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Semihypergroup Actions by using the Generalized Permutations

B. Davvaz and M. Mostafavi

Abstract. An ordered hypergroupoid is a hypergroupoid together with a partial order such that satisfies the monotone condition. In this paper, we introduce the notion of semihypergroup actions on ordered hypergroupoids. Some results in this respect are investigated. In particular, we prove that if S is a commutative semihypergroup acting on an ordered hypergroupoid H, then there exists a commutative semihypergroup $\widetilde{S}$ acting on the ordered hypergroupoid $\widetilde{H} := (H \times S)/\bar{\rho}$ in such a way that H is embedded in $\widetilde{H}$.

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1. Introduction

The concept of algebraic hyperstructures was introduced in 1934 by Marty [21] and has been studied in the following decades and nowadays by many mathematicians. Marty published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non-commutative groups. Several books have been written on hyperstructure theory, see [4, 5, 8, 10, 11, 22]. The concept of a semihypergroup is a generalization of the concept of a semigroup. As we know, in a semigroup, the composition of two elements is an element, while in a semihypergroup, the composition of two elements is a non-empty set. Indeed, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Semihypergroups are studied by many authors, for example, Bonansinga and Corsini [1], Davvaz [6, 7], De Salvo et al. [13], Freni [15], Hila et al. [17], Leoreanu [19], and many others. The concept of ordering hypergroups investigated by Chvalina [3] as a special class of hypergroups and studied by him and many others. In [16], Heidari and Davvaz studied a semihypergroup $(H, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Several mathematicians considered actions of algebraic hyperstructures, for example see [9, 20, 23]. In [20], Madanshekaf and Ashrafi considered a generalized action of a hypergroup H on a non-empty set X and obtained some results in this respect. In [18], Kehayopulu and Tsingelis are studied semigroup actions on ordered groupoids. Now, in this paper, we apply their results to algebraic hyperstructures and introduce the notion of semihypergroup actions on ordered hypergroupoids.

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2. Semihypergroups

Let H be a non-empty set. A mapping $\circ : H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the family of all non-empty subsets of H, is called a *hyperoperation* on H. The couple $(H, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two non-empty subsets of H and $x \in H$, then we denote

$$A \circ B = \bigcup_{\substack{a\in A\\ b\in B}} a \circ b, \;\; x \circ A = \{x\} \circ A \;\text{ and }\; A \circ x = A \circ \{x\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for every $x, y, z \in H$, $x \circ (y \circ z) = (x \circ y) \circ z$, that is

$$\bigcup_{u\in y\circ z} x \circ u = \bigcup_{v\in x\circ y} v \circ z.$$

A semihypergroup H is *commutative* if it satisfies $x \circ y = y \circ x$, for all $x, y, z \in H$. The associativity for semihypergroups can be applied for subsets. A non-empty subset A of H is called a *subsemihypergroup* if $x \circ y \subseteq A$ for all x, y in A.

Remark 2.1. Every semigroup is a semihypergroup.

A hypergroupoid $(H, \circ)$ called a *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$. This condition is called the reproduction axiom. The couple $(H, \circ)$ is called a hypergroup if it is a semihypergroup and a quasihypergroup.

Definition 2.1. [16] An *ordered hypergroupoid* $(G, \circ, \leq)$ is a hypergroupoid $(H, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation, meaning that for any x, y, z in H,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

Example 2.1. [2] We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	a	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
b	a	$\{a,b\}$	$\{a,c\}$	$\{a,d\}$
c	a	b	c	d
d	a	b	c	d

$$\leq \; = \{(a,a), (b,b), (c,c), (d,d), (a,b)\}.$$

Example 2.2. [12] Suppose that $S = \{x, y, z, r, s, t\}$. We consider the ordered semihypergroup $(S, \circ, \leq)$, where the hyperoperation $\circ$ is defined by the following table:

$\circ$	x	y	z	r	s	t
x	r	$\{r,s\}$	$\{r,t\}$	x	$\{x,y\}$	$\{x,z\}$
y	r	s	$\{r,t\}$	x	y	$\{x,z\}$
z	r	$\{r,s\}$	t	x	$\{x,y\}$	z
r	x	$\{x,y\}$	$\{x,z\}$	r	$\{r,s\}$	$\{r,t\}$
s	x	y	$\{x,z\}$	r	s	$\{r,t\}$
t	x	$\{x,y\}$	z	r	$\{r,s\}$	t

and the order $\leq$ is defined by:

$$\leq := \{(x,x),(y,y),(z,z),(r,r),(s,s),(t,t),(s,r),(t,r),(y,x),(z,x)\}.$$

Let $(H,+,\leq_H)$ and $(H',+,\leq_{H'})$ be two ordered hypergroupoids. Let $f : H \to H'$ be a mapping. Then,

- f is called *isotone* if $x, y \in H$, $x \leq_H y$ implies that $f(x) \leq_{H'} f(y)$.
- f is called *reverse isotone* if $x, y \in H$, $f(x) \leq_{H'} f(y)$ implies that $x \leq_H y$; (each reverse isotone mapping is one-to-one.)
- f is called a *homomorphism* if it is isotone and satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in H$;
- f is called an *isomorphism* if it is homomorphism, onto and reverse isotone.

We say H is embedded in H' if there exists a mapping $f : H \to H'$ which is homomorphism and reverse isotone.

3. Generalized permutations and semihypergroup actions

According to [14, 22], we can consider a *generalized permutation* on a non-empty set X as a map $f : X \to \mathcal{P}^*(X)$ such that the *reproductive axiom* holds, i.e.,

$$\bigcup_{x\in X} f(x) = f(X) = X.$$

We denote the set of all generalized permutations by M_X. A generalized permutation f is said to satisfy the condition θ [22] if

$$x \in X \text{ and } z \in f(x) \;\Rightarrow\; f(z) = f(x).$$

We denote the set of all generalized permutations that satisfies the condition θ by M_θ.

Theorem 3.1. [22, Theorem 6.2.9] *Let $f \in M_\theta$ and $M_f = \{g \in M_X \mid g \subseteq f\}$. Then, M_f is a hypergroup with respect to the hyperoperation $\star$ defined by*

$$f_1 \star f_2 = \{p \in M_X \mid p \subseteq f_1 \circ f_2\},$$

where $f_1 \circ f_2$ is defined by

$$f_1 \circ f_2 = \bigcup_{y\in f_2(x)} f_1(y).$$

Now, we give the definition of an action of a commutative semihypergroup on an ordered hypergroupoid.

Definition 3.1. Let $(H,+,\leq_H)$ be an ordered hypergroupoid and $(S,\cdot)$ be a commutative semihypergroup. The map $f : S \times H \to \mathcal{P}^*(H)$ is called an *action* of S on H and denoted by (H,S,f), if the following axioms hold:

(1) For all $\alpha \in S$, for all $x, y \in H$, $f(\alpha, x + y) = f(\alpha, x) + f(\alpha, y)$, where for all $g \in H$ and $X \in \mathcal{P}^*(H)$,

$$f(g,X) = \bigcup_{x\in X} f(g,x).$$

(2) For all $\alpha, \beta \in S$ and for all $x \in H$, $f(\alpha\beta, x) = f(\alpha, f(\beta, x))$, where for all $B \in \mathcal{P}^*(S)$ and $g \in H$,

$$f(B,g) = \bigcup_{b\in B} f(b,g).$$

(3) For all $\alpha \in S$ and for all $x, y \in H$, $f(\alpha, x) \leq_H f(\alpha, y) \Leftrightarrow x \leq_H y$.

Lemma 3.2. *Consider* (H, S, f). *Then,*

(1) *For all* $A, B \in \mathcal{P}^*(H)$ *and for all* $m \in S$, *we have*
$$f(m, A + B) = f(m, A) + f(m, B).$$

(2) *For all* $a, b \in H$ *and for all* $M \in \mathcal{P}^*(S)$, *we have*
$$f(M, a + b) = f(M, a) + f(M, b).$$

(3) *For all* $A, B \in \mathcal{P}^*(H)$ *and* $M \in \mathcal{P}^*(S)$, *we have*
$$f(C, A + B) = f(C, A) + f(C, B).$$

Proof. (1) We have
$$\begin{aligned}
f(m, A + B) &= f\Big(m, \Big(\bigcup_{\substack{a \in A \\ b \in B}} a + b\Big)\Big) \\
&= \bigcup_{\substack{a \in A \\ b \in B}} f(m, a + b) \\
&= \bigcup_{\substack{a \in A \\ b \in B}} \Big(f(m, a) + f(m, b)\Big) \\
&= \bigcup_{\substack{a \in A \\ b \in B}} \Big(\bigcup_{\substack{i \in f(m,a) \\ j \in f(m,b)}} i + j\Big) \\
&= \bigcup_{\substack{i \in f(m,A) \\ j \in f(m,B)}} i + j \\
&= f(m, A) + f(m, B).
\end{aligned}$$

(2) We have
$$\begin{aligned}
f(M, a + b) &= \bigcup_{m \in M} f(m, a + b) \\
&= \bigcup_{m \in M} \Big(f(m, a) + f(m, b)\Big) \\
&= \bigcup_{m \in M} \Big(\bigcup_{\substack{\alpha \in f(m,a) \\ \beta \in f(m,b)}} \alpha + \beta\Big) \\
&= \bigcup_{\substack{\alpha \in f(M,a) \\ \beta \in f(M,b)}} \alpha + \beta \\
&= f(M, a) + f(M, b).
\end{aligned}$$

(3) We have
$$\begin{aligned}
f(C, A + B) &= \bigcup_{c \in C} f(c, A + B) \\
&= \bigcup_{c \in C} \Big(f(c, A) + f(c, B)\Big) \\
&= \bigcup_{c \in C} \Big(\bigcup_{a \in A} f(c, a) + \bigcup_{b \in B} f(c, b)\Big) \\
&= \bigcup_{\substack{c \in C \\ a \in A}} f(c, a) + \bigcup_{\substack{c \in C \\ b \in B}} f(c, b) \\
&= f(C, A) + f(C, B).
\end{aligned}$$

□

Let H be a set and $\widetilde{\leq_H}$ be a relation on $\mathcal{P}^*(H) \times \mathcal{P}^*(H)$. Suppose that A, B and C are arbitrary non-empty subsets of H. The following properties can be considered
(1) $A\widetilde{\leq_H}A$;
(2) if $A\widetilde{\leq_H}B$ and $B\widetilde{\leq_H}A$, then $A = B$;
(3) if $A\widetilde{\leq_H}B$, then $B\widetilde{\leq_H}A$;
(4) if $A\widetilde{\leq_H}B$ and $B\widetilde{\leq_H}C$, then $A\widetilde{\leq_H}C$.
If "$\widetilde{\leq_H}$" has the properties 1, 2 and 4, then it is a partial order relation on the non-empty subsets of H. If "$\widetilde{\leq_H}$" has the properties 1, 3 and 4, then it is an equivalence relation on the non-empty subsets of H.

A relation σ on H is called a *quasi-order* on H if it is reflexive and transitive. If σ is a quasi-order on H, then the relation $\sigma^* = \sigma \cap \sigma^{-1}$ is an equivalence relation on H.

Lemma 3.3. *Let σ be a quasi-order on $\mathcal{P}^*(H)$, the non-empty subsets of H. Then, each of the following equivalent definitions defines an order on the set $\mathcal{P}^*(H)/\sigma^*$.*
(1) $\leq_\sigma = \{(\sigma^*(A), \sigma^*(B)) \mid \exists\ A_1 \in \sigma^*(A),\ \exists\ B_1 \in \sigma^*(B), \ni: (A_1, B_1) \in \sigma\}$.
(2) $\leq_\sigma = \{\sigma^*((A), \sigma^*(B)) \mid \forall\ A_1 \in \sigma^*(A),\ \forall\ B_1 \in \sigma^*(B), \ni: (A_1, B_1) \in \sigma\}$.
(3) $\sigma^*(A) \leq_\sigma \sigma^*(B) \ \Leftrightarrow (A, B) \in \sigma$.

Proof. It is straightforward. □

Definition 3.2. Consider (H, S, f), $A, B \in \mathcal{P}^*(H)$ and $C \in \mathcal{P}^*(S)$. Then,

$$A\widetilde{\leq_H}B \ \Leftrightarrow \ f(C, A)\widetilde{\leq_H}f(C, B).$$

In particular, if $C := \{c\}$, then $A\widetilde{\leq_H}B \ \Leftrightarrow \ f(c, A)\widetilde{\leq_H}f(c, B)$. Moreover, if $A := \{a\}$ and $B := \{b\}$, then $a \leq_H b \ \Leftrightarrow \ f(C, a)\widetilde{\leq_H}f(C, b)$.

Remark 3.1. If (H, S, f), $\alpha, \beta \in S$ and $x \in H$, then $f(\beta, f(\alpha, x)) = f(\alpha, f(\beta, x))$.

Lemma 3.4. (1) *If (H, S, f), $\alpha, \beta, \gamma \in S$ and $x \in H$, then $f(\alpha\beta, f(\gamma, x)) = f(\alpha\beta\gamma, x)$.*
(2) *If (H, S, f), $A, B \in \mathcal{P}^*(S)$ and $C \in \mathcal{P}^*(H)$, then $f(A, f(B, C)) = f(B, f(A, C))$.*

Proof. (1) We have

$$f(\alpha\beta, f(\gamma, x)) = \bigcup_{a \in \alpha\beta} f(a, f(\gamma, x)) = \bigcup_{a \in \alpha\beta} f((a\gamma), x) = f((\alpha\beta\gamma), x).$$

(2) We have

$$\begin{aligned} f(A, f(B, C)) &= \bigcup_{a \in A, b \in B, x \in C} f(a, f(b, x)) \\ &= \bigcup_{a \in A, b \in B, x \in C} f((ab), x) \\ &= \bigcup_{a \in A, b \in B, x \in C} f((ba), x) \\ &= \bigcup_{a \in A, b \in B, x \in C} f(b, f(a, x)) \\ &= f(B, f(A, C)). \end{aligned}$$

□

In the following we generalize [18, Theorem 3].

Theorem 3.5. *Consider (H, S, f). Then, there exists a $(\bar{H}, \bar{S}, \boxplus)$ such that*
(1) *H is embedded in $\bar{H}$ under a mapping ψ.*

(2) *There exists a homomorphism* $Q : S \to \bar{S}$ *satisfying the conditions:*
(i) *For every* $\beta \in S$ *and every* $x \in H$, $\psi(f(\beta, x)) = Q(\beta) \boxplus \psi(x)$.
(ii) *For each* $\beta \in S$ *and each* $x \in H$ *there exists* $q^x_\beta \in \bar{H}$ *such that*

$$Q(\beta) \boxplus q^x_\beta = \psi(x).$$

Moreover, if H *is a semihypergroup, then* $\bar{H}$ *is also a semihypergroup. If* H *is a commutative hypergroupoid, then so is* $\bar{H}$. *If* H *is a cancellative ordered hypergroupoid, then* $\bar{H}$ *is also so. Conversely, suppose* $(\bar{\bar{H}}, \bar{\bar{S}}, \boxdot)$ *such that*
(1) H *is embedded in* $\bar{\bar{H}}$ *under a mapping* ψ'.
(2) *There exists a homomorphism* $Q' : S \to \bar{\bar{S}}$ *having the properties:*
(i) *For every* $\alpha \in S$ *and every* $x \in H$, $\psi'(f(\alpha, x)) = Q'(\alpha) \boxdot \psi'(x)$
(ii) *For each* $\beta \in S$ *and each* $x \in H$ *there exists* $q^x_\alpha \in \bar{\bar{H}}$ *such that*

$$Q(\beta) \boxdot q^x_\beta = \psi(x).$$

Then, $\bar{H}$ *is embedded in* $\bar{\bar{H}}$.

Proof. The proof consists of a series of constructions and steps. We shall complete the proof in the rest of paper. □

Proposition 3.6. *Consider* (H, S, f). *Define a relation* σ *by*

$$(f(\beta, x) + f(\alpha, y), \alpha\beta)\sigma(f(\beta_1, x_1) + f(\alpha_1, y_1), \alpha_1\beta_1)$$
$$\Leftrightarrow f(\alpha_1\beta_1, f(\beta, x) + f(\alpha, y)) \widetilde{\leq_H} f(\alpha\beta, f(\beta_1, x_1) + f(\alpha_1, y_1))$$

on the non-empty subsets of $H \times S$. *Then,* σ *is a quasi-order on the non-empty subsets of* $H \times S$.

Proof. If $(f(\beta, x) + f(\alpha, y), \alpha\beta) \subseteq H \times S$ then, since $f(\alpha\beta, f(\beta, x) + f(\alpha, y)) \subseteq H$ and $\widetilde{\leq_H}$ is an order on $\mathcal{P}^*(H)$, we have $f(\alpha\beta, f(\beta, x) + f(\alpha, y)) \widetilde{\leq_H} f(\alpha\beta, f(\beta, x) + f(\alpha, y))$, i.e., $((f(\beta, x) + f(\alpha, y), \alpha\beta)), (f(\beta, x) + f(\alpha, y), \alpha\beta)) \in \sigma$, so σ is reflexive. To show the transitivity, suppose that $(f(\beta, x) + f(\alpha, y), \alpha\beta)\sigma(f(\beta_1, x_1) + f(\alpha_1, y_1), \alpha_1\beta_1)$ and $(f(\beta_1, x_1) + f(\alpha_1, y_1), \alpha_1\beta_1)\sigma(f(\beta_2, x_2) + f(\alpha_2, y_2), \alpha_2\beta_2)$. Then,

$$(f(\beta, x) + f(\alpha, y), \alpha\beta)\sigma(f(\beta_1, x_1) + f(\alpha_1, y_1), \alpha_1\beta_1)$$
$$\Rightarrow f(\alpha_1\beta_1, f(\beta, x) + f(\alpha, y)) \widetilde{\leq_H} f(\alpha\beta, f(\beta_1, x_1) + f(\alpha_1, y_1))$$
$$\Rightarrow f(\alpha_2\beta_2, f(\alpha_1\beta_1, f(\beta, x) + f(\alpha, y))) \widetilde{\leq_H} f(\alpha_2\beta_2, f(\alpha\beta, f(\beta_1, x_1) + f(\alpha_1, y_1)))$$
$$\Rightarrow f(\alpha_1\beta_1, f(\alpha_2\beta_2, f(\beta, x) + f(\alpha, y))) \widetilde{\leq_H} f(\alpha\beta, f(\alpha_2\beta_2, f(\beta_1, x_1) + f(\alpha_1, y_1))).$$

Moreover, we have

$$(f(\beta_1, x_1) + f(\alpha_1, y_1), \alpha_1\beta_1)\sigma(f(\beta_2, x_2) + f(\alpha_2, y_2), \alpha_2\beta_2)$$
$$\Rightarrow f(\alpha_2\beta_2, f(\beta_1, x_1) + f(\alpha_1, y_1)) \widetilde{\leq_H} f(\alpha_1\beta_1, f(\beta_2, x_2) + f(\alpha_2, y_2))$$
$$\Rightarrow f(\alpha\beta, f(\alpha_2\beta_2, f(\beta_1, x_1) + f(\alpha_1, y_1))) \widetilde{\leq_H} f(\alpha\beta, f(\alpha_1\beta_1, f(\beta_2, x_2) + f(\alpha_2, y_2))).$$

Consequently, we obtain

$$f(\alpha_1\beta_1, f(\alpha_2\beta_2, f(\beta, x) + f(\alpha, y))) \widetilde{\leq_H} f(\alpha\beta, f(\alpha_1\beta_1, f(\beta_2, x_2) + f(\alpha_2, y_2)))$$
$$\Rightarrow f(\alpha_1\beta_1, f(\alpha_2\beta_2, f(\beta, x) + f(\alpha, y))) \widetilde{\leq_H} f(\alpha_1\beta_1, f(\alpha\beta, f(\beta_2, x_2) + f(\alpha_2, y_2)))$$
$$\Rightarrow f(\alpha_2\beta_2, f(\beta, x) + f(\alpha, y)) \widetilde{\leq_H} f(\alpha\beta, f(\beta_2, x_2) + f(\alpha_2, y_2)).$$

This completes the proof. □

In the following we always denote by σ the quasi-order on the non-empty subsets of $H \times S$ defined in Proposition 3.6. Since σ is a quasi-order on $\mathcal{P}^*(H \times S)$, it follows that σ^* is an equivalence relation on $\mathcal{P}^*(H \times S)$. Clearly, σ^* is an equivalence relation on $H \times S$ too.

Proposition 3.7. *Consider (H, S, f). Then, for every $\alpha, \beta \in S$ and $x, y \in H$, we have*
(1) $\sigma^*((x, \alpha)) = \sigma^*((y, \beta)) \Leftrightarrow f(\beta, x) = f(\alpha, y)$,
(2) $\sigma^*((f(\alpha, x), \alpha)) = \sigma^*((f(\beta, x), \beta))$,
(3) $\sigma^*((f(\alpha, x), \alpha\beta)) = \sigma^*((x, \beta))$.

Proof. (1) Since σ^* is an equivalence relation, it follows that

$$\begin{aligned}(x, \alpha)\sigma^*(y, \beta) &\Leftrightarrow (x, \alpha)\sigma(y, \beta) \text{ and } (y, \beta)\sigma^{-1}(x, \alpha)\\ &\Leftrightarrow (x, \alpha)\sigma(y, \beta) \text{ and } (y, \beta)\sigma(x, \alpha)\\ &\Leftrightarrow f(\beta, x)\sigma f(\alpha, y) \text{ and } f(\alpha, y)\sigma f(\beta, x)\\ &\Leftrightarrow f(\beta, x) = f(\alpha, y).\end{aligned}$$

The proofs of (2) and (3) are similar to the proof of (1). $\square$

Proposition 3.8. *Consider (H, S, f). Define the hyperoperation "$\odot$" and the order "$\preceq_\sigma$" on $(H \times S)/\sigma^*$ by*

$$\begin{aligned}&\odot : \Big((H \times S)/\sigma^*\Big) \times \Big((H \times S)/\sigma^*\Big) \to \mathcal{P}^*\Big((H \times S)/\sigma^*\Big)\\ &\Big(\sigma^*((x, \alpha)), \sigma^*((y, \beta))\Big) \mapsto \sigma^*((f(\beta, x) + f(\alpha, y), \alpha\beta))\\ &\sigma^*((x, \alpha)) \preceq_\sigma \sigma((y, \beta)) \Leftrightarrow f(\beta, x)\widetilde{\leq_H} f(\alpha, y).\end{aligned}$$

Then, $\Big((H \times S)/\sigma^*, \odot, \preceq_\sigma\Big)$ is an ordered hypergroupoid.

Proof. The hyperoperation is single-valued. Let (x, α), (x', α'), (y, β), $(y', \beta') \in H \times S$ be such that $\sigma^*((x, \alpha)) = \sigma^*((x', \alpha'))$ and $\sigma^*((y, \beta)) = \sigma^*(y', \beta'))$. Then, by Proposition 3.7 (1), we have $f(\alpha', x) = f(\alpha, x')$ and $f(\beta', y) = f(\beta, y')$. Hence,

$$\sigma^*((f(\beta, x) + f(\alpha, y), \alpha\beta)) = \sigma^*((f(\beta', x') + f(\alpha', y'), \alpha'\beta')),$$

or equivalently

$$f(\alpha'\beta', f(\beta, x) + f(\alpha, y)) = f(\alpha\beta, f(\beta', x') + f(\alpha', y')).$$

Thus, we conclude that

$$\begin{aligned}&f(\alpha'\beta', f(\beta, x) + f(\alpha, y)) = f(\alpha'\beta', f(\beta, x)) + f(\alpha'\beta', f(\alpha, y))\\ &= f(\beta\beta', f(\alpha', x)) + f(\alpha\alpha', f(\beta', y)) = f(\beta\beta', f(\alpha, x')) + f(\alpha\alpha', f(\beta, y'))\\ &= f(\alpha\beta, f(\beta', x') + f(\alpha', y')).\end{aligned}$$

Consequently, $(H \times S)/\sigma^*$ is a hypergroupoid.

Since σ is a quasi-order on $H \times S$, it follows that σ^* is an equivalence relation on non-empty subsets of $H \times S$. Then, by Lemma 3.3, the relation

$$\sigma^*((x, \alpha)) \preceq_\sigma \sigma^*((y, \beta)) \Leftrightarrow (x, \alpha)\sigma(y, \beta)$$

is an order relation on $(H \times S)/\sigma^*$. On the other hand,

$$(x, \alpha)\sigma(y, \beta) \Leftrightarrow f(\beta, x)\widetilde{\leq_H} f(\alpha, y).$$

Finally, we prove that the operation $\odot$ is compatible with the ordering. Suppose that $\sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((y,\beta))$ and $(z,\gamma) \in H \times S$. Then,

$$\sigma^*((z,\gamma)) \odot \sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((z,\gamma)) \odot \sigma^*((y,\beta)).$$

In fact, we have to prove that

$$\sigma^*((f(\alpha,z) + f(\gamma,x), \gamma\alpha)) \preceq_\sigma ((f(\beta,z) + f(\gamma,y), \gamma\beta)),$$

or equivalently,

$$f(\gamma\beta, [f(\alpha,z) + f(\gamma,x)]) \widetilde{\leq_H} f(\gamma\alpha, [f(\beta,z) + f(\gamma,y)]).$$

Since $\sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((y,\beta))$, it follows that $f(\beta,x)\widetilde{\leq_H} f(\alpha,y)$. Now, we have

$$\begin{aligned}&f((\gamma\beta),[f(\alpha,z)+f(\gamma,x)]) = [(f(\gamma\beta),f(\alpha,z))] + [f((\gamma\beta),f(\gamma,x))]\\ &= [f((\gamma\beta\alpha),z)] + [f((\gamma\beta\gamma),x)] = [f((\gamma\alpha\beta),z)] + [f((\gamma\gamma\beta),x)]\\ &= [f((\gamma\alpha),f(\beta,z))] + [f((\gamma\gamma),f(\beta,x))].\end{aligned}$$

Since $f(\beta,x)\widetilde{\leq_H} f(\alpha,y)$ and $\gamma\gamma \in \mathcal{P}^*(S)$, by Definition 3.2, we obtain

$$f(\gamma\gamma, f(\beta,x))\widetilde{\leq_H} f(\gamma\gamma, f(\alpha,y)).$$

Since $(H,+,\leq_H)$ is an ordered hypergroupoid and $f(\gamma\alpha, f(\beta,z)) \subseteq H$, we have

$$[f((\gamma\alpha),f(\beta,z))] + [f((\gamma\gamma),f(\beta,x))]\widetilde{\leq_H}[f((\gamma\alpha),f(\beta,z))] + [f((\gamma\gamma),f(\alpha,y))].$$

Hence, we conclude that

$$\begin{aligned}&f((\gamma\beta),[f(\alpha,z)+f(\gamma,x)])\tilde{\leq}_H[f((\gamma\alpha),f(\beta,z)]) + [f((\gamma\gamma),f(\alpha,y))]\\ &= [f((\gamma\alpha),f(\beta,z))] + [f((\gamma\gamma\alpha),y)] = [f((\gamma\alpha),f(\beta,z))] + [f((\gamma\alpha\gamma),y)]\\ &= [f((\gamma\alpha),f(\beta,z))] + [f((\gamma\alpha),f(\gamma,y))]\\ &= f((\gamma\alpha),[f(\beta,z)+f(\gamma,y)]).\end{aligned}$$

Similarly, it can be proved that $\sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((y,\beta))$ and $(z,\gamma) \in H \times S$ imply

$$\sigma^*((x,\alpha)) \odot \sigma^*((z,\gamma)) \preceq_\sigma \sigma^*((y,\beta)) \odot \sigma^*((z,\gamma)).$$

□

Proposition 3.9. *Let* (H,S,f). *Define* $\phi_\alpha : (H \times S)/\sigma^* \to \mathcal{P}^*(H \times S)/\sigma^*)$ *by*

$$\phi_\alpha\big(\sigma^*((x,\beta))\big) = \sigma^*((f(\alpha,x),\beta)) := \{\sigma^*((t,\beta)) \mid t \in f(\alpha,x)\}.$$

is an isomorphism and a generalized permutation.

Proof. The mapping ϕ_α is well defined, since

$$\begin{aligned}&\sigma^*((x,\beta)) = \sigma^*((x',\beta')) \Rightarrow f(\beta',x) = f(\beta,x') \Rightarrow f(\alpha,f(\beta',x)) = f(\alpha,f(\beta,x'))\\ &\Rightarrow f(\beta',f(\alpha,x)) = f(\beta,f(\alpha,x') \Rightarrow \sigma^*((f(\alpha,x),\beta)) = \sigma^*((f(\alpha,x'),\beta')).\end{aligned}$$

The mapping ϕ_α is a homomorphism, since

$$\begin{aligned}
\phi_\alpha[\sigma^*((x,\beta))\odot\sigma^*((y,\gamma))] &= \{\phi_\alpha\big(\sigma^*((z,\theta))\big)\mid z\in f(\gamma,x)+f(\beta,y),\theta\in\beta\gamma\}\\
&= \{\sigma^*((f(\alpha,z),\theta))\mid z\in f(\gamma,x)+f(\beta,y),\theta\in\beta\gamma\}\\
&= \{\sigma^*((t,\theta))\mid t\in f(\alpha,f(\gamma,x)+f(\beta,y)),\theta\in\beta\gamma\}\\
&= \{\sigma^*((t,\theta))\mid t\in f(\alpha,f(\gamma,x))+f(\alpha,f(\beta,y)),\theta\in\beta\gamma\}\\
&= \{\sigma^*((t,\theta))\mid t\in f(\gamma,f(\alpha,x))+f(\beta,f(\alpha,y)),\theta\in\beta\gamma\}\\
&= \{\sigma^*((t,\theta))\mid t\in f(\gamma,m)+f(\beta,n),m\in f(\alpha,x),n\in f(\alpha,y),\theta\in\beta\gamma\}\\
&= \{\sigma^*((f(\gamma,m)+f(\beta,n),\beta\gamma))\mid m\in f(\alpha,x),n\in f(\alpha,y)\}\\
&= \{\sigma^*((m,\beta))\odot\sigma^*((n,\gamma))\mid m\in f(\alpha,x),n\in f(\alpha,y)\}\\
&= \sigma^*((f(\alpha,x),\beta))\widetilde{\odot}\sigma^*((f(\alpha,y),\gamma))\\
&= \phi_\alpha\big(\sigma^*((x,\beta))\big)\widetilde{\odot}\phi_\alpha\big(\sigma^*((y,\gamma))\big).
\end{aligned}$$

The mapping ϕ_α is isotone and reverse isotone, since

$$\begin{aligned}
\sigma^*((x,\beta))\preceq_\sigma\sigma^*((x',\beta')) &\Leftrightarrow f(\beta',x)\widetilde{\leq_H}f(\beta,x')\\
&\Leftrightarrow f(\alpha,f(\beta',x))\widetilde{\leq_H}f(\alpha,f(\beta,x'))\\
&\Leftrightarrow f(\beta',f(\alpha,x))\widetilde{\leq_H}f(\beta,f(\alpha,x'))\\
&\Leftrightarrow \sigma^*((f(\alpha,x),\beta))\widetilde{\preceq_\sigma}\sigma^*((f(\alpha,x'),\beta'))\\
&\Leftrightarrow \phi_\alpha\big(\sigma^*((x,\beta))\big)\widetilde{\preceq_\sigma}\phi_\alpha\big(\sigma^*((x'\beta'))\big).
\end{aligned}$$

Clearly, ϕ_α is onto. The mapping ϕ_α is a generalized permutation, since

$$\begin{aligned}
\bigcup_{\sigma^*((x,\beta))\in(H\times S)/\sigma^*}\phi_\alpha\sigma^*((x,\beta)) &= \bigcup_{\sigma^*((x,\beta))\in(H\times S)/\sigma^*}\sigma^*((f(\alpha,x),\beta))\\
&= \bigcup_{a\in f(x,\alpha),x\in H}\sigma^*((a,\beta)).
\end{aligned}$$

On the other hand, $\sigma^*((f(x,\alpha),\beta))\subseteq(H\times S)/\sigma^*$, so $\sigma^*((a,\beta))\in(H\times S)/\sigma^*$. Therefore, we get

$$\bigcup_{\sigma^*((a,\beta))\in(H\times S)/\sigma^*}\sigma^*((a,\beta))=(H\times S)/\sigma^*.$$

This completes the proof. □

In the following, for (H,S,f), we denote $\bar{S}$ the set of all generalized permutation on $(H\times S)/\sigma^*$. We can define operation "$\bullet$" the usual composition on $\bar{S}$, i.e., if $\phi_\alpha,\phi_\beta\in\bar{S}$, then

$$\phi_\alpha\bullet\phi_\beta\big(\sigma^*((x,\alpha))\big)=\bigcup_{y\in\phi_\beta\left(\sigma^*((x,\alpha))\right)}\phi_\alpha(y),$$

for all $\sigma^*((x,\alpha))\in(H\times S)/\sigma^*$.

Proposition 3.10. *Let $\phi_\alpha,\phi_\beta\in\bar{S}$. We define $*:\bar{S}\times\bar{S}\to\mathcal{P}^*(\bar{S})$ by*

$$\phi_\alpha*\phi_\beta=\Big\{\phi_\gamma\mid\phi_\gamma\subseteq\phi_\alpha\bullet\phi_\beta,\bigcup_{\sigma^*((x,\alpha))\in(H\times S)/\sigma^*}\phi_\gamma\big(\sigma^*((x,\alpha))\big)=(H\times S)/\sigma^*\Big\}.$$

Then, $(\bar{S},*)$ is a commutative semihypergroup.

Proof. The hyperoperation "$*$" is associative, because the operation "$\bullet$" is associative. More precisely for each $\phi_\alpha, \phi_\beta, \phi_\gamma \in \bar{S}$ we have

$$\phi_\alpha * (\phi_\beta * \phi_\gamma) = \{\phi_\lambda \in \bar{S} \mid \phi_\lambda \subseteq \phi_\alpha \bullet \phi_\beta \bullet \phi_\gamma\} = (\phi_\alpha * \phi_\beta) * \phi_\gamma.$$

We show $\phi_\alpha \bullet \phi_\beta = \phi_{\alpha\beta}$. In fact, if $(x, \gamma) \in H \times S$, then

$$\begin{aligned}(\phi_\alpha \bullet \phi_\beta)\big(\sigma^*((x,\gamma))\big) &= \bigcup_{r \in \phi_\beta\left(\sigma^*((x,\gamma))\right)} \phi_\alpha(r) \\ &= \bigcup_{\substack{r\in\sigma^*((t,\gamma)) \\ t\in f(\beta,x)}} \phi_\alpha(r) \\ &= \bigcup_{t\in f(\beta,x)} \phi_\alpha\big(\sigma^*((t,\gamma))\big) \\ &= \bigcup_{t\in f(\beta,x)} \{\sigma^*((z,\gamma)) \mid z \in f(\alpha,t)\} \\ &= \{\sigma^*((z,\gamma)) \mid z \in f(\alpha\beta, x)\} \\ &= \{\sigma^*((z,\gamma)) \mid z \in f(k,x), k \in \alpha\beta\} \\ &= \{\sigma^*((f(k,x),\gamma)) \mid k \in \alpha\beta\} \\ &= \{\phi_k\big(\sigma^*((x,\gamma))\big) \mid k \in \alpha\beta\} \\ &= \phi_{\alpha\beta}\big(\sigma^*((x,\gamma))\big).\end{aligned}$$

The semihypergroup $\bar{S}$ is commutative, since

$$\begin{aligned}\phi_\alpha * \phi_\beta &= \{\phi_\gamma \mid \phi_\gamma \subseteq \phi_\alpha \bullet \phi_\beta\} = \{\phi_\gamma \mid \phi_\gamma \subseteq \phi_{\alpha\beta}\} \\ &= \{\phi_\gamma \mid \phi_\gamma \subseteq \phi_{\beta\alpha}\} = \{\phi_\gamma \mid \phi_\gamma \subseteq \phi_\beta \bullet \phi_\alpha\} \\ &= \phi_\beta * \phi_\alpha.\end{aligned}$$

□

Proposition 3.11. *Consider (H, S, f). We define*

$$\boxplus : (\bar{S}, *) \times ((H \times S)/\sigma^*, \odot, \preceq_\sigma) \to (\mathcal{P}^*((H \times S)/\sigma^*), \widetilde{\odot}, \widetilde{\preceq_\sigma})$$

by $\phi_\alpha \boxplus \sigma^*((x,\beta)) = \phi_\alpha\big(\sigma^*((x,\beta))\big)$. Then, for every $\alpha, \beta, \gamma \in S$ and for every $x, y \in H$. We have

(1) $\phi_\alpha \boxplus [\sigma^*((x,\beta)) \odot \sigma^*((y,\gamma))] = [\phi_\alpha \boxplus \sigma^*((x,\beta))] \ \widetilde{\odot}\ [\phi_\alpha \boxplus \sigma^*((y,\gamma))]$, *where for any $\phi_\alpha \in \bar{S}$ and $\sigma^*((A,B)) \subseteq (H \times S)/\sigma^*$,*
$$\phi_\alpha \boxplus \sigma^*((A,B)) = \bigcup_{\sigma^*((x,\alpha))\in\sigma^*((A,B))} \phi_\alpha \boxplus \sigma^*((x,\alpha)).$$
(2) $(\phi_\alpha * \phi_\beta) \boxplus \sigma^*((x,\gamma)) = \phi_\alpha \boxplus (\phi_\beta \boxplus \sigma^*((x,\gamma)))$, *where for any $K \subseteq \bar{S}$ and $\sigma^*((x,\beta)) \in (H \times S)/\sigma^*$,*
$$K \diamondsuit \sigma^*((x,\beta)) = \bigcup_{\phi_\alpha \in K} \phi_\alpha \boxplus \sigma^*((x,\beta)).$$
(3) $\phi_\alpha \boxplus \sigma^*((x,\beta)) \widetilde{\preceq}_\sigma \phi_\alpha \sigma^*((y,\gamma))$ *if and only if* $\sigma^*((x,\beta)) \preceq_\sigma \sigma^*((y,\gamma))$.

Proof. The proof follows from By Proposition 3.9. □

Theorem 3.12. *If (H, S, f), then $((H \times S)/\sigma^*, \bar{S}, \boxplus)$.*

Proof. By Proposition 3.11, the proof is straightforward. □

Definition 3.3. Consider (H, S, f). If $C \in \mathcal{P}^*(S)$ and $A, B \in \mathcal{P}^*(H)$, then $f(C, A) = f(C, B) \Leftrightarrow A = B$. In particular, if $A := \{x\}, B := \{y\}$, then $f(C, x) = f(C, y) \Leftrightarrow x = y$. If $A := \{x\}, B := \{y\}$ and $C := \{\alpha\}$, then $f(\alpha, x) = f(\alpha, y) \Leftrightarrow x = y$.

Theorem 3.13. *Let (H,S,f). We define $\psi_\alpha : (H,+,\leq_H) \to (\mathcal{P}^*(H\times S/\sigma^*), \widetilde{\odot}, \widetilde{\preceq}_\sigma)$ by*

$$\psi_\alpha(x) = \sigma^*((f(\alpha,x),\alpha)) := \{\sigma^*((t,\alpha)) \mid t \in f(\alpha,x)\}.$$

Then, ψ_α is a reverse isotone homomorphism.

Proof. We have

$$\begin{aligned} x = y \quad &\Rightarrow f(\beta,x) = f(\beta,y) \\ &\Rightarrow f(\alpha,f(\beta,x)) = f(\alpha,f(\beta,y)) \\ &\Rightarrow f(\beta,f(\alpha,x)) = f(\alpha,f(\beta,y)) \\ &\Rightarrow \sigma^*((f(\alpha,x),\alpha)) = \sigma^*((f(\beta,y),\beta)) \\ &\Rightarrow \psi_\alpha(x) = \psi_\alpha(y). \end{aligned}$$

So, ψ_α is well defined. Since

$$\begin{aligned} \psi_\alpha(x+y) \quad &= \{\psi_\alpha(k) \mid k \in x+y\} \\ &= \{\sigma^*((f(\alpha,k),\alpha)) \mid k \in x+y\} \\ &= \{\sigma^*((f(\alpha^2,k),\alpha^2)) \mid k \in x+y\} \\ &= \{\sigma^*((z,\alpha^2)) \mid z \in f(\alpha^2,x+y)\} \\ &= \{\sigma^*((z,\alpha^2)) \mid z \in f(\alpha^2,x)+f(\alpha^2,y)\} \\ &= \{\sigma^*((z,\alpha^2)) \mid z \in f(\alpha,f(\alpha,x))+f(\alpha,f(\alpha,y))\} \\ &= \sigma^*((f(\alpha,x),\alpha))\widetilde{\odot}\sigma^*((f(\alpha,y),\alpha)) \\ &= \psi_\alpha(x)\widetilde{\odot}\psi_\alpha(y), \end{aligned}$$

it follows that ψ_α is a homomorphism.

Now, let $x \leq_H y$. Since $\alpha^2 \in \mathcal{P}^*(S)$, by Definition 3.2, we have $f(\alpha^2,x)\widetilde{\leq_H} f(\alpha^2,y)$, that is $f(\alpha\alpha,x)\widetilde{\leq_H} f(\alpha\alpha,y)$. Then, by Definition 3.1 (2), $f(\alpha,f(\alpha,x))\widetilde{\leq_H} f(\alpha,f(\alpha,y))$, from which $\sigma^*((f(\alpha,x),\alpha))\widetilde{\preceq}_\sigma\sigma^*((f(\alpha,y),\alpha))$, i.e., $\psi_\alpha(x)\widetilde{\preceq}_\sigma\psi_\alpha(y)$. Thus, the mapping ψ_α is isotone.

Finally, if $x,y \in H$ such that $\psi_\alpha(x)\widetilde{\preceq}_\sigma\psi_\alpha(y)$, i.e., $\sigma^*((f(\alpha,x),\alpha))\widetilde{\preceq}_\sigma\sigma^*((f(\alpha,y),\alpha))$, then $f(\alpha,f(\alpha,x))\widetilde{\leq_H} f(\alpha,f(\alpha,y))$ then, by Definition 3.1 (2) $f(\alpha\alpha,x)\widetilde{\leq_H} f(\alpha\alpha,y)$, and by Definition 3.2, $x \leq_H y$. Hence, the mapping ψ_α is reverse isotone. □

We remark that if (H,S,f) and $\alpha,\beta \in S$, then $\psi_\alpha = \psi_\beta$. If $x \in H$ then, by Proposition 3.7 (2), we obtain $\psi_\alpha(x) := \sigma^*((f(\alpha,x),\alpha)) = \sigma^*((f(\beta,x),x)) := \psi_\beta(x)$.

Proposition 3.14. *Consider (H,S,f). Then, $Q : (S,.) \to (\bar{S},*)$ with $\alpha \mapsto \phi_\alpha$ is an onto homomorphism.*

Proof. If $\alpha,\beta \in S$ then, by Proposition 3.10, we have

$$\begin{aligned} Q(\alpha\beta) \quad &= \{Q(\lambda) \mid \lambda \in \alpha\beta\} = \{\phi_\lambda \mid \phi_\lambda \subseteq \phi_{\alpha\beta}\} \\ &= \{\phi_\lambda \mid \phi_\lambda \subseteq \phi_\alpha \bullet \phi_\beta\} = Q(\alpha) * Q(\beta), \end{aligned}$$

because $(\lambda \in \alpha\beta \Rightarrow Q(\lambda) \subseteq Q(\alpha\beta) = \phi_{\alpha\beta} = \phi_\alpha \bullet \phi_\beta)$. So, Q is a homomorphism. Clearly, Q is onto. □

Proposition 3.15. *If (H,S,f), then*

$$\forall\beta \in S, \forall x \in H;\ \psi_\alpha(f(\beta,x)) = Q(\beta) \boxplus \psi_\alpha(x).$$

Proof. We have

$$\begin{aligned}
\psi_\alpha(f(\beta,x)) &= \{\psi_\alpha(y) \mid y \in f(\beta,x)\} \\
&= \{\sigma^*((p,\alpha)) \mid y \in f(\beta,x), p \in f(\alpha,y)\} \\
&= \{\sigma^*((p,\alpha)) \mid p \in f(\alpha, f(\beta,x))\} \\
&= \{\sigma^*((p,\alpha)) \mid p \in f(\beta, f(\alpha,x))\} \\
&= \{\sigma^*((p,\alpha)) \mid p \in f(\beta,z), z \in f(\alpha,x)\} \\
&= \{\sigma^*((f(\beta,z),\alpha)) \mid z \in f(\alpha,x)\} \\
&= \{\phi_\beta\big(\sigma^*((z,\alpha))\big) \mid z \in f(\alpha,x)\} \\
&= \{\phi_\beta \boxplus \sigma^*((z,\alpha)) \mid z \in f(\alpha,x)\} \\
&= \phi_\beta \boxplus \sigma^*((f(\alpha,x),\alpha)) \\
&= \phi_\beta \boxplus \psi_\alpha(x) \\
&= Q(\beta) \boxplus \psi_\alpha(x).
\end{aligned}$$

□

Proposition 3.16. *Consider (H,S,f) and $\alpha \in S$. Then, for each $\beta \in S$ and each $x \in H$ there exists $(y,\gamma) \in H \times S$ such that $Q(\beta) \boxplus \sigma^*((y,\gamma)) = \psi_\alpha(x)$.*

Proof. Suppose that $\beta \in S$, $x \in H$. Since $\psi_\alpha(x) \in \mathcal{P}^*((H \times S)/\sigma^*)$ and F_β is a mapping of $(H \times S)/\sigma^*$ onto $\mathcal{P}^*((H \times S)/\sigma^*)$, there exists $(y,\gamma) \in H \times S$ so that $\phi_\beta\big(\sigma^*((y,\gamma))\big) = \psi_\alpha(x)$. On the other hand, since $\psi_\beta\big(\sigma^*((y,\gamma))\big) := \phi_\beta \boxplus \sigma^*((y,\gamma))$ and $\phi_\beta := Q(\beta)$, we have $Q(\beta) \boxplus \sigma^*((y,\gamma)) = \psi_\alpha(x)$. □

Definition 3.4. An ordered hypergroupoid $(H,\cdot,\leq_H)$ is said to be s-cancellative if for every $A,B,C \in \mathcal{P}^*(H)$, we have
(1) $AB\widetilde{\leq_H}AC$ implies $B\widetilde{\leq_H}C$;
(2) $BA\widetilde{\leq_H}CA$ implies $B\widetilde{\leq_H}C$.

Proposition 3.17. *Consider (H,S,f). Then, we have the following statements.*
(1) *If $(H,+)$ is a semihypergroup, then $((H \times S)/\sigma^*, \odot)$ is a semihypergroup.*
(2) *If $(H,+)$ is a commutative hypergroupoid, then $((H\times S)/\sigma^*, \odot)$ is a commutative hypergroupoid.*
(3) *If $(H,+,\leq_H)$ is an s-cancellative ordered hypergroupoid, then $((H\times S)/\sigma^*, \odot, \preceq_\sigma$ $)$ is an s-cancellative ordered hypergroupoid as well.*

Proof. (1) Let $(x,\alpha),(y,\beta),(z,\gamma) \in H \times S$. Then, we show that

$$[\sigma^*((x,\alpha)) \odot \sigma^*((y,\beta))] \odot \sigma^*((z,\gamma)) = \sigma^*((x,\alpha)) \odot [\sigma^*((y,\beta)) \odot \sigma^*((z,\gamma))].$$

We have

$$\begin{aligned}
&[\sigma^*((x,\alpha)) \odot \sigma^*((y,\beta))] \odot \sigma^*((z,\gamma)) \\
&= \{\sigma^*((t,\lambda)) \mid t \in f(\beta,x) + f(\alpha,y), \lambda \in \alpha\beta\} \odot \sigma^*((z,\gamma)) \\
&= \{\sigma^*((p,\theta)) \mid p \in f(\gamma,t) + f(\lambda,z), \theta \in \lambda\gamma, \lambda \in \alpha\beta, t \in f(\beta,x) + f(\alpha,y)\} \\
&= \{\sigma^*((p,\theta)) \mid p \in f(\gamma, f(\beta,x) + f(\alpha,y)) + f(\alpha\beta,z)\} \\
&= \{\sigma^*((p,\theta)) \mid p \in [f(\gamma,f(\beta,x)) + f(\gamma,f(\alpha,y))] + f(\alpha,f(\beta,z)), \theta \in (\alpha\beta)\gamma\}.
\end{aligned}$$

Since $(H,+)$ is a semihypergroup and $f(\alpha,f(\beta,z)), f(\gamma,f(\alpha,y), f(\gamma,f(\beta,x)) \subseteq H$, we obtain

$$\begin{aligned}
&\{\sigma^*((p,\theta)) \mid p \in [f(\gamma, f(\beta,x)) + f(\gamma, f(\alpha,y))] + f(\alpha, f(\beta,z)), \theta \in (\alpha\beta)\gamma\} \\
&= \{\sigma^*((p,\theta)) \mid p \in f(\gamma, f(\beta,x)) + [f(\gamma, f(\alpha,y)) + f(\alpha, f(\beta,z))], \theta \in \alpha(\beta\gamma)\} \\
&= \{\sigma^*((p,\theta)) \mid p \in f(\gamma\beta, x) + [f(\alpha, (f(\gamma,y) + f(\beta,z))], \theta \in \alpha(\beta\gamma)\} \\
&= \{\sigma^*((p,\theta)) \mid p \in f(k,x) + f(\alpha,q), k \in \gamma\beta, q \in f(\gamma,y) + f(\beta,z), \theta \in \alpha k\} \\
&= \{\sigma^*((x,\alpha)) \odot \sigma^*((q,k)) \mid q \in f(\gamma,y) + f(\beta,z), k \in \gamma\beta, \theta \in \alpha k\} \\
&= \sigma^*((x,\alpha)) \odot [\sigma^*((y,\beta)) \odot \sigma^*((z,\gamma))].
\end{aligned}$$

(2) Let $(x,\alpha),(y,\beta) \in H \times S$. We show that

$$\sigma^*((x,\alpha)) \odot \sigma^*((y,\beta)) = \sigma^*((y,\beta)) \odot \sigma^*((x,\alpha)).$$

Since $(H,+)$ is a commutative hypergroupoid and $f(\alpha,y), f(\beta,x) \subseteq H$, we obtain

$$\begin{aligned}
\sigma^*((x,\alpha)) \odot \sigma^*((y,\beta)) &= \{\sigma^*((z,\gamma)) \mid z \in f(\beta,x) + f(\alpha,y), \gamma \in \alpha\beta\} \\
&= \{\sigma^*((z,\gamma)) \mid z \in f(\alpha,y) + f(\beta,x), \gamma \in \beta\alpha\} \\
&= \sigma^*((y,\beta)) \odot \sigma^*((x,\alpha)).
\end{aligned}$$

(3) Suppose that $(x,\alpha),(y,\beta),(z,\gamma) \in H \times S$ such that

$$\sigma^*((x,\alpha)) \odot \sigma^*((y,\beta)) \widetilde{\preceq_\sigma} \sigma^*((x,\alpha)) \odot \sigma^*((z,\gamma)) \Rightarrow \sigma^*((y,\beta)) \preceq_\sigma \sigma^*((z,\gamma)).$$

Since

$$\begin{aligned}
\sigma^*((x,\alpha)) \odot \sigma^*((y,\beta)) &:= \sigma^*((f(\beta,x) + f(\alpha,y), \alpha\beta)), \\
\sigma^*((x,\alpha)) \odot \sigma^*((z,\gamma)) &:= \sigma^*((f(\gamma,x) + f(\alpha,z), \alpha\gamma)),
\end{aligned}$$

we have

$$\begin{aligned}
&\sigma^*((f(\beta,x) + f(\alpha,y), \alpha\beta)) \widetilde{\preceq_\sigma} \sigma^*((f(\gamma,x) + f(\alpha,z), \alpha\gamma)) \\
&\Rightarrow f((\alpha\gamma), [f(\beta,x) + f(\alpha,y)]) \ \widetilde{\leq_H}\ f((\alpha\beta), [f(\gamma,x) + f(\alpha,z)]) \\
&\Rightarrow [f((\alpha\gamma), f(\beta,x))] + [f((\alpha\gamma), f(\alpha,y))] \ \widetilde{\leq_H}\ [f(\alpha\beta, f(\gamma,x))] + [f((\alpha\beta), f((\alpha,z)] \\
&\Rightarrow [f((\alpha\gamma\beta),x)] + [f((\alpha\gamma\alpha),y)] \ \widetilde{\leq_H}\ [f((\alpha\beta\gamma),x)] + [f((\alpha\beta\alpha),z)] \\
&\Rightarrow [f((\alpha\beta\gamma),x)] + [f((\alpha^2\gamma),y)] \ \widetilde{\leq_H}\ [f((\alpha\beta\gamma),x)] + [f((\alpha^2\beta),z)].
\end{aligned}$$

Since $(H,+,\leq_H)$ is an s-cancellative ordered hypergroupoid on the subsets and $f((\alpha\beta\gamma),x)$, $f((\alpha^2\gamma),y)$, $f((\alpha^2\beta),z) \subseteq H$, we obtain $f((\alpha^2\gamma),y) \ \widetilde{\leq_H}\ f((\alpha^2\beta),z)$. Therefore, we have

$$\begin{aligned}
f(\alpha^2, f(\gamma,y)) \ \widetilde{\leq_H}\ f(\alpha^2, f(\beta,z)) &\Rightarrow f(\gamma,y) \ \widetilde{\leq_H}\ f(\beta,z) \\
&\Rightarrow \sigma^*((y,\beta)) \preceq_\sigma \sigma^*((z,\gamma)).
\end{aligned}$$

This completes the proof. □

Theorem 3.18. *Let (H,S,f) and $(\bar{\bar{H}}, \bar{\bar{S}}, \boxdot)$ having the following properties.*

(1) *There exists $\psi' : (H,+,\leq_H) \to (\bar{\bar{H}}, \bar{\bar{+}}, \leq_{\bar{\bar{H}}})$ reverse isotone homomorphism.*

(2) *There exists $Q' : (S,.) \to (\bar{\bar{S}}, \bar{\bar{.}})$ homomorphism such that*

(i) *for all $\alpha \in S$, for all $x \in H$, $\psi'(f(\alpha,x)) = Q'(\alpha) \boxdot \psi'(x)$,*

(ii) *for all $\alpha \in S$, for all $x \in H$, exists $q^x_\alpha \in \bar{\bar{H}}$, $Q'(\alpha) \boxdot q^x_\alpha = \psi'(x)$.*

Then, $\chi : ((H \times S)/\sigma^*, \odot, \preceq_\sigma) \to (\bar{\bar{H}}, \bar{\bar{+}}, \leq_{\bar{\bar{H}}})$ by $\chi\big(\sigma^*((x,\alpha))\big) = q^x_\alpha$ is a reverse isotone homomorphism.

Proof. The mapping χ is well defined. Indeed, we have

$$\begin{aligned}\sigma^*((x,\alpha)) =&\sigma^*((y,\beta)) \Rightarrow f(\beta,x) = f(\alpha,y)\\ &\Rightarrow \psi'(f(\beta,x)) = \psi'(f(\alpha,y))\\ &\Rightarrow Q'(\beta) \boxdot \psi'(x) = Q'(\alpha) \boxdot \psi'(y)\\ &\Rightarrow Q'(\beta) \boxdot (Q'(\alpha) \boxdot q^x_\alpha) = Q'(\alpha) \boxdot (Q'(\beta) \boxdot q^y_\beta)\\ &\Rightarrow Q'(\beta) \boxdot (Q'(\alpha) \boxdot q^x_\alpha) = Q'(\beta) \boxdot (Q'(\alpha) \boxdot q^y_\beta)\\ &\Rightarrow (Q'(\beta) \overset{=}{\cdot} Q'(\alpha)) \boxdot q^x_\alpha = (Q'(\alpha) \overset{=}{\cdot} (Q'(\beta)) \boxdot q^y_\beta\\ &\Rightarrow Q'(\beta\alpha) \boxdot q^x_\alpha = Q'(\beta\alpha) \boxdot q^y_\beta.\end{aligned}$$

Because $q^x_\alpha \in \bar{\bar{H}}$ and $Q'(\alpha), Q'(\beta) \subseteq \bar{\bar{S}}$, we have $Q'(\alpha)\overset{=}{\cdot}Q'(\beta) = Q'(\alpha\beta) \subseteq \bar{\bar{S}}$. As well as $(\bar{\bar{H}}, \bar{\bar{S}}, \boxdot)$, thus by Definition 3.3, we have $q^x_\alpha = q^y_\beta$.

Suppose that $\chi\big(\sigma^*((x,\alpha))\big) = q^x_\alpha, \chi\big(\sigma^*((y,\beta))\big) = q^y_\beta$ and $\chi\big(\sigma^*((x,\alpha))\odot\sigma^*((y,\beta))\big) = q^{(f(\beta,x)+f(\alpha,y))}_{\alpha\beta}$. Then, we show

$$q^x_\alpha \overset{=}{+} q^y_\beta = q^{f(\beta,x)+f(\alpha,y)}_{\alpha\beta}.$$

Indeed, we have

$$\begin{aligned}&(Q'(\alpha) \overset{=}{\cdot} Q'(\beta)) \boxdot (q^x_\alpha \overset{=}{+} q^y_\beta)\\ &\quad= (Q'(\alpha) \overset{=}{\cdot} Q'(\beta)) \boxdot q^x_\alpha \overset{=}{+} (Q'(\alpha) \overset{=}{\cdot} Q'(\beta)) \boxdot q^y_\beta\\ &\quad= (Q'(\beta) \overset{=}{\cdot} Q'(\alpha)) \boxdot q^x_\alpha \overset{=}{+} (Q'(\alpha) \overset{=}{\cdot} Q'(\beta)) \boxdot q^y_\beta\\ &\quad= Q'(\beta) \boxdot (Q'(\alpha) \boxdot q^x_\alpha) \overset{=}{+} Q'(\alpha) \boxdot (Q'(\beta) \boxdot q^y_\beta)\\ &\quad= Q'(\beta) \boxdot \psi'(x) \overset{=}{+} Q'(\alpha) \boxdot \psi'(y)\\ &\quad= \psi'(f(\beta,x)) \overset{=}{+} \psi'(f(\alpha,y))\\ &\quad= \psi'(f(\beta,x) + f(\alpha,y)).\end{aligned}$$

On the other hand,

$$\begin{aligned}(Q'(\alpha) \overset{=}{\cdot} Q'(\beta)) \boxdot q^{(f(\beta,x)+f(\alpha,y))}_{\alpha\beta} &= Q'(\alpha\beta) \boxdot q^{(f(\beta,x)+f(\alpha,y))}_{\alpha\beta}\\ &= \psi'(f(\beta,x) + f(\alpha,y)).\end{aligned}$$

So, $(Q'(\alpha)\overset{=}{\cdot}Q'(\beta)) \boxdot (q^x_\alpha \overset{=}{+} q^y_\beta) = (Q'(\alpha) \overset{=}{\cdot} Q'(\beta)) \boxdot q^{(f(\beta,x)+f(\alpha,y))}_{\alpha\beta}$. Since $(\bar{\bar{H}}, \bar{\bar{S}}, \boxdot)$, by Definition 3.3, we have

$$q^{(f(\beta,x)+f(\alpha,y))}_{\alpha\beta} = (q^x_\alpha \overset{=}{+} h^y_\beta).$$

Therefore, the mapping χ is a homomorphism.

Suppose that $(x,\alpha), (y,\beta) \in H \times S$ so that $\sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((y,\beta))$. Then, $q^x_\alpha \leq_{\bar{\bar{H}}} q^y_\beta$. By condition (B), we have $Q'(\alpha) \boxdot q^x_\alpha = \psi'(x)$ and $Q'(\beta) \boxdot q^y_\beta = \psi'(y)$. Since $\sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((y,\beta))$, it follows that we have $f(\beta,x) \widetilde{\leq_H} f(\alpha,y)$. On the other hand,

$$\begin{aligned} Q'(\alpha\beta) \boxdot q_\alpha^x &= (Q'(\alpha) \bar{\bar{\cdot}} Q'(\beta)) \boxdot q_\alpha^x \\ &= Q'(\alpha) \boxdot (Q'(\beta) \boxdot q_\alpha^x) \\ &= Q'(\beta) \boxdot (Q'(\alpha) \boxdot q_\alpha^x) \\ &= Q'(\beta) \boxdot \psi'(x) \\ &= \psi'(f(\beta,x)) \widetilde{\leq_{\bar{\bar{H}}}} \psi'(f(\alpha,y)) \\ &= Q'(\alpha) \boxdot \psi'(y) = Q'(\alpha) \boxdot (Q'(\beta) \boxdot q_\beta^y) \\ &= (Q'(\alpha) \bar{\bar{\cdot}} Q'(\beta)) \boxdot q_\beta^y \\ &= Q'(\alpha\beta) \boxdot q_\beta^y. \end{aligned}$$

Since $Q'(\alpha\beta) \boxdot q_\alpha^x \widetilde{\leq_{\bar{\bar{H}}}} Q'(\alpha\beta) \boxdot q_\beta^y$, by Definition 3.2, we have $q_\alpha^x \leq_{\bar{\bar{H}}} q_\beta^y$. Thus, the mapping χ is isotone.

Now, suppose that $(x,\alpha),(y,\beta) \in H \times S$. Then, $q_\alpha^x \leq_{\bar{\bar{H}}} q_\beta^y$. So, $\sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((y,\beta))$. Consequently, we have

$$\begin{aligned} q_\alpha^x \leq_{\bar{\bar{H}}} q_\beta^y &\Rightarrow Q'(\alpha\beta) \boxdot q_\alpha^x \widetilde{\leq_{\bar{\bar{H}}}} Q'(\alpha\beta) \boxdot q_\beta^y \\ &\Rightarrow (Q'(\alpha) \bar{\bar{\cdot}} Q'(\beta)) \boxdot q_\alpha^x \widetilde{\leq_{\bar{\bar{H}}}} (Q'(\alpha) \bar{\bar{\cdot}} Q'(\beta)) \boxdot q_\beta^y \\ &\Rightarrow Q'(\alpha) \boxdot (Q'(\beta) \boxdot q_\alpha^x) \widetilde{\leq_{\bar{\bar{H}}}} Q'(\alpha) \boxdot (Q'(\beta) \boxdot q_\beta^y) \\ &\Rightarrow Q'(\beta) \boxdot (Q'(\alpha) \boxdot q_\alpha^x) \widetilde{\leq_{\bar{\bar{H}}}} Q'(\alpha) \boxdot (Q'(\beta) \boxdot q_\beta^y) \\ &\Rightarrow Q'(\beta) \boxdot \psi'(x) \widetilde{\leq_{\bar{\bar{H}}}} Q'(\alpha) \boxdot \psi'(y) \\ &\Rightarrow \psi'(f(\beta,x)) \widetilde{\leq_{\bar{\bar{H}}}} \psi'(f(\alpha,y)) \\ &\Rightarrow f(\beta,x) \widetilde{\leq_{\bar{\bar{H}}}} f(\alpha,y) \\ &\Rightarrow \sigma^*((x,\alpha)) \preceq_\sigma \sigma^*((y,\beta)). \end{aligned}$$

This completes the proof. □

Now we complete the proof of Theorem 3.5. We put $\bar{H} = (H \times S)/\sigma^*$. By Theorem 3.12, we have $(\bar{H}, \bar{S}, \boxplus)$. By Corollary 3.13, H is embedded in $\mathcal{P}^*(\bar{H})$ under the mapping ψ_α (where α is an arbitrary element of S). By Proposition 3.14, the mapping $Q: S \to \bar{S}$ is a homomorphism, by Propositions 3.15 and 3.16 conditions (i) and (ii) of the first part of Theorem are satisfied. By Proposition 3.17, if H is a semihypergroup, then so is $\bar{H}$; if H is a commutative hypergroupoid, then so is $\bar{H}$, if H is an s-cancellative ordered hypergroupoid, then $\bar{H}$ is also so. As far as the converse statement is concerned, under the hypotheses of Theorem 3.18, $\mathcal{P}^*(\bar{H})$ is embedded in $\mathcal{P}^*(\bar{\bar{H}})$. This completes the proof of theorem.

In continue, assuming that the commutative semihypergroup S considered in Definition 3.1 is an ordered semihypergroup under the order " $\leq_S$ ", we add a new condition in Definition 3.1, and we consider actions (H,S,f) for which the following condition also holds:

(4) For all $x \in H, \alpha \leq_S \beta \Rightarrow f(\alpha,x) \widetilde{\leq_H} f(\beta,x)$.

Such an action is called a complete action and it is denoted by $\widetilde{(H,S,f)}$. We prove that if $\widetilde{(H,S,f)}$, then the semihypergroup $(\bar{\bar{S}},*)$ is an ordered semihypergroup.

Proposition 3.19. *Let* $\widetilde{(H,S,f)}$. *Then,*

$$\alpha \leq_S \beta, x \leq_H y \Rightarrow f(\alpha,x) \widetilde{\leq_H} f(\beta,y).$$

Proof. Since $\alpha \leq_S \beta$ and $x \in H$, by Definition 3.1 (4), we have $f(\alpha,x) \leq_H f(\beta,x)$. Since $x \leq_H y$ and $\beta \in S$, by Definition 3.1 (3), we have $f(\beta,x)\widetilde{\leq_H} f(\beta,y)$ Thus, we get $f(\alpha,x)\widetilde{\leq_H} f(\beta,y)$. □

Lemma 3.20. *Let $(A,\leq)$ be an ordered set and F a set of isotone mappings of A into $\mathcal{P}^*(A)$, closed under the composition "$\bullet$" of mappings. Let be the order on F defined by $f \preceq g$ if and only if for all $x \in A, f(x) \leq g(x)$. Then, $(F,\bullet,\preceq)$ is an ordered semihypergroup.*

Proposition 3.21. *Let $(\widetilde{H,S,f})$. Then, the semihypergroup $(\bar{S},*)$ endowed with the relation*

$$\phi_\alpha \preceq \phi_\beta \Leftrightarrow \text{for all } \sigma^*((x,\gamma)) \in (H\times S)/\sigma^*,\ \phi_\alpha\big(\sigma^*((x,\gamma))\big)\widetilde{\preccurlyeq_\sigma}\phi_\beta\big(\sigma^*((x,\gamma))\big)$$

is an ordered semihypergroup.

Proof. By Proposition 3.8, $((H\times S)/\sigma^*, \preccurlyeq_\sigma)$ is an ordered set, by Proposition 3.9, the set $\bar{S}$ is a nonempty family of isotone mappings of $(H\times S)/\sigma^*$ into $\mathcal{P}^*((H\times S)/\sigma^*)$. Moreover, $\phi_\alpha \bullet \phi_\beta \subseteq \bar{S}$ for all $\phi_\alpha, \phi_\beta \in \bar{S}$. According to Lemma 3.20, $(\bar{S},*,\preceq)$ is an ordered semihypergroup. □

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(B. Davvaz) Department of Mathematical Sciences, Yazd University, Yazd, Iran
E-mail address: davvaz@yazd.ac.ir

(M. Mostafavi) Department of Mathematical Sciences, Yazd University, Yazd, Iran
E-mail address: mahbob.mostafavi@gmail.com