Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 51(1), 2024, Pages 118–132, DOI: 10.52846/ami.v51i1.1750
ISSN: 1223-6934

Coupled Systems of Conformable Fractional Differential Equations

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Abstract. This paper deals with some existence of solutions for some classes of coupled systems of conformable fractional differential equations with initial and boundary conditions in Banach and Fréchet spaces. Our results are based on some fixed point theorems. Some illustrative examples are presented in the last section.

2020 *Mathematics Subject Classification.* Primary 26A33.
Key words and phrases. Conformable fractional differential equation; conformable integral of fractional order; conformable fractional derivative; coupled system; Banach space; Fréchet space; fixed point.

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1. Introduction

In recent years, fractional differential equations have found applications in diverse fields such as engineering, mathematics, and physics, as well as other applied sciences. There has been a significant focus on studying the existence of solutions for initial and boundary value problems related to fractional differential equations. To this end, several monographs [1, 2, 17, 23, 24, 28] and papers [8, 9, 19, 20, 22] have explored this area in depth.

In a recent publication by Khalil *et al.* [16], a novel definition of the fractional derivative was introduced. This definition, known as the conformable fractional derivative, is a natural extension of the standard first derivative. The conformable fractional derivative possesses several desirable properties, such as linearity, product rule, quotient rule, power rule, and chain rule, similar to those of the classical integral derivative. Its adoption has greatly facilitated the modeling of various physical problems, resulting in an extensive literature on the topic [3, 4, 6, 5, 7, 10, 13, 14, 15, 26, 27].

In [18], the authors considered the following conformable impulsive problem:

$$\begin{cases} \mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta) = \aleph\left(\zeta, \chi_\zeta, \mathcal{T}^{\vartheta}_{\jmath}\chi(\zeta)\right), & \zeta \in \gimel_\jmath; \jmath = 0, 1, \ldots, \varsigma, \\ \Delta\chi|_{\zeta=\zeta_\jmath} = \Upsilon_\jmath(\chi_{\zeta_\jmath^-}), & \jmath = 1, 2, \ldots, \varsigma, \\ \chi(\zeta) = \mu(\zeta), & \zeta \in (-\infty, \varkappa], \end{cases}$$

where $0 \leq \varkappa = \zeta_0 < \zeta_1 < \cdots < \zeta_\varsigma < \zeta_{\varsigma+1} = \bar{\varkappa} < \infty$, $\mathcal{T}^{\vartheta}_{\zeta_\jmath}\chi(\zeta)$ is the conformable fractional derivative of order $0 < \vartheta < 1$, $\aleph : \gimel \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $\gimel := [\varkappa, \bar{\varkappa}]$, $\gimel_0 := [\varkappa, \zeta_1]$, $\gimel_\jmath := (\zeta_\jmath, \zeta_{\jmath+1}]; \jmath = 1, 2, \ldots, \varsigma$, $\mu : (-\infty, \varkappa] \to \mathbb{R}$ and $\Upsilon_\jmath : \mathcal{Q} \to \mathbb{R}$ are given continuous functions, and $\mathcal{Q}$ is called a phase space.

Received April 1, 2023. Accepted October 7, 2023.

In this paper we investigate the existence of solutions for the following coupled conformable fractional differential system:

$$\begin{cases} (\mathcal{T}_{0^+}^{\mu_1}\chi)(\theta) = \aleph_1(\theta, \chi(\theta), \xi(\theta)) \\ (\mathcal{T}_{0^+}^{\mu_2}\xi)(\theta) = \aleph_2(\theta, \chi(\theta), \xi(\theta)) \end{cases} ; \ \theta \in \mho, \tag{1}$$

with the following coupled boundary conditions:

$$(\chi(0), \xi(0)) = (\delta_1\xi(\varkappa), \delta_2\chi(\varkappa)), \tag{2}$$

where $\varkappa > 0$, $\mho := [0, \varkappa]$, $\mu_\jmath \in (0, 1]$; $\jmath = 1, 2$ $\aleph_\jmath : \mho \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$; $\jmath = 1, 2$ are given continuous functions, $\mathcal{T}_a^{\mu_\jmath}$ is the conformable fractional derivative of order $\mu_\jmath$; $\jmath = 1, 2$, and δ_1, δ_2 are real numbers with $\delta_1\delta_2 \neq 1$.

Next, we investigate the following coupled conformable fractional differential system:

$$\begin{cases} (\mathcal{T}_{a^+}^{\mu_1}\chi)(\theta) = \aleph_1(\theta, \chi(\theta), \xi(\theta)) \\ (\mathcal{T}_{a^+}^{\mu_2}\xi)(\theta) = \aleph_2(\theta, \chi(\theta), \xi(\theta)) \end{cases} ; \ \theta \in [a, \infty), \tag{3}$$

with the coupled initial conditions:

$$(\chi(a), \xi(a)) = (\chi_a, \xi_a), \tag{4}$$

where $a > 0$, $\mu_\jmath \in (0, 1]$; $\jmath = 1, 2$, $(\Xi, \|\cdot\|)$ is a (real or complex) Banach space, $\chi_a, \xi_a \in \Xi$ and $\aleph_\jmath : \mathbb{R}_+ \times \Xi \times \Xi \to \Xi$; $\jmath = 1, 2$ are given continuous functions.

2. Preliminaries

First, let us introduce some basic lemmas and definitions that are needed throughout all the manuscript.
Let $C := \mathrm{C}(\mho, \Xi)$ be the Banach space equipped with the norm defined by

$$\|\chi\|_\infty := \sup_{\theta \in \mho} \|\chi(\theta)\|.$$

In the case when $\Xi := \mathbb{R}$ we have $\|\chi\|_\infty := \sup_{\theta \in \mho} |\chi(\theta)|$.

By $\Bbbk := C \times C$, we denote the complete metric space with the usual metric

$$D((\chi_1, \xi_1), (\chi_2, \xi_2)) := d(\chi_1, \chi_2) + d(\xi_1, \xi_2).$$

$\Bbbk$ is a Banach space with the norm

$$\|(\chi, \xi)\|_\Bbbk = \|\chi\|_\infty + \|\xi\|_\infty.$$

By $L^1(\mho, \Xi)$ we denote the Banach space of measurable functions $\chi : \mho \to \Xi$, which are Bochner integrable, equipped with the norm

$$\|\chi\|_1 = \int_0^\varkappa \|\chi(\theta)\| d\theta.$$

Let $\daleth := \mathrm{C}(\mathbb{R}_+, \Xi)$ be the Fréchet space of all continuous functions χ from $\mathbb{R}_+$ into Ξ, equipped with the family of semi norms

$$\|\chi\|_\imath = \sup_{\theta \in [0, \imath]} \{\|\chi(\theta)\| : \imath \in \mathbb{N}\},$$

and the distance

$$d(\chi, \xi) = \sum_{\imath=0}^{\infty} \frac{2^{-\imath}\|\chi - \xi\|_\imath}{1 + \|\chi - \xi\|_\imath}; \ \chi, \xi \in \daleth.$$

Definition 2.1 ([25])**.** A nonempty subset $\nabla \subset \daleth$ is said to be bounded if

$$\sup_{\chi\in\nabla} \|\chi\|_\imath < \infty; \, for \ \imath \in \mathbb{N}.$$

Definition 2.2. Let $\mathfrak{Y}\daleth$ be the family of all nonempty and bounded subsets of a Fréchet space $\daleth$. A family of functions $\{\zeta_\imath\}_\imath \in \mathbb{N}$ where $\zeta_\imath : \mathfrak{Y}\daleth \to [0,\infty)$ is said to be a family of measures of noncompactness in the real Fréchet space $\daleth$ if it satisfies the following conditions for all $\nabla, \nabla_1, \nabla_2 \in \mathfrak{Y}\daleth$:

- (a) $\{\zeta_\imath\}_\imath \in \mathbb{N}$ is full, that is: $\zeta_\imath(\nabla) = 0$ for $\imath \in \mathbb{N}$ and only if ∇ is precompact,
- (b) $\zeta_\imath(\nabla_1) < \zeta_\imath(\nabla_2)$ for $\nabla_1 \subset \nabla_2$ and $\imath \in \mathbb{N}$,
- (c) $\zeta_\imath(Conv\nabla) = \zeta_\imath(\nabla)$ for $\imath \in \mathbb{N}$.

If $\{\nabla_\jmath\}_{\jmath=1}$ is a sequence of closed sets from $\mathfrak{Y}\daleth$ such that $\nabla_{\jmath+1} \subset \nabla_\jmath$ and if $\lim_{\jmath\to\infty} \zeta_\imath(\nabla_\jmath) = 0$, for each $\imath \in \mathbb{N}$, then the intersection set $\nabla_\infty := \cap_{\jmath=1}^{\infty} \nabla_\jmath$ is nonempty.

Property 2.1. We have the following properties:

1. We call the family of measures of noncompactness $\{\zeta_\imath\}_\imath \in \mathbb{N}$ to be homogeneous if $\zeta_\imath(\varpi\nabla) = |\varpi|\zeta_\imath(\nabla)$; for $\varpi \in \mathbb{R}$ and $\imath \in \mathbb{N}$.
2. If the family $\{\zeta_\imath\}_\imath \in \mathbb{N}$ satisfied the condition $\zeta_\imath(\nabla_1 \cup \nabla_2) < \zeta_\imath(\nabla_1) + \zeta_\imath(\nabla_2)$, for $\imath \in N$, it is called subadditive.
3. We say that the family of measures $\{\zeta_\imath\}_\imath \in \mathbb{N}$ has the maximum property if $\zeta_\imath(\nabla_1 \cup \nabla_2) = \max\{\zeta_\imath(\nabla_1), \zeta_\imath(\nabla_2)\}$.
4. The family of measures of noncompactness $\{\zeta_\imath\}_\imath \in \mathbb{N}$ is said to be regular if if the conditions (a), (3) and (4) hold; (full sublinear and has maximum property.

Example 2.1 ([25, 12])**.** For $\nabla \in \mathfrak{Y}\daleth$, $\psi \in \nabla, \imath \in \mathbb{N}$ and $\gamma > 0$, let us denote by $\Im^\imath(\psi,\gamma)$ the modulus of continuity of the function ψ on the interval $[0,\imath]$, that is

$$\Im^\imath(\psi,\gamma) = \sup\{\|\psi(\theta) - \psi(\varrho)\| : \theta, \varrho \in [0,\imath], |\theta - \varrho| < \gamma\}.$$

Further, let us put

$$\begin{aligned} \Im^\imath(\nabla,\gamma) &= \sup\{\Im^\imath(\psi,\gamma) : \psi \in \nabla\}, \\ \Im_0^\imath(\nabla) &= \lim_{\gamma\to 0^+} \Im^\imath(\nabla,\gamma), \\ \zeta^{-\imath}(\nabla) = \sup_{\theta\in[0,\imath]} \zeta(\nabla(\theta)) &= \sup_{\theta\in[0,\imath]} \zeta(\{\psi(\theta) : \psi \in \nabla\}), \end{aligned}$$

and

$$\varsigma_\imath(\nabla) = \Im_0^\imath(\nabla) + \zeta^{-\imath}(\nabla).$$

The family of mappings $\{\varsigma_\imath\}_{\imath\in\mathbb{N}}$ where $\varsigma_\imath : \mathfrak{Y}\daleth \to [0,\infty)$ satisfies the conditions (a)-(d) from definition 2.2.

Lemma 2.2 ([21])**.** *If Y is a bounded subset of a Banach space $\daleth$, there is a sequence $(y_k)_{k=1}^{\infty} \subset Y$ such that*

$$\zeta(Y) \leq 2\zeta((y_k)_{k=1}^{\infty}) + \gamma, \quad for \ each \quad \gamma > 0,$$

where ζ is the Kuratowskii measure of noncompactness.

Lemma 2.3 ([21])**.** *If $\{\chi_k\}_{k=1}^{\infty} \subset L^1([0,\varkappa])$ is uniformly integrable, then $\zeta(\{\chi_k\}_{k=1}^{\infty})$ is measurable and*

$$\zeta\left(\left\{\int_1^\theta \chi_k(\varrho)\mathrm{d}\varrho\right\}_{k=1}^{\infty}\right) \leq 2\int_1^\theta \zeta(\{\chi_k(\varrho)\}_{k=1}^{\infty})\mathrm{d}\varrho, \quad for \ each \quad \theta \in [0,\varkappa].$$

Definition 2.3. Let $\gimel$ be a nonempty subset of a Fréchet space $\daleth$ and let be A a continuous operator which transforms bounded subsets of onto bounded ones. One says that A satisfies the Darbo condition with constants $(k_\imath)_{\imath\in\mathbb{N}}$ with respect to a family of measures of noncompactness $(\zeta_\imath)_{\imath\in\mathbb{N}}$, if

$$\zeta_\imath(A(\nabla)) \leq k_\imath \zeta_\imath(\nabla), \quad \text{for each bounded set} \quad \nabla \subset \gimel \quad \text{and} \quad \imath \in \mathbb{N}.$$

If $k_\imath < 1$; $\imath \in \mathbb{N}$, then A is called a contraction with respect to $(\zeta_\imath)_{\imath\in\mathbb{N}}$.

Let us now recall some essential definitions on conformable derivatives that can be found in [16, 3].

Let $\imath < \mu < \imath + 1$, and set $\varsigma = \mu - \imath$. For a function $\aleph : [a, \infty) \to \mathbb{R}$, let

$$\mathcal{J}_a^\mu \aleph(\theta) = \int_a^\theta (\varrho - a)^{\mu-1} \aleph(\varrho) \mathrm{d}\varrho, \imath = 0,$$

and

$$\mathcal{J}_a^\mu \aleph(\theta) = \frac{1}{\imath!} \int_a^\theta (\theta - \varrho)^\imath \aleph(\varrho) d\varsigma(\varrho, a) = \frac{1}{\imath!} \int_a^\theta (\theta - \varrho)^\imath (\varrho - a)^{\varsigma-1} \aleph(\varrho) \mathrm{d}\varrho; \imath \geq 1.$$

Remark 2.1. Since $0 < \varsigma < 1$, $\mathcal{J}_a^\mu \aleph(\theta)$ is the Lebesgue-Stieltjes integral of the function $(\theta - \varrho)^\imath \aleph(\varrho)$ on $[a, \theta]$ and $d\varsigma(\varrho, a) = (\varrho - a)^{\varsigma-1} \mathrm{d}\varrho$ is an absolutely continuous measure with respect to the Lebesgue measure on the real line, generated by the absolutely continuous function $(\theta - a)^\varsigma$ and the weight function $(\varrho - a)^{\varsigma-1} \in L^1[a, b]$ is its Radon- Nikodym derivative according to the Lebesgue measure.

The conformable derivative of order $0 < \mu < 1$, of a function $\aleph : [a, \infty) \to R$ is defined by

$$\mathcal{T}_a^\mu \aleph(\theta) = \lim_{\gamma \to 0} \frac{\aleph(\theta + \gamma(\theta - a)^{1-\mu} - \aleph(\theta)}{\gamma}, \theta > a.$$

If $\mathcal{T}_a^\mu \aleph(\theta)$ exists on $(a, b), b > a$ and $\lim\limits_{\theta \to a^+} \mathcal{T}_a^\mu \aleph(\theta)$ exists, then we define

$$\mathcal{T}_a^\mu \aleph(a) = \lim_{\theta \to a^+} \mathcal{T}_a^\mu \aleph(\theta).$$

The conformable derivative of order $\imath < \mu < \imath + 1$ of a function $\aleph : [a, \infty) \to \mathbb{R}$, when $\aleph^{(\imath)}$ exists, is defined by $\mathcal{T}_a^\mu \aleph(\theta) = \mathcal{T}_a^\varsigma \aleph^{(\imath)}(\theta)$, where $\varsigma = \mu - \imath \in (0, 1)$.

Lemma 2.4. *For the properties of the conformable derivative, we mention the following:*

- *Let* $\imath < \mu < \imath + 1$ *and* $\aleph$ *be an* $(\imath + 1)$*-differentiable at* $\theta > a$*, then we have*

$$\mathcal{T}_a^\mu \aleph(\theta) = (\theta - a)^{\imath+1-\mu} \aleph^{(\imath+1)}(\theta),$$

and

$$\mathcal{J}_a^\mu \mathcal{T}_a^\mu \aleph(\theta) = \aleph(\theta) - \sum_{k=0}^{\imath} \frac{\aleph^{(\imath)}(a)(\theta - a)^k}{k!}.$$

- *In particular, if* $0 < \mu < 1$*, then we have*

$$\mathcal{J}_a^\mu \mathcal{T}_a^\mu \aleph(\theta) = \chi(\theta) - \chi(a).$$

Remark 2.2. We provide the following remarks:

- For $0 < \mu < 1$, using Lemma 2.4 it follows that, if a function $\aleph$ is differentiable at $\theta > a$, then one has
$$\lim_{\mu \to 1} \mathcal{T}_a^{\mu} \aleph(\theta) = \aleph'(\theta)$$
and
$$\lim_{\mu \to 0} \mathcal{T}_a^{\mu} \aleph(\theta) = (\theta - a)\aleph'(\theta),$$
i.e. the zero order derivative of a differentiable function does not return to the function itself.
- Let $\imath < \mu < \imath + 1$, if $\aleph$ is $(\imath + 1)$-differentiable on $(a, b), b > a$ and $\lim\limits_{\theta \to a^+} \aleph^{(\imath+1)}$ exists, then from Lemma 2.4, we get $\mathcal{T}_a^{\mu} \aleph(a) = \lim\limits_{\theta \to a^+} \mathcal{T}_a^{\mu} \aleph(\theta) = 0$.
- Let $\imath < \mu < \imath + 1$, if $\aleph$ is $(\imath + 1)$-differentiable at $\theta > a$, then we can show that $\mathcal{T}_a^{\mu} \aleph(\theta) = \mathcal{T}_a^{\mu - k} \aleph^{(k)}(\theta)$ for all positive integer $k < \mu$.

Proposition 2.5. *Let $1 < \mu < 2$, if a function $\aleph \in C^1[a, b]$ attains a global maximum (respectively minimum) at some point $\theta \in (a, b)$, then $\mathcal{T}_a^{\mu} \aleph(\theta) \leq 0$ (respectively $\mathcal{T}_a^{\mu} \aleph(\theta) \geq 0$.*

Proof. The result follows from the fact that
$$\mathcal{T}_a^{\mu} \aleph(\theta) = \mathcal{T}_a^{\mu - 1} \aleph'(\theta) = \lim_{\gamma \to 0} \frac{\aleph'(\theta + \gamma(\theta - a)^{2-\mu})}{\gamma}.$$
□

Theorem 2.6 (Schaefer's fixed point theorem [11])**.** *Let U be a Banach space and $T : U \to U$ be continuous and compact mapping (completely continuous mapping). Moreover, suppose*
$$S = \{\chi \in U : \chi = \varpi T u, \quad \textit{for some} \quad \varpi \in (0, 1)\}$$
be a bounded set. Then, T has at least one fixed point in U.

Theorem 2.7 ([25])**.** *Let $\gimel$ be a nonempty, bounded, closed, and convex subset of a Fréchet space $\daleth$ and let $V : \gimel \to \gimel$ be a continuous mapping. Suppose that V is a contraction with respect to a family of measures of noncompactness $\zeta_{\imath_{\imath \in \mathbb{N}}}$. Then the mapping V has at least one fixed point in the set $\gimel$.*

Lemma 2.8. *Let $\psi, \widehat{\psi} \in C$, and $\delta_1 \delta_2 \neq 1$ Then the unique solution (χ, ξ) of problem*
$$\begin{cases} \mathcal{T}_a^{\mu_1} \chi(\theta) = \psi(\theta);\ \theta \in \mho := [0, \varkappa],\ \mu_1 \in (0, 1], \\ \mathcal{T}_a^{\mu_2} \xi(\theta) = \widehat{\psi}(\theta);\ \theta \in \mho := [0, \varkappa],\ \mu_2 \in (0, 1], \\ \chi(0) = \delta_1 \xi(\varkappa), \\ \xi(0) = \delta_2 \chi(\varkappa), \end{cases} \tag{5}$$
is given by
$$\chi(\theta) = \frac{\delta_1}{1 - \delta_1 \delta_2} \left[\delta_2 \int_0^{\varkappa} \varrho^{\mu_1 - 1} \psi(\varrho) ds + \int_0^{\varkappa} \varrho^{\mu_2 - 1} \widehat{\psi}(\varrho) \mathrm{d}\varrho \right] + \int_0^{\theta} \varrho^{\mu_1 - 1} \psi(\varrho) \mathrm{d}\varrho,$$
$$\xi(\theta) = \frac{\delta_2}{1 - \delta_1 \delta_2} \left[\delta_1 \int_0^{\varkappa} \varrho^{\mu_2 - 1} \widehat{\psi}(\varrho) ds + \int_0^{\varkappa} \varrho^{\mu_1 - 1} \psi(\varrho) \mathrm{d}\varrho \right] + \int_0^{\theta} \varrho^{\mu_2 - 1} \widehat{\psi}(\varrho) \mathrm{d}\varrho.$$

Proof. By Lemma 2.4, solving the linear fractional differential equation

$$\mathcal{T}_0^{\mu_1}\chi(\theta)=\psi(\theta),$$

we find that

$$\mathcal{J}_0^{\mu_1}\mathcal{T}_0^{\mu_1}\chi(\theta)=\mathcal{J}_0^{\mu_1}\psi(\theta).$$

Hence,

$$\chi(\theta)=\chi(0)+\int_0^{\theta}\varrho^{\mu_1-1}\psi(\varrho)\mathrm{d}\varrho, \tag{6}$$

$$\xi(\theta)=\xi(0)+\int_0^{\theta}\varrho^{\mu_2-1}\widehat{\psi}(\varrho)\mathrm{d}\varrho. \tag{7}$$

By using the boundary conditions $\chi(0)=\delta_1\xi(\varkappa)$, and $\xi(0)=\delta_2\chi(\varkappa)$, we obtain

$$\chi(0)=\delta_1\left[\xi(0)+\int_0^{\varkappa}\varrho^{\mu_2-1}\widehat{\psi}(\varrho)\mathrm{d}\varrho\right], \tag{8}$$

and

$$\xi(0)=\delta_2\left[\chi(0)+\int_0^{\varkappa}\varrho^{\mu_1-1}\psi(\varrho)\mathrm{d}\varrho\right]. \tag{9}$$

It follows from (8) and (9) that

$$\chi(0)=\frac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^{\varkappa}\varrho^{\mu_1-1}\psi(\varrho)\mathrm{d}\varrho+\int_0^{\varkappa}\varrho^{\mu_2-1}\widehat{\psi}(\varrho)\mathrm{d}\varrho\right],$$

and

$$\xi(0)=\frac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\int_0^{\varkappa}\varrho^{\mu_2-1}\widehat{\psi}(\varrho)\mathrm{d}\varrho+\int_0^{\varkappa}\varrho^{\mu_1-1}\psi(\varrho)\mathrm{d}\varrho\right].$$

Thus,

$$\begin{cases}\chi(\theta)=\dfrac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\displaystyle\int_0^{\varkappa}\varrho^{\mu_1-1}\psi(\varrho)\mathrm{d}\varrho+\int_0^{\varkappa}\varrho^{\mu_2-1}\widehat{\psi}(\varrho)\mathrm{d}\varrho\right]+\displaystyle\int_0^{\theta}\varrho^{\mu_1-1}\psi(\varrho)\mathrm{d}\varrho,\\[2ex] \xi(\theta)=\dfrac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\displaystyle\int_0^{\varkappa}\varrho^{\mu_2-1}\widehat{\psi}(\varrho)\mathrm{d}\varrho+\int_0^{\varkappa}\varrho^{\mu_1-1}\psi(\varrho)\mathrm{d}\varrho\right]+\displaystyle\int_0^{\theta}\varrho^{\mu_2-1}\widehat{\psi}(\varrho)\mathrm{d}\varrho.\end{cases}$$

□

The following lemma is a direct conclusion of Lemma 2.8.

Lemma 2.9. *Let* $\aleph_\jmath:\mho\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\jmath=1,2$, *be such that* $\aleph_\jmath(\cdot,\chi,\xi)\in\mathrm{C}(\mho)$ *for each* $\chi,\xi\in\mathrm{C}(\mho)$. *Then the coupled system (1)-(2) is equivalent to the coupled system of integral equations*

$$\begin{cases}\chi(\theta)=\dfrac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\displaystyle\int_0^{\varkappa}\varrho^{\mu_1-1}\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))ds+\int_0^{\varkappa}\varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho\right]\\ \qquad+\displaystyle\int_0^{\theta}\varrho^{\mu_1-1}\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho,\\ \xi(\theta)=\dfrac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\displaystyle\int_0^{\varkappa}\varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho+\int_0^{\varkappa}\varrho^{\mu_1-1}\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho\right]\\ \qquad+\displaystyle\int_0^{\theta}\varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho.\end{cases}$$

3. Existence results in Banach spaces

Now, we shall prove the main results concerning the existence of solutions of our first problem by applying Schaefer's fixed point theorem.

Let us introduce the following hypothesis:

(H) there exist real constants $L_\jmath, K_\jmath, \mathfrak{Y}_\jmath > 0$; $\jmath = 1, 2$, such that

$$|\aleph_\jmath(\theta, \chi_1, \chi_2)| \leq L_\jmath + K_\jmath|\chi_1| + \mathfrak{Y}_\jmath|\chi_2|; \quad \text{for} \quad \theta \in \mho \quad \text{and} \quad \chi_\jmath \in \mathbb{R}.$$

Set

$$W_1 = \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|} + 1\right]\frac{\mathcal{T}^{\mu_1}}{\mu_1}, \quad W_2 = \left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\mu_2}}{\mu_2},$$
$$W_3 = \left[\frac{|\delta_2|}{|1-\delta_1\delta_2|}\right]\frac{\mathcal{T}^{\mu_1}}{\mu_1}, \quad W_4 = \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|} + 1\right]\frac{\mathcal{T}^{\mu_2}}{\mu_2}.$$

Theorem 3.1. *Assume that the hypothesis (H) is satisfies. If*

$$(W_1 + W_3)(K_1 + \mathfrak{Y}_1) + (W_2 + W_4)(K_2 + \mathfrak{Y}_2) < 1, \tag{10}$$

then the problem (1)-(2) has at least one solution.

Proof. Define the operator $\Psi : \Bbbk \to \Bbbk$ by

$$(\Psi(\chi, \xi))(\theta) = ((\Psi_1\chi)(\theta), (\Psi_2\xi)(\theta)), \tag{11}$$

where $\Psi_1, \Psi_2 : C \to C$ are given by

$$\begin{aligned}(\Psi_1\chi)(\theta) &= \tfrac{\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^{\varkappa}\varrho^{\mu_1-1}\aleph_1(\varrho, \chi(\varrho), \xi(\varrho))ds + \int_0^{\varkappa}\varrho^{\mu_2-1}\aleph_2(\varrho, \chi(\varrho), \xi(\varrho))ds\right]\\ &+ \int_0^{\theta}\varrho^{\mu_1-1}\aleph_1(\varrho, \chi(\varrho), \xi(\varrho))ds,\end{aligned}$$

and

$$\begin{aligned}(\Psi_2\xi)(\theta) &= \tfrac{\delta_2}{1-\delta_1\delta_2}\left[\delta_1\int_0^{\varkappa}\varrho^{\mu_2-1}\aleph_2(\varrho, \chi(\varrho), \xi(\varrho))ds + \int_0^{\varkappa}\varrho^{\mu_1-1}\aleph_1(\varrho, \chi(\varrho), \xi(\varrho))ds\right]\\ &+ \int_0^{\theta}\varrho^{\mu_2-1}\aleph_2(\varrho, \chi(\varrho), \xi(\varrho))ds.\end{aligned}$$

Set

$$R \geq \frac{(W_1 + W_3)L_1 + (W_2 + W_4)L_2}{1 - (W_1 + W_3)(K_1 + \mathfrak{Y}_1) - (W_2 + W_4)(K_2 + \mathfrak{Y}_2)},$$

and consider the closed and convex ball

$$\nabla_R = \{(\chi, \xi) \in \Bbbk : \ \|(\chi, \xi)\|_{\Bbbk} \leq R\}.$$

Let $(\chi, \xi) \in \nabla_R$. Then, for each $\theta \in \mho$ and any $\jmath = 1, 2$, we have

$$\begin{aligned}|(\Psi_1\chi)(\theta)| = \Bigg|&\frac{\delta_1\delta_2}{1-\delta_1\delta_2}\int_0^{\varkappa}\varrho^{\mu_1-1}\aleph_1(\varrho, \chi(\varrho), \xi(\varrho))\mathrm{d}\varrho\\ &+ \frac{\delta_1}{1-\delta_1\delta_2}\int_0^{\varkappa}\varrho^{\mu_2-1}\aleph_2(\varrho, \chi(\varrho), \chi(\varrho))\mathrm{d}\varrho\\ &+ \int_0^{\theta}\varrho^{\mu_1-1}\aleph_1(\varrho, \chi(\varrho), \xi(\varrho))\mathrm{d}\varrho\Bigg|\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\frac{\delta_1\delta_2}{1-\delta_1\delta_2}\right| \int_0^{\varkappa} \varrho^{\mu_1-1}|\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))|\mathrm{d}\varrho \\
&\quad + \left|\frac{\delta_1}{1-\delta_1\delta_2}\right| \int_0^{\varkappa} \varrho^{\mu_2-1}|\aleph_2(\varrho,\chi(\varrho),\chi(\varrho))|\mathrm{d}\varrho \\
&\quad + \int_0^{\varkappa} \varrho^{\mu_1-1}|\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))|\mathrm{d}\varrho \\
&\leq \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right] \int_0^{\varkappa} \varrho^{\mu_1-1}(L_1+K_1|\chi(\varrho)|+\mathfrak{Y}_1|\xi(\varrho)|)\mathrm{d}\varrho \\
&\quad + \frac{|\delta_1|}{|1-\delta_1\delta_2|} \int_0^{\varkappa} \varrho^{\mu_2-1}(L_2+K_2|\chi(\varrho)|+\mathfrak{Y}_2|\xi(\varrho)|)\mathrm{d}\varrho \\
&\leq \left[\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1\right] \frac{\varkappa^{\mu_1}}{\mu_1}(L_1+(K_1+\mathfrak{Y}_1)R) \\
&\quad + \left[\frac{|\delta_1|}{|1-\delta_1\delta_2|}\right] \frac{\varkappa^{\mu_2}}{\mu_2}(L_2+(K_2+\mathfrak{Y}_2)R) \\
&\leq W_1(L_1+(K_1+\mathfrak{Y}_1)R)+W_2(L_2+(K_2+\mathfrak{Y}_2)R).
\end{aligned}
$$

Also,

$$
\begin{aligned}
|(\Psi_2\xi)(\theta)| &= \left| \frac{\delta_2\delta_1}{1-\delta_2\delta_1} \int_0^{\varkappa} \varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho \right. \\
&\quad + \frac{\delta_2}{1-\delta_2\delta_1} \int_0^{\varkappa} \varrho^{\mu_1-1}\aleph_1(\varrho,\chi(\varrho),\chi(\varrho))\mathrm{d}\varrho \\
&\quad \left. + \int_0^{\theta} \varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho \right| \\
&\leq \left| \frac{\delta_2\delta_1}{1-\delta_2\delta_1} \int_0^{\varkappa} \varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho \right| \\
&\quad + \left| \frac{\delta_2}{1-\delta_2\delta_1} \int_0^{\varkappa} \varrho^{\mu_2-1}\aleph_1(\varrho,\chi(\varrho),\chi(\varrho))\mathrm{d}\varrho \right| \\
&\quad + \left| \int_0^{\varkappa} \varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho \right| \\
&\leq W_3(L_1+(K_1+\mathfrak{Y}_1)R)+W_4(L_2+(K_2+\mathfrak{Y}_2)R).
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
|\Psi(\chi,\xi)(\theta)| \leq{}& ((W_1+W_3)(K_1+\mathfrak{Y}_1)+(W_2+W_4)(K_2+\mathfrak{Y}_2))R \\
&+ (W_1+W_3)L_1+(W_2+W_4)L_2.
\end{aligned}
$$

Thus,

$$
\|\Psi(\chi,\xi)\|_{\Bbbk} \leq R.
$$

Hence, Ψ maps the ball ∇_R into itself. We shall show that the operator $\Psi : \nabla_R \to \nabla_R$ satisfies the assumptions of Schaefer's fixed point theorem. The proof will be given in several steps.

Step 1. We show that Ψ is continuous. Let $\{(\chi_\imath,\xi_\imath)\}$ be a sequence such that $(\chi_\imath,\xi_\imath) \to (\chi,\xi)$ in ∇_R. Then, for each $\theta \in \mho$, we have

$$
|\Psi_1(\chi_\imath,\xi_\imath)(\theta)-\Psi_1(\chi,\xi)(\theta)|
$$

$$
\begin{aligned}
&\leq [\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1]\int_0^{\varkappa}\varrho^{\mu_1-1}|[\aleph_1(\varrho,\chi_\imath(\varrho),\xi_\imath(\varrho))-\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))]|\mathrm{d}\varrho\\
&+\frac{|\delta_1|}{|1-\delta_1\delta_2|}\int_0^{\varkappa}\varrho^{\mu_2-1}|[\aleph_2(\varrho,\chi_\imath(\varrho),\xi_\imath(\varrho))-\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))]|\mathrm{d}\varrho.
\end{aligned}
$$

Analogously, we get

$$
\begin{aligned}
&|\Psi_2(\chi_\imath,\xi_\imath)(\theta)-\Psi_2(\chi,\xi)(\theta)|\\
&\leq [\frac{|\delta_1\delta_2|}{|1-\delta_1\delta_2|}+1]\int_0^{\varkappa}\varrho^{\mu_1-1}|[\aleph_2(\varrho,\chi_\imath(\varrho),\xi_\imath(\varrho))-\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))]|\mathrm{d}\varrho\\
&+\frac{|\delta_2|}{|1-\delta_1\delta_2|}\int_0^{\varkappa}\varrho^{\mu_2-1}|[\aleph_1(\varrho,\chi_\imath(\varrho),\xi_\imath(\varrho))-\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))]|\mathrm{d}\varrho.
\end{aligned}
$$

Since $(\chi_\imath,\xi_\imath)\to(\chi,\xi)$ as $\imath\to\infty$ and $\aleph_\jmath$, $\jmath=1,2$, are continuous, by the Lebesgue dominated convergence theorem

$$
\|\Psi(\chi_\imath,\xi_\imath)-\Psi(\chi,\xi)\|_{\Bbbk}\to 0 \quad \text{as} \quad \imath\to\infty.
$$

Step 2. We show that Ψ maps bounded sets into bounded and equicontinuous sets in ∇_R. $\Psi(\nabla_R)$ is bounded. This is clear since $\Psi:\nabla_R\to\nabla_R$ and ∇_R is bounded.

Now, let $\theta_1,\theta_2\in[0,\varkappa]$ be such that $\theta_1<\theta_2$. and let $(\chi_1;\chi_2)\in\nabla_R$. Then, we have

$$
\begin{aligned}
|(\Psi_1\chi)(\theta_2)-(\Psi_1\chi)(\theta_1)| &\leq \int_0^{\theta_2}\varrho^{\mu_1-1}|\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))|\mathrm{d}\varrho-\int_0^{\theta_1}\varrho^{\mu_1-1}|\aleph_1(\varrho,\chi(\varrho),\xi(\varrho)|\mathrm{d}\varrho\\
&\leq \int_{\theta_1}^{\theta_2}\varrho^{\mu_1-1}|\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))|\mathrm{d}\varrho\\
&\leq \frac{L_1+K_1R+\mathfrak{Y}_1R}{\mu_1}(\theta_2^{\mu_1}-\theta_1^{\mu_1}).
\end{aligned}
$$

Thus,

$$
|(\Psi_1\chi)(\theta_2)-(\Psi_1\chi)(\theta_1)|\leq\frac{L_1+K_1R+\mathfrak{Y}_1R}{\mu_1}(\theta_2^{\mu_1}-\theta_1^{\mu_1}). \tag{12}
$$

In a similar manner, we can easily get

$$
|(\Psi_2\xi)(\theta_2)-(\Psi_2\xi)(\theta_1)|\leq\frac{L_1+K_2R+\mathfrak{Y}_2R}{\mu_2}(\theta_2^{\mu_2}-\theta_1^{\mu_2}). \tag{13}
$$

The right-hand sides of the inequalities (12) and (13) tend to zero as $\theta_2\to\theta_1$. Therefore, the operator $\Psi(\chi,\xi)$ is equicontinuous. By collecting the above steps along with the Arzela-Ascoli theorem, we deduce that $\Psi:\nabla_R\to\nabla_R$ is completely continuous mapping.

Step 3. The set $\gimel=\{(\chi,\xi)\in\Bbbk:(\chi,\xi)=\varpi\Psi(\chi,\xi);\ 0\leq\varpi\leq1\}$ is bounded. Let $(\chi,\xi)\in\gimel$ such that $(\chi,\xi)=\varpi\Psi(\chi,\xi)$. Then for any $\theta\in\mho$, we have

$$
\chi(\theta)=\varpi(\Psi_1\chi)(\theta),\ \mathit{and}\ \xi(\theta)=\varpi(\Psi_2\xi)(\theta).
$$

Hence,

$$
\chi(\theta)=\frac{\varpi\delta_1}{1-\delta_1\delta_2}\left[\delta_2\int_0^{\varkappa}\varrho^{\mu_1-1}\aleph_1(\varrho,\chi,\xi)ds+\int_0^{\varkappa}\varrho^{\mu_2-1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho\right]
$$

$$+\varpi\int_0^{\theta}\varrho^{\mu_1-1}\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))\mathrm{d}\varrho.$$

From the assumption (H), we obtain

$$|\chi(\theta)|\le W_1(L_1+(K_1+\mathfrak{Y}_1)(|\chi(\theta)|+|\xi(\theta)|))+W_2(L_2+(K_2+\mathfrak{Y}_2)(|\chi(\theta)|+|\xi(\theta)|)).$$

By the same approach, we have

$$|\xi(\theta)|\le W_3(L_1+(K_1+\mathfrak{Y}_1)(|\chi(\theta)|+|\xi(\theta)|))+W_4(L_2+(K_2+\mathfrak{Y}_2)(|\chi(\theta)|+|\xi(\theta)|)).$$

Thus, we obtain

$$|\chi(\theta)|+|\xi(\theta)|\le((W_1+W_3)(K_1+\mathfrak{Y}_1)+(W_2+W_4)(K_2+\mathfrak{Y}_2))(|\chi(\theta)|+|\xi(\theta)|) \tag{14}$$

$$+(W_1+W_3)L_1+(W_2+W_4)L_2. \tag{15}$$

This gives

$$|\chi(\theta)|+|\xi(\theta)|\le\frac{(W_1+W_3)L_1+(W_2+W_4)L_2}{1-((W_1+W_3)(K_1+\mathfrak{Y}_1)+(W_2+W_4)(K_2+\mathfrak{Y}_2))}:=\nu.$$

Hence,

$$\|(\chi,\xi)\|_{\Bbbk}\le\nu.$$

Therefore, the set $\daleth$ is bounded.

As a consequence of Theorem 2.6, we conclude that Ψ has at least one fixed point. This confirms that there exists at least one solution of the coupled system (1)-(2). ☐

4. Existence results in Fréchet spaces

Let us introduce the following hypotheses:

(H$_1$) The functions $\aleph_\jmath;\ \jmath=1,2$ are measurable on $\mathbb{R}_+$; for each $\theta\in\mho$ *and* $\chi_\jmath,\xi_\jmath\in\Xi$, and the the functions $(\chi,\xi)\to\aleph_\jmath(\theta,\chi,\xi)$ are continuous on Ξ for a.e.$\theta\in\mathbb{R}_+;\ \jmath=1,2$.

(H$_2$) There exist continuous functions $h_\jmath,p_\jmath,q_\jmath:\mathbb{R}_+\to\mathbb{R}_+$ and $0<k_\jmath<1;\ \jmath=1,2$, such that

$$\|\aleph_\jmath(\theta,\chi_1,\chi_2)\|\le h_\jmath(\theta)+p_\jmath(\theta)\|\chi_1\|+q_\jmath(\theta)\|\chi_2\|;\quad\text{for}\quad\theta\in\mathbb{R}_+,\quad\text{and}\quad\chi_\jmath,\xi_\jmath\in\Xi.$$

(H_3) For each bounded sets $\nabla_\jmath\subset\Xi;\ \jmath=1,2$ and for each $\theta\in\mathbb{R}_+$, we have

$$\zeta(\aleph_\jmath(\theta,\nabla_1,\nabla_2))\le p_\jmath(\theta)\zeta(\nabla_1)+q_\jmath(\theta)\zeta(\nabla_2),$$

where ζ is a measure of noncompactness on the Banach space Ξ.

For $\imath\in\mathbb{N}$, set

$$p_\jmath^*=\sup_{\theta\in[0,\imath]}p_\jmath(\theta),\ q_\jmath^*=\sup_{\theta\in[0,\imath]}q_\jmath(\theta),\ h_\jmath^*=\sup_{\theta\in[0,\imath]}h_\jmath(\theta).$$

Theorem 4.1. *Assume that* (H_1)-(H_3) *are satisfied. If*

$$(p_1^*+q_1^*)\frac{(\imath-a)^{\mu_1}}{\mu_1}+(p_2^*+q_2^*)\frac{(\imath-a)^{\mu_2}}{\mu_2}<\frac{1}{2},$$

for each $\imath\in\mathbb{N}^*$, *then the problem (3)-(4) has at least one solution.*

Proof. Define the operator $\Psi : \Bbbk \to \Bbbk$ by

$$(\Psi(\chi,\xi))(\theta) = ((\Psi_1\chi)(\theta),(\Psi_2\xi)(\theta)), \tag{16}$$

where $\Psi_1, \Psi_2 : C \to C$ with

$$(\Psi_1\chi))(\theta) = \chi_a + \int_1^\theta (\varrho - a)^{\mu_1 - 1}\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))ds, \tag{17}$$

and

$$(\Psi_2\xi))(\theta) = \xi_a + \int_1^\theta (\varrho - a)^{\mu_2 - 1}\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))ds. \tag{18}$$

Clearly, the fixed points of the operator Ψ are solutions of the coupled system (3)-(4).

For any $\imath \in \mathbb{N}^*$, we set

$$R_\imath \geq \frac{\|\chi_a\| + \|\xi_a\| + h_1^*\frac{(\imath-a)^{\mu_1}}{\mu_1} + h_2^*\frac{(\imath-a)^{\mu_2}}{\mu_2}}{1 - ((p_1^*+q_1^*)\frac{(\imath-a)^{\mu_1}}{\mu_1} + (p_2^*+q_2^*)\frac{(\imath-a)^{\mu_2}}{\mu_2})}.$$

Consider the ball

$$\nabla_{R_\imath} := \nabla(0,R_\imath) = \{(\chi,\xi) \in \daleth : \|\chi\|_\imath \leq R_\imath, \|\xi\|_\imath \leq R_\imath\}.$$

For any $\imath \in \mathbb{N}^*$, and each $\chi, \xi \in \nabla_{R_\imath}$ and $\theta \in [0,\imath]$ we have

$$\begin{aligned}
\|(\Psi_1\chi)(\theta)\| &\leq \|\chi_a\| + \int_1^\theta (\varrho - a)^{\mu_1-1}\left\|\aleph_1(\varrho,\chi(\varrho),\xi(\varrho))\right\| \mathrm{d}\varrho \\
&\leq \|\chi_a\| + \int_1^\theta (\varrho - a)^{\mu_1-1}(h_1(\varrho) + p_1(\varrho)\|\chi_1\| + q_1(\varrho)\|\chi_2\|)\mathrm{d}\varrho \\
&\leq \|\chi_a\| + (h_1^* + (p_1^*+q_1^*)R_\imath)\int_1^\theta (\varrho - a)^{\mu_1-1}\mathrm{d}\varrho \\
&\leq \|\chi_a\| + (h_1^* + (p_1^*+q_1^*)R_\imath)\frac{(\imath - a)^{\mu_1}}{\mu_1},
\end{aligned}$$

and

$$\begin{aligned}
\|(\Psi_2\xi)(\theta)\| &\leq \|\xi_a\| + \int_1^\theta (\varrho - a)^{\mu_2-1}\left\|\aleph_2(\varrho,\chi(\varrho),\xi(\varrho))\right\| \mathrm{d}\varrho \\
&\leq \|\xi_a\| + \int_1^\theta (\varrho - a)^{\mu_2-1}(h_2(\varrho) + p_2(\varrho)\|\chi_1\| + q_2(\varrho)\|\chi_2\|)\mathrm{d}\varrho \\
&\leq \|\xi_a\| + (h_2^* + (p_2^*+q_2^*)R_\imath)\int_1^\theta (\varrho - a)^{\mu_2-1}\mathrm{d}\varrho \\
&\leq \|\xi_a\| + (h_2^* + (p_2^*+q_2^*)R_\imath)\frac{(\imath - a)^{\mu_2}}{\mu_2}.
\end{aligned}$$

Then,

$$\begin{aligned}
\|(\Psi(\chi,\xi))(\theta)\| &\leq \|\chi_a\| + \|\xi_a\| + h_1^*\frac{(\imath - a)^{\mu_1}}{\mu_1} + h_2^*\frac{(\imath - a)^{\mu_2}}{\mu_2} + ((p_1^*+q_1^*)\frac{(\imath - a)^{\mu_1}}{\mu_1} \\
&\quad + (p_2^*+q_2^*)\frac{(\imath - a)^{\mu_2}}{\mu_2})R_\imath \\
&\leq R_\imath.
\end{aligned}$$

Thus,

$$\|(\Psi(\chi,\xi)\|_{\imath} \leq R_{\imath}. \tag{19}$$

This proves that Ψ transforms the ball $\nabla_{R_\imath}$ into itself. We shall show that the operator $\Psi : \nabla_{R_\imath} \to \nabla_{R_\imath}$ satisfies all the assumptions of Theorem 2.7. The proof will be given in two steps.

Step 1: $\Psi(\nabla_{R_\imath})$ is bounded and $\Psi : \Psi(\nabla_{R_\imath}) \to \Psi(\nabla_{R_\imath})$ is continuous. Since $\Psi(\nabla_{R_\imath}) \subset \nabla_{R_\imath}$ and $\nabla_{R_\imath}$ is bounded, $\Psi(\nabla_{R_\imath})$ is bounded. Let $\{(\chi_k,\xi_k)\}_{k\in\mathbb{N}}$ be a sequence such that $(\chi_k,\xi_k) \to (\chi,\xi)$ in $\nabla_{R_\imath}$. Then, for each $\theta \in [0,\imath]$, we have

$$\begin{aligned}
&\|(\Psi(\chi_\imath,\xi_\imath))(\theta) - (\Psi(\chi,\xi))(\theta)\| \\
&\leq \sum_{\jmath=1}^{2}\int_a^\theta \left\|(\varrho-a)^{\mu_\jmath-1}[\aleph_\jmath(\varrho,\chi_\imath(\varrho),\xi_\imath(\varrho) - \aleph_\jmath(\varrho,(\chi(\varrho),\xi(\varrho))]\right\| d\varrho \\
&\leq \sum_{\jmath=1}^{2}\int_a^\theta (\varrho-a)^{\mu_\jmath-1}\left\|[\aleph_\jmath(\varrho,\chi_\imath(\varrho),\xi_\imath(\varrho) - \aleph_\jmath(\varrho,(\chi(\varrho),\xi(\varrho))]\right\| d\varrho.
\end{aligned}$$

Since $(\chi_k,\xi_k) \to (\chi,\xi)$ as $k \to \infty$ and $\aleph_\jmath$, $\jmath = 1,2$, are continuous, by the Lebesgue dominated convergence theorem

$$\|\Psi(\chi_\imath,\xi_\imath) - \Psi(\chi,\xi)\|_\imath \to 0 \quad \text{as} \quad k \to \infty.$$

Step 2: For each bounded equicontinuous subset D of $\nabla_{R_\imath}, \zeta_\imath(\Psi(D)) < \ell_\imath\zeta_\imath(D)$. From Lemmas 2.2 and 2.3, for any $D \subset \nabla_{R_\imath}$ and any $\gamma > 0$, there exists a sequence $\{\chi_k,\xi_k\}_{k=0}^\infty \subset D$, such that for all $\theta \in [a,\imath]$, we have

$$\begin{aligned}
\zeta((ND)(\theta)) &= \sum_{\jmath=1}^{2}\zeta(\{\chi_{ia} + \int_a^\theta (\varrho-a)^{\mu_\jmath-1}\aleph_\jmath(\varrho,(\chi(\varrho),\xi(\varrho))ds; (\chi,\xi)\in D\}) \\
&\leq \sum_{\jmath=1}^{2}\zeta(\{\int_a^\theta (\varrho-a)^{\mu_\jmath-1}\aleph_\jmath(\varrho,(\chi_k(\varrho),\xi_k(\varrho))ds\}_{k=1}^\infty) + \gamma \\
&\leq 2\sum_{\jmath=1}^{2}\int_a^\theta (\varrho-a)^{\mu_\jmath-1}\zeta(\{\aleph_\jmath(\varrho,(\chi_k(\varrho),\xi_k(\varrho))\}_{k=1}^\infty)ds + \gamma \\
&\leq 2\sum_{\jmath=1}^{2}\int_a^\theta (\varrho-a)^{\mu_\jmath-1}p_\jmath(\varrho)\zeta(\{(\chi_k(\varrho))\}_{k=1}^\infty) + q_\jmath(\varrho)\zeta(\{\xi_k(\varrho))\}_{k=1}^\infty)ds + \gamma \\
&\leq 2((p_1^*+q_1^*)\frac{(\imath-a)^{\mu_1}}{\mu_1} + (p_2^*+q_2^*)\frac{(\imath-a)^{\mu_2}}{\mu_2})\zeta_\imath(D) + \gamma.
\end{aligned}$$

Since $\gamma > 0$ is arbitrary, then

$$\zeta((ND)(\theta)) \leq 2((p_1^*+q_1^*)\frac{(\imath-a)^{\mu_1}}{\mu_1} + (p_2^*+q_2^*)\frac{(\imath-a)^{\mu_2}}{\mu_2})\zeta_\imath(D).$$

Thus,

$$\zeta_\imath(ND) \leq 2((p_1^*+q_1^*)\frac{(\imath-a)^{\mu_1}}{\mu_1} + (p_2^*+q_2^*)\frac{(\imath-a)^{\mu_2}}{\mu_2})\zeta_\imath(D).$$

By combining steps 1 and 2 with Theorem 2.7, it follows that there exists a fixed point of Ψ within $\nabla_{R_\imath}$, which serves as a solution to problem (3)-(4). □

5. Examples

Example 5.1. Consider the coupled system of Conformable fractional differential equations

$$\begin{cases} (\mathcal{T}_{0^+}^{\frac{1}{2}}\chi)(\theta) = \aleph_1(\theta, \chi(\theta), \xi(\theta)) \\ (\mathcal{T}_{0^+}^{\frac{1}{2}}\xi)(\theta) = \aleph_2(\theta, \chi(\theta), \xi(\theta)) \end{cases} ; \quad \theta \in [0, 1], \tag{20}$$

with the following coupled boundary conditions:

$$(\chi(0), \xi(0)) = (1, 2), \tag{21}$$

where

$$\aleph_1(\theta, \chi, \xi) = \frac{\sin(\chi + \xi)}{40(e^{\theta} + 1)},$$

$$\aleph_2(\theta, \chi, \xi) = \frac{\tan \chi}{10 + |\chi| + |\xi|}, \quad \theta \in [0, 1]; \quad \chi, \xi \in \mathbb{R}.$$

The hypothesis (H) and the condition (10) are satisfied with

$$\mathfrak{Y}_1 = K_1 = \frac{1}{80}, \quad K_2 = \frac{1}{10}, \quad \delta_1 = \delta_2 = \frac{1}{2},$$

$$W_1 = W_4 = \frac{8}{3}, W_2 = W_3 = \frac{4}{3}.$$

Hence, Theorem 3.1 implies that the system (20)–(21) has at least one solution defined on $[0, 1]$.

Example 5.2. Let

$$l^1 = \left\{ \chi = (\chi_1, \chi_2, \dots, \chi_\imath, \dots), \sum_{k=1}^{\infty} |\chi_k| < \infty \right\}$$

be the Banach space with the norm

$$\|\chi\| = \sum_{k=1}^{\infty} |\chi_k|,$$

and $C(\mathbb{R}_+, l^1)$ be the Fréchet space of all continuous functions ξ from $\mathbb{R}_+$ into l^1, equipped with the family of seminorms

$$\|\xi\|_\imath = \sup_{\theta \in [0,\imath]} \|\xi(\theta)\|; \ \imath \in \mathbb{N}.$$

Consider the coupled system of Conformable fractional differential equations

$$\begin{cases} (\mathcal{T}_{0^+}^{\frac{1}{5}}\chi_k)(\theta) = \aleph_k(\theta, \chi(\theta), \xi(\theta)) \\ (\mathcal{T}_{0^+}^{\frac{1}{5}}\xi_k)(\theta) = \widehat{\aleph}_k(\theta, \chi(\theta), \xi(\theta)) \end{cases} ; \quad \theta \in [1, \infty), \ k = 1, 2, \dots, \tag{22}$$

with the following initial coupled conditions:

$$(\chi(1), \xi(1)) = (0, 0), \tag{23}$$

where

$$\aleph_k(\theta, \chi, \xi) = \frac{c}{1 + \|\chi\| + \|\xi\|}(e^{-7} + \frac{1}{e^{\theta+5}})(2^{-k} + \chi_k(\theta)), \quad \theta \in [1, \infty),$$

$$\widehat{\aleph}_k(\theta, \chi, \xi) = \frac{c}{e^{\theta+5}(1 + \|\chi\| + \|\xi\|)}(2^{-k} + \xi_k(\theta)), \quad \theta \in [1, \infty), \ k = 1, 2, \cdots, \ c > 0,$$

for each $\theta \in [1, \imath]$; $\imath \in \mathbb{N}$, with

$$\aleph = (\aleph_1, \aleph_2, \ldots, \aleph_k, \ldots),\ \widehat{\aleph} = (\widehat{\aleph}_1, \widehat{\aleph}_2, \ldots, \widehat{\aleph}_k, \ldots),\ \textit{and}\ \chi = (\chi_1, \chi_2, \ldots, \chi_k, \ldots).$$

We can show that all hypotheses of Theorem 4.1 are satisfied with

$$h_1(\theta) = p_1(\theta) = c(e^{-7} + \frac{1}{e^{\theta+5}}), \quad q_1(\theta) = p_2(\theta) = 0, \quad h_2(\theta) = q_2(\theta) = \frac{c}{e^{\theta+5}}.$$

So,

$$h_1^* = p_1^* = c(e^{-7} + e^{-6}), \quad h_2^* = q_2^* = ce^{-6}.$$

Therefore, Theorem 4.1 implies that the system (22)–(23) has at least one solution defined on $[1, \infty)$.

Declarations

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

Competing interests: It is declared that authors has no competing interests.

Author's contributions: The study was carried out in collaboration of all authors. All authors read and approved the final manuscript.

Funding: Not available.

Availability of data and materials: Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

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