviour of the coupled system). Notice however that in the case of a homogeneous flow ($u(t,x) = \kappa(t)x$), the stress tensor τ defined by (9) does not depend on space, so that it is divergence free. Therefore, in this special case, it is natural to suppose that the velocity field is given independently of X_t. Homogeneous flows are indeed used in practice to study the rheology of polymeric fluids.*

3. Mathematical analysis

3.1. Convergence in the case of a Lipschitz function ϕ.

We want here to study the long-time behaviour of $\mathbb{E}(\phi(X_t))$ for some smooth functions ϕ. The idea is to introduce the stationary process X_t^∞ solution to:

$$\left\{\begin{array}{l} dX_t^\infty = -\nabla\Pi(X_t^\infty)\,dt + dW_t, \\ X_0^\infty \sim \mu_\infty. \end{array}\right. \tag{11}$$

Following for example the proof by Rogers in [10], one can prove that X_t^∞ satisfies the detailed balance property, so that for any time t, the law of X_t^∞ is μ_∞ (see [5] for the case of an explosive potential Π).

When ϕ is a globally Lipschitz function, we have, for $t \geq 0$:

$$\begin{aligned} \left|\mathbb{E}(\phi(X_t)) - \int \phi(X)\,d\mu_\infty(X)\right| &= \left|\mathbb{E}(\phi(X_t)) - \mathbb{E}(\phi(X_t^\infty))\right|, \\ &\leq [\phi]_{\mathrm{lip}}\mathbb{E}|X_t - X_t^\infty|, \end{aligned} \tag{12}$$

where $[\phi]_{\mathrm{lip}}$ denotes the Lipschitz constant of function ϕ. We can now conclude by proving the convergence of $(X_t - X_t^\infty)$ to 0 in L^1-norm. This is where the α-convexity of Π plays a role.

We can actually prove the convergence of $(X_t - X_t^\infty)$ to 0 in L^k-norm, k being any positive integer. Indeed, we have (using (2)-(3) and assuming that $\mathbb{E}|X_0|^k < \infty$ and $\int |X|^k d\mu_\infty(X) < \infty$)[1]:

$$\begin{aligned} \frac{d}{dt}\mathbb{E}|X_t - X_t^\infty|^k &= -k\mathbb{E}\left(1_{X_t \neq X_t^\infty}|X_t - X_t^\infty|^{k-2}\left(\nabla\Pi(X_t) - \nabla\Pi(X_t^\infty)\right).(X_t - X_t^\infty)\right) \\ &\quad + k\mathbb{E}\left(1_{X_t \neq X_t^\infty}|X_t - X_t^\infty|^{k-2}b(t,X_t).(X_t - X_t^\infty)\right), \\ &\leq -\alpha k\mathbb{E}|X_t - X_t^\infty|^k + kCe^{-\beta t}\mathbb{E}\left(|X_t||X_t - X_t^\infty|^{k-1}\right) \\ &\quad + kCe^{-\beta t}\mathbb{E}|X_t - X_t^\infty|^{k-1}, \\ &\leq -\alpha k\mathbb{E}|X_t - X_t^\infty|^k + kCe^{-\beta t}\mathbb{E}|X_t - X_t^\infty|^k \\ &\quad + kCe^{-\beta t}\mathbb{E}\left((|X_t^\infty| + 1)\,|X_t - X_t^\infty|^{k-1}\right), \\ &\leq -\alpha k\mathbb{E}|X_t - X_t^\infty|^k + kCe^{-\beta t}\mathbb{E}|X_t - X_t^\infty|^k \\ &\quad + kCe^{-\beta t}\left(\mathbb{E}\left(|X_t^\infty| + 1\right)^k\right)^{1/k}\left(\mathbb{E}|X_t - X_t^\infty|^k\right)^{1-1/k}. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\left(\mathbb{E}|X_t - X_t^\infty|^k\right)^{1/k} \leq -\alpha\left(\mathbb{E}|X_t - X_t^\infty|^k\right)^{1/k} + Ce^{-\beta t}\left(\mathbb{E}|X_t - X_t^\infty|^k\right)^{1/k} + C'e^{-\beta t},$$

where we have used the fact that $\mathbb{E}\left(|X_t^\infty| + 1\right)^k = \int(|X| + 1)^k d\mu_\infty(X) < \infty$. From this we deduce that:

$$\begin{aligned} \left(\mathbb{E}|X_t - X_t^\infty|^k\right)^{1/k} &\leq \left(\mathbb{E}|X_0 - X_0^\infty|^k\right)^{1/k} e^{-\alpha t}\exp\left(C\int_0^t e^{-\beta s}\,ds\right) \\ &\quad + C'\int_0^t e^{-\beta s}e^{-\alpha(t-s)}\exp\left(C\int_s^t e^{-\beta r}\,dr\right)ds. \end{aligned}$$

Using the fact that $\exp\left(C\int_0^\infty e^{-\beta r}\,dr\right) < \infty$, we finally have:

$$\left(\mathbb{E}|X_t - X_t^\infty|^k\right)^{1/k} \leq C\left(\mathbb{E}|X_0 - X_0^\infty|^k\right)^{1/k} e^{-\alpha t} + \frac{C\,C'}{\alpha - \beta}\left(e^{-\beta t} - e^{-\alpha t}\right). \tag{13}$$

[1]In the following, C denotes a positive constant which may change from one line to another, and which does not depend on t, α, β nor k.

In particular, using (12) and (13) with $k = 1$, we have

$$\left| \mathbb{E}(\phi(X_t)) - \int \phi(X) \, d\mu_\infty(X) \right| \leq C'' [\phi]_{\text{lip}} e^{-\alpha \wedge \beta \, t}. \quad (14)$$

We have thus shown the following result:

Proposition 3.1. *We suppose (2) and (3).*

- *If we assume that $\mathbb{E}|X_0|^k < \infty$ and $\int |X|^k d\mu_\infty(X) < \infty$ where k is a positive integer, then $(X_t - X_t^\infty)$ converges exponentially fast (with rate $\alpha \wedge \beta$) to 0 in L^k-norm.*
- *In particular, if $\mathbb{E}|X_0| < \infty$ and $\int |X| d\mu_\infty(X) < \infty$, for any Lipschitz function ϕ, we have:*

$$\left| \mathbb{E}(\phi(X_t)) - \int \phi(X) \, d\mu_\infty(X) \right| \leq C'' [\phi]_{\text{lip}} e^{-\alpha \wedge \beta \, t}. \quad (15)$$

Remark 3.1 (Almost sure inequality)**.**
By a similar computation, one can obtain that a.s.,

$$|X_t - X_t^\infty| \leq C |X_0 - X_0^\infty| e^{-\alpha t} + C \int_0^t |X_s^\infty| e^{-\beta s} e^{-\alpha(t-s)} \, ds.$$

This could be used to obtain a.s. estimates on the process $(X_t - X_t^\infty)$.

Remark 3.2 (Convergence for a locally Lipschitz function ϕ with polynomial growth)**.**
We have shown that the L^k-norm of $(X_t - X_t^\infty)$ converges exponentially fast to 0 (under the assumption that $\mathbb{E}|X_0|^k < \infty$ and $\int |X|^k d\mu_\infty(X) < \infty$). In particular, for any integer $k \geq 1$, $\mathbb{E}|X_t|^k$ is bounded from above by a constant which does not depend on time and converges exponentially fast to $\int |X|^k d\mu_\infty(X)$. More generally, if ϕ is a locally Lipschitz function with polynomial growth, which means that there exists a positive integer k and a positive constant C such that: $\forall X, Y$,

$$|\phi(X) - \phi(Y)| \leq C(1 + |X|^k + |Y|^k)|X - Y|,$$

then $\mathbb{E}(\phi(X_t))$ converges exponentially fast to $\int \phi(X) d\mu_\infty(X)$, provided that $\mathbb{E}|X_0|^{k+1} < \infty$ and $\int |X|^{k+1} d\mu_\infty(X) < \infty$.

Remark 3.3 (Convergence in the Wasserstein distance W_k, $k \geq 1$)**.**
We can state the result of Proposition 3.1 in terms of the law of the process X_t by introducing the Wasserstein distance W_k, k being a positive integer (see [1]*). For $k = 1$, the Wasserstein distance W_1 between two measures μ and ν can be defined as the dual of the $[.]_{\text{lip}}$-norm:*

$$W_1(\mu, \nu) = \sup \left\{ \int f d\mu - \int f d\nu, \, [f]_{\text{lip}} \leq 1 \right\}.$$

We recall that, more generally, for $k \geq 1$, the Wasserstein distance W_k between two measures μ and ν can be defined as:

$$W_k(\mu, \nu) = \left(\sup \left\{ \int f d\mu - \int g d\nu, \;\; f, g \; L^\infty \text{ and Lipschitz, } f(x) - g(y) \leq \frac{1}{k} |x - y|^k \right\} \right)^{1/k},$$

or, equivalently,

$$W_k(\mu, \nu) = \left(\inf \left\{ \int \frac{1}{k} |x - y|^k d\lambda(x, y), \, \lambda \text{ with marginals } \mu \text{ and } \nu \right\} \right)^{1/k}.$$

We have shown that for any $k \geq 1$, if $\mathbb{E}|X_0|^k < \infty$ and $\int |X|^k d\mu_\infty(X) < \infty$, the L^k-norm of $(X_t - X_t^\infty)$ converges exponentially fast to 0. Therefore, we have actually *proven that the convergence of the law of X_t to μ_∞ holds for the Wasserstein distance W_k, for any $k \geq 1$.*

Remark 3.4 (Convergence in variation)**.** *To obtain the convergence of the law of X_t to μ_∞ in variation (the Wasserstein distance W_0), at least when $b = 0$, one can use more sophisticated coupling techniques (see for example* [8]*), based on the fact that*

$$W_0(\mu, \nu) = \inf \left\{ \int 1_{x \neq y} d\lambda(x, y),\ \lambda \textit{ with marginals } \mu \textit{ and } \nu \right\}.$$

3.2. Back to polymeric fluids: long-time behaviour of $\mathbb{E}(X_t \otimes \nabla\Pi(X_t))$. As noticed in Section 2.2, one problem of practical interest in the context of polymeric fluids is the long-time behaviour of $\mathbb{E}(X_t \otimes \nabla\Pi(X_t))$. In this framework, it is natural to restrict ourselves to the case of a radial potential Π (see (4)). The problem here is therefore to prove the convergence of $\mathbb{E}(\phi(X_t))$ to $\int \phi(X) d\mu_\infty(X)$ for a non globally Lipschitz function ϕ.

We have:

$$\begin{aligned}
|\mathbb{E}(X_t \otimes \nabla\Pi(X_t)) - \mathbb{E}(X_t^\infty \otimes \nabla\Pi(X_t^\infty))| &= \left| \mathbb{E}\left(\frac{\pi'(|X_t|)}{|X_t|}(X_t \otimes X_t) - \frac{\pi'(|X_t^\infty|)}{|X_t^\infty|}(X_t^\infty \otimes X_t^\infty) \right) \right|, \\
\leq\ & \left| \mathbb{E}\left(\left(\frac{\pi'(|X_t|)}{|X_t|} - \frac{\pi'(|X_t^\infty|)}{|X_t^\infty|} \right)(X_t \otimes X_t) \right) \right| + \left| \mathbb{E}\left(\frac{\pi'(|X_t^\infty|)}{|X_t^\infty|}(X_t \otimes X_t - X_t^\infty \otimes X_t^\infty) \right) \right|, \\
\leq\ & \mathbb{E}\left(\left| \frac{\pi'(|X_t|)}{|X_t|} - \frac{\pi'(|X_t^\infty|)}{|X_t^\infty|} \right| |X_t|^2 \right) + \mathbb{E}\left(\frac{\pi'(|X_t^\infty|)}{|X_t^\infty|}(|X_t| + |X_t^\infty|)|X_t - X_t^\infty| \right).
\end{aligned}$$

Using the fact that we have a uniform-in-time control of the moments of X_t and exponential convergence of the L^k-norm of $(X_t - X_t^\infty)$ to 0 (see Proposition 3.1), we see that sufficient conditions to obtain the exponential convergence of $\mathbb{E}(X_t \otimes \nabla\Pi(X_t))$ to $\displaystyle\int (X \otimes \nabla\Pi(X)) d\mu_\infty(X)$ are:

$$\begin{cases}
(1) & \text{the exponential convergence of } \mathbb{E}\left| \dfrac{\pi'(|X_t|)}{|X_t|} - \dfrac{\pi'(|X_t^\infty|)}{|X_t^\infty|} \right|^p \text{ to 0 for some } p > 1, \\
(2) & \text{a bound from above for } \displaystyle\int \left(\frac{\pi'(|X|)}{|X|} \right)^q \exp(-2\pi(|X|)) \text{ for some } q > 1.
\end{cases}$$

If we consider the Hookean case ($\Pi(X) = \frac{|X|^2}{4}$) or the polynomial case ($\Pi(X) = \frac{|X|^2}{4} + \frac{|X|^n}{2n}$), it is then easy to check that both these conditions are fulfilled (we are actually in the case of a locally Lipschitz function with polynomial growth, see Remark 3.2).

Let us now turn to the FENE case. Here, the potential Π is explosive, since π is defined on an interval $(0, \sqrt{b})$ and equal to $+\infty$ on $(\sqrt{b}, +\infty)$. In this case, we cannot apply the general results obtained above to prove the convergence of the stress tensor.

Concerning the existence of a global-in-time solution to (1) in this case, one can show that, if $b \geq 2$, the potential π explodes sufficiently fast when $|X|^2 \to b$ so that there exists a unique solution to (1), which is such that $\mathbb{P}(\exists t \geq 0, |X_t|^2 = b) = 0$ (see [6, 5]). In particular, if $b \geq 2$, $\sqrt{|X_t|} < b$ a.s., so that any moment of X_t is bounded by b.

Moreover, in the FENE framework, we have:

$$\frac{\pi'(l)}{l} = \frac{1}{2} \frac{1}{1 - l^2/b}.$$

The second condition therefore writes $\int_{|X|^2<b} \left(\frac{1}{1-|X|^2/b} \right)^q \left(1-|X|^2/b\right)^{b/2} < \infty$ and is clearly fulfilled for any q if $b > 2(q-1)$.

For the first condition, we have:

$$\begin{aligned}
\left| \frac{\pi'(|X_t|)}{|X_t|} - \frac{\pi'(|X_t^\infty|)}{|X_t^\infty|} \right| &= \frac{1}{2} \left| \frac{1}{(1-|X_t|^2/b)} - \frac{1}{(1-|X_t^\infty|^2/b)} \right|, \\
&= \frac{1}{2} \frac{1}{(1-|X_t|^2/b)} \frac{1}{(1-|X_t^\infty|^2/b)} \frac{\left| |X_t^\infty|^2 - |X_t|^2 \right|}{b}, \\
&\leq \frac{1}{(1-|X_t|^2/b)} \frac{1}{(1-|X_t^\infty|^2/b)} \left| |X_t^\infty| - |X_t| \right|.
\end{aligned}$$

One can then conclude using the fact that $\mathbb{E}\left(\frac{1}{1-|X_t^\infty|^2/b} \right)^q < C < \infty$ as soon as $b > 2(q-1)$ (where C does not depend on time) and $\left(\mathbb{E}\left(\frac{1}{1-|X_t|^2/b} \right)^r \right)^{1/r} < C + Mt$ as soon as $b > 2(r+1)$ and $\mathbb{E}\left(\frac{1}{1-|X_0|^2/b} \right)^r < \infty$ (apply Itô's formula to compute $\left(\frac{1}{1-|X_t|^2/b} \right)^r$ and conclude like in the end of the proof of Lemma 2 in [6]). It is then easy to prove that:

Proposition 3.2. *In the case of the FENE force, provided that $b > 2(1+\sqrt{2})$ and $\mathbb{E}\left(\frac{1}{1-|X_0|^2/b} \right)^{\sqrt{2}} < \infty$, we have exponential convergence of the stress tensor: $\forall \gamma < 1/2 \wedge \beta$, $\exists C > 0$, $\forall t \geq 0$,*

$$\left| \mathbb{E}\left(\frac{X_t \otimes X_t}{1-|X_t|^2/b} \right) - \int \frac{X \otimes X}{1-|X|^2/b} d\mu_\infty(X) \right| \leq C e^{-\gamma t}.$$

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(Benjamin Jourdain, Claude Le Bris) CERMICS,
Ecole Nationale des Ponts et Chaussées,
6 & 8 Av. Pascal, 77455 Champs-sur-Marne,
France

(Tony Lelièvre) CRM,
Université de Montréal,
C.P. 6128, Succ. Centre-ville Montréal, Québec H3C 3J7,
Canada
E-mail address: {jourdain,lebris,lelievre}@cermics.enpc.fr