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On Lacunary $\mathcal{I}$-Convergence Almost Surely of Complex Uncertain Sequences

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Abstract. In this paper, as a part of uncertainty theory, we explore the concepts of lacunary $\mathcal{I}$ and $\mathcal{I}^*$-convergence almost surely in complex uncertain sequences and study some of their properties and identify the relationships between them. Also, we introduced the notions of lacunary $\mathcal{I}$ and lacunary $\mathcal{I}^*$-Cauchy sequence almost surely of complex uncertain sequences and analyze a few of their characteristics and try to determine how they relate to one another.

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1. Introduction

Since the most vital part of summability theory is the convergence of the related sequences and series, the new concepts relevant to the convergence were introduced such as absolutely, uniformly, and statistical convergence, etc. The concept of statistical convergence, which is an extension of the usual idea of convergence, was introduced by Fast [9] and Steinhaus [27], individually in the year 1951. But the research on this concept got flourish soon after the works of Šalát [24] and Fridy [11] came into literature. As an extension of statistical convergence, the idea of $\mathcal{I}$ and $\mathcal{I}^*$-convergence was introduced by Kostyrko et al. [16]. Then various generalizations, extensions, and applications of these idea have been studied in this field by different prominent authors so far (see, for detail, [12, 18–20, 23, 25, 26], etc.)

The initial work on lacunary sequence is found in Freedman et al. [10]. Further lacunary sequences have been investigated by many famous researchers. In the year 2012, Tripathy et al. [29] introduced the concepts of lacunary $\mathcal{I}$-convergence of a real sequence. After that, this idea have been widely studied and made significant progress in this field such as [2, 7, 8, 13]

We often encounter situations in our daily lives where there is scant or no evidence of the events, not only for the technical and economic problem but also for the unexpected events. Those inadequate data make it difficult to apply the probability distribution of events. Consequently, some domain experts are consulted to give a belief degree that each event will take place while making a decision. To address some aspects of these uncertain events, Liu [17] introduced initially the uncertainty theory in the year 2007. Also Liu defined different types of convergence of a sequence of real uncertain variables and identify the relationships among them. Then it has

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been extended to the complex uncertain variable by Peng [21]. Chen et al. [1] subsequently investigated the notion of convergence of the complex uncertain sequences using complex uncertain variables. The notion of statistical convergence of complex uncertain sequences in the field of uncertainty theory was introduced by Tripathy and Nath [30]. Thereafter, lots of interesting developments have occurred in this field like Das et al. [3], Debnath and Das [4, 5], Dowari and Tripathy [6], Kişi [14], Kişi and G$\ddot{u}$rdal [15], Roy et al. [22], Tripathy and Dowari [28], etc.

In the context of complex uncertain sequence, the idea of lacunary $\mathcal{I}$-convergence has not before been investigated. Motivated by this fact and inspired by the above works, in this paper, we introduce the notion of lacunary $\mathcal{I}$ and $\mathcal{I}^*$-convergence almost surely in complex uncertain sequences, examined several properties, and identify the relationships between them. Moreover, we define lacunary $\mathcal{I}$ and lacunary $\mathcal{I}^*$-Cauchy sequence almost surely of complex uncertain sequence. It can be observe that if $\mathcal{I}$ be an admissible ideal, then lacunary $\mathcal{I}^*$-convergence almost surely coincide with lacunary $\mathcal{I}$-convergence almost surely but the converse is not true.

2. Definitions and preliminaries

In this section, we provide some basic definitions and results on generalized convergence concepts and the theory of uncertainty which will be used throughout the article.

Definition 2.1. [16] Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if
(i) $\phi \in \mathcal{I}$;
(ii) for each $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$;
(iii) for each $A \in \mathcal{I}$ and $B \subset A \implies B \in \mathcal{I}$.
An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $X \notin \mathcal{I}$.
A non-trivial ideal $\mathcal{I}$ is called an admissible ideal in X if and only if $\{\{x\} : x \in X\} \subset \mathcal{I}$.

Example 2.1. (i) $\mathcal{I}_f :=$ The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.
(ii) $\mathcal{I}_d :=$ The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.

Definition 2.2. [16] Let X be a non-empty set. A family of subsets $\mathcal{F} \subset P(X)$ is called a filter on X if and only if
(i) $\phi \notin \mathcal{F}$;
(ii) for each $A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$;
(iii) for each $A \in \mathcal{F}$ and $B \supset A \implies B \in \mathcal{F}$.
Let $\mathcal{I}$ is an admissible ideal. Then the filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$ is called the filter associated with the ideal $\mathcal{I}$.

Definition 2.3. [11] A sequence $x = (x_n)$ is said to be statistically convergent to ℓ provided that for each $\varepsilon > 0$ such that $\lim\limits_{n\to\infty} \frac{1}{n}\big|\{k \le n : |x_k - \ell| \ge \varepsilon\}\big| = 0$, $n \in \mathbb{N}$.

Definition 2.4. [16] A sequence $x = (x_n)$ is said to be $\mathcal{I}$-convergent to ℓ, if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - \ell| \ge \varepsilon\} \in \mathcal{I}$. The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$-convergence. In this case, $\mathcal{I}=\mathcal{I}_d = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \frac{|A\cap\{1,2,\ldots,n\}|}{n} = 0\}$, where $|A|$ is the cardinality of the set A.

Definition 2.5. [16] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_n)$ is said to be $\mathcal{I}^*$-convergent to ℓ, if there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{n\to\infty} |x_{m_n} - \ell| = 0.$$

Definition 2.6. [16] A sequence $x = (x_n)$ is said to be $\mathcal{I}$-Cauchy, if for every $\varepsilon > 0$, there exists a 'N'$\in \mathbb{N}$ such that $\{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\} \in \mathcal{I}$.

Definition 2.7. [20] A sequence (x_n) is said to be $\mathcal{I}^*$-Cauchy, if there exists a set $A = \{m_1 < m_2 < \cdots < m_n < \cdots\} \subset \mathbb{N}$, $A \in \mathcal{F}(\mathcal{I})$ such that the subsequence (x_{m_n}) is a Cauchy sequence i.e., $\lim_{i,j\to\infty} \left|x_{m_i} - x_{m_j}\right| = 0$.

Definition 2.8. [16] An admissible ideal $\mathcal{I}$ of $\mathbb{N}$ is said to satisfy the condition AP, if for every countable family of mutually disjoint sets $\{C_n\}_{n\in\mathbb{N}}$ from $\mathcal{I}$, there exists a countable family of sets $\{B_n\}_{n\in\mathbb{N}}$ such that the symmetric difference $C_j \triangle B_j$ is finite for every $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Definition 2.9. [10] By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.
Throughout this paper the intervals determined by θ will be denoted by

$$I_r = (k_{r-1}, k_r]$$

and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Definition 2.10. [29] Let $\theta = (k_r)$ be a lacunary sequence. Then a sequence (x_k) is said to be lacunary $\mathcal{I}$-null if for every $\varepsilon > 0$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} |x_k| \geq \varepsilon\right\} \in \mathcal{I}.$$

We write $\mathcal{I}_\theta$-$\lim x_k = 0$.

Definition 2.11. [17] Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$,

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event. To obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu as

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Definition 2.12. [21] A variable $\zeta = \xi + i\eta$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers is a complex uncertain variable if and only if ξ and η are uncertain variables, where ξ and η are the real and imaginary parts of ζ, respectively.

Definition 2.13. [1] A complex uncertain sequence (ζ_k) is said to be convergent almost surely (a.s) to ζ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} || \zeta_k(\gamma) - \zeta(\gamma) ||= 0, \ for\ every\ \gamma \in \Lambda.$$

Symbolically we write $\zeta_k \xrightarrow{A_s} \zeta$.

Definition 2.14. [28] A complex uncertain sequence (ζ_k) is said to be statistically convergent almost surely to ζ if for every $\varepsilon > 0$ there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n : \|\zeta_k(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right\}\right| = 0, \ for\ every\ \gamma \in \Lambda.$$

Symbolically we write $\zeta_k \xrightarrow{S^{A_s}} \zeta$.

Definition 2.15. A complex uncertain sequence (ζ_k) is said to be $\mathcal{I}$-convergent almost surely to ζ if, for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\left\{n \in \mathbb{N} : \|\zeta_k(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right\} \in \mathcal{I}, \ for\ every\ \gamma \in \Lambda.$$

Symbolically we write $\zeta_k \xrightarrow{A_s(\mathcal{I})} \zeta$.

Throughout the paper, we consider $\mathcal{I}$ is a non-trivial admissible ideal of $\mathbb{N}$.

3. Main results

Definition 3.1. Let $\theta = (k_r)$ be a lacunary sequence. A complex uncertain sequence (ζ_k) is said to be lacunary $\mathcal{I}$-convergent almost surely to ζ if for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \ge \varepsilon\right\} \in \mathcal{I}, \ for\ every\ \gamma \in \Lambda.$$

Symbolically we write $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta$.

Example 3.1. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_k\in\Lambda} \frac{k}{(2k+1)}, & if\ \sup\limits_{\gamma_k\in\Lambda} \frac{k}{(2k+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_k\in\Lambda^c} \frac{k}{(2k+1)}, & if\ \sup\limits_{\gamma_k\in\Lambda^c} \frac{k}{(2k+1)} < \frac{1}{2} \\ \frac{1}{2}, \ otherwise \end{cases} \quad for\ k = 1, 2, 3, \cdots.$$

Let $\theta = (2^r)$, $r = 1, 2, 3, \cdots$ be a lacunary sequence and take $\mathcal{I}$=$\mathcal{I}_d$.
Also, the complex uncertain variables defined by

$$\zeta_k(\gamma) = \begin{cases} i\beta_k, \ if\ \gamma \in \{\gamma_1, \gamma_4, \gamma_9, \cdots\} \\ 0, \ \ otherwise \end{cases} \quad for\ k = 1, 2, 3, \cdots,$$

where $\beta_k = \begin{cases} k, \ if\ k \in J_{r^2} \\ 0, \ \ otherwise \end{cases}$

and $\zeta \equiv 0$.

For any $\varepsilon > 0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right\}$$
$$= \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma)\| \geq \varepsilon\right\} \in \mathcal{I} \text{ for every } \gamma \in \Lambda \text{ with } \mathcal{M}\{\Lambda\} = 1.$$

Thus the sequence (ζ_k) is lacunary $\mathcal{I}$-convergent almost surely to ζ.

Theorem 3.1. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be complex uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. If $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta$, then ζ is uniquely determined.*

Proof. If possible let $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta$, and $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta^*$ for some $\zeta(\gamma) \neq \zeta^*(\gamma)$. Let $\varepsilon > 0$ be arbitrary. Then for any $\varepsilon > 0$, we have

$$A = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}) \text{ and}$$
$$B = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta^*(\gamma)\| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I}).$$

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Let $r \in A \cap B$.

Then $\frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| < \frac{\varepsilon}{2}$ and $\frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta^*(\gamma)\| < \frac{\varepsilon}{2}$

$$\begin{aligned}
\text{Therefore } \|\zeta(\gamma) - \zeta^*(\gamma)\| &= \frac{1}{h_r} \sum_{k \in J_r} \|\zeta(\gamma) - \zeta^*(\gamma)\| \\
&= \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta^*(\gamma) + \zeta(\gamma) - \zeta_k(\gamma)\| \\
&\leq \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta^*(\gamma)\| + \frac{1}{h_r} \sum_{k \in J_r} \|\zeta(\gamma) - \zeta_k(\gamma)\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Hence the theorem is proved. □

Theorem 3.2. *If the complex uncertain sequence (ζ_k) and (ζ_k^*) are lacunary $\mathcal{I}$-convergent almost surely to ζ and ζ^*, respectively, then*

(i) $(\zeta_k + \zeta_k^*)$ is lacunary $\mathcal{I}$-convergent almost surely to $\zeta + \zeta^*$.
(ii) $(\zeta_k - \zeta_k^*)$ is lacunary $\mathcal{I}$-convergent almost surely to $\zeta - \zeta^*$.
(iii) $(c\zeta_k)$ is lacunary $\mathcal{I}$-convergent almost surely to $c\zeta$, where $c \in \mathbb{C}$.

Proof. (*i*) Let $\varepsilon > 0$, then $A = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I})$

and $B = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k^*(\gamma) - \zeta^*(\gamma)\| < \frac{\varepsilon}{2}\right\} \in \mathcal{F}(\mathcal{I})$.

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Therefore for all $r \in A \cap B$ we have,

$$\begin{aligned}
&\frac{1}{h_r} \sum_{k \in J_r} \|(\zeta_k(\gamma) + \zeta_k^*(\gamma)) - (\zeta(\gamma) + \zeta^*(\gamma))\| \\
&\quad \leq \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| + \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k^*(\gamma) - \zeta^*(\gamma)\| < \varepsilon.
\end{aligned}$$

i.e, $\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|(\zeta_k + \zeta_k^*) - (\zeta + \zeta^*)\| < \varepsilon\right\} \in \mathcal{F}(\mathcal{I})$.

Hence $(\zeta_k + \zeta_k^*)$ is lacunary $\mathcal{I}$-convergent almost surely to $\zeta + \zeta^*$.

(ii) It is similar to the proof of (i) above and therefore omitted.
(iii) The proof is easy so omitted. □

Theorem 3.3. *If the complex uncertain sequences* (ζ_k) *and* (ζ_k^*) *are lacunary* $\mathcal{I}$*-convergent almost surely to* ζ *and* ζ^* *respectively, and there exist positive numbers* $p_1, p, q_1,$ *and* q *such that* $p_1 \leq \|\zeta_k\|, \|\zeta\| \leq p$ *and* $q_1 \leq \|\zeta_k^*\|, \|\zeta^*\| \leq q$ *for any* k*, then*
(i) $(\zeta_k\zeta_k^*)$ *is lacunary* $\mathcal{I}$*-convergent almost surely to* $\zeta\zeta^*$*.*
(ii) $(\frac{\zeta_k}{\zeta_k^*})$ *is lacunary* $\mathcal{I}$*-convergent almost surely to* $\frac{\zeta}{\zeta^*}$*.*

Proof. (i) Let $\varepsilon > 0$, and $p, q > 0$ then

$$A = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k - \zeta\| < \frac{\varepsilon}{2q}\right\} \in \mathcal{F}(\mathcal{I})$$

$$\text{and } B = \left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k^* - \zeta^*\| < \tfrac{\varepsilon}{2p}\right\} \in \mathcal{F}(\mathcal{I}).$$

Since $A \cap B \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$ this implies $A \cap B \neq \phi$. Therefore for all $r \in A \cap B$ we have,

$$\begin{aligned}
\tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k\zeta_k^* - \zeta\zeta^*\| &= \tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k\zeta_k^* - \zeta_k\zeta^* + \zeta_k\zeta^* - \zeta\zeta^*\| \\
&\leq \tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k\zeta_k^* - \zeta_k\zeta^*\| + \tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k\zeta^* - \zeta\zeta^*\| \\
&\leq p\Big(\tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k^* - \zeta^*\|\Big) + q\Big(\tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k - \zeta\|\Big). \\
&< \varepsilon
\end{aligned}$$

i.e, $\left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in J_r} \|\zeta_k\zeta_k^* - \zeta\zeta^*\| < \varepsilon\right\} \in \mathcal{F}(\mathcal{I})$.
Hence $(\zeta_k\zeta_k^*)$ is lacunary $\mathcal{I}$-convergent almost surely to $\zeta\zeta^*$.

(ii) It is similar to the proof of (i) above and therefore omitted. □

Theorem 3.4. *If every subsequence of a complex uncertain sequence* (ζ_k) *is lacunary* $\mathcal{I}$*-convergent almost surely to* ζ*, then* (ζ_k) *is lacunary* $\mathcal{I}$*-convergent almost surely to* ζ*.*

Proof. If possible let, every subsequence of a complex uncertain sequence (ζ_k) is lacunary $\mathcal{I}$-convergent almost surely to ζ, but (ζ_k) is not lacunary $\mathcal{I}$-convergent almost surely to ζ. Then there exists some $\varepsilon > 0$ such that

$$A = \left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\right\} \notin \mathcal{I}.$$

So A must be an infinite set. Let $A = \{r_1 < r_2 < \cdots < r_j < \cdots\}$. Now we define a sequence (ζ_k^*) as $\zeta_k^* = \zeta_k$ for $k \in J_r, r \in A$. Then (ζ_k^*) is a subsequence of (ζ_k) which is not lacunary $\mathcal{I}$-convergent almost surely to ζ, a contradiction. □

Remark 3.1. But the converse of Theorem 3.4 is not true in general. Example 3.2 illustrates this fact.

Example 3.2. From Example 3.1, we see that the complex uncertain sequence (ζ_k) is lacunary $\mathcal{I}$-convergent almost surely to $\zeta \equiv 0$.
Now we define a subsequence (ζ_m^*) of (ζ_k) by

$(\zeta^*_m) = (\zeta_{m_k})$, where $m_k \in J_{r^2}, r \in \mathbb{N}$ which is not lacunary $\mathcal{I}$-convergent almost surely to $\zeta \equiv 0$.

Definition 3.2. Suppose that $\zeta, \zeta_1, \zeta_2, \cdots$ are complex uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $\theta = (k_r)$ be a lacunary sequence. A complex uncertain sequence (ζ_k) is said to be a lacunary $\mathcal{I}$-Cauchy sequence almost surely if for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ and an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that

$$\Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| \geq \varepsilon\Big\} \in \mathcal{I},\ \textit{for every } \gamma \in \Lambda.$$

Theorem 3.5. *If a complex uncertain sequence (ζ_k) is lacunary $\mathcal{I}$-convergent almost surely to ζ then it is a lacunary $\mathcal{I}$-Cauchy sequence almost surely.*

Proof. Let the complex uncertain sequence (ζ_k) be lacunary $\mathcal{I}$-convergent almost surely to ζ. Then for every $\varepsilon > 0$ and there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$, we have

$$U(\gamma, \varepsilon) = \Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon\Big\} \in \mathcal{I} \text{ for every } \gamma \in \Lambda.$$

Clearly, $\mathbb{N} \setminus U(\gamma, \varepsilon) \in \mathcal{F}(\mathcal{I})$ and therefore it is non-empty. So, we can choose a positive integer r such that $r \in \mathbb{N} \setminus U(\gamma, \varepsilon)$. Then there exists a $k_0 \in \mathbb{N}$, we have

$$\frac{1}{h_r} \sum_{k_0 \in J_r} \|\zeta_{k_0}(\gamma) - \zeta(\gamma)\| < \varepsilon \text{ for every } \gamma \in \Lambda. \tag{1}$$

$$\Longrightarrow \frac{1}{h_r} \|\zeta_{k_0}(\gamma) - \zeta(\gamma)\| < \varepsilon \text{ for every } \gamma \in \Lambda.$$

$$\Longrightarrow \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_{k_0}(\gamma) - \zeta(\gamma)\| < \varepsilon \text{ for every } \gamma \in \Lambda. \tag{2}$$

Case 1. When k,k_0 belongs to same J_r.
We define the set $V_1(\gamma, \varepsilon) = \Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k, k_0 \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| \geq 2\varepsilon\Big\} \in \mathcal{I}$ for every $\gamma \in \Lambda$. Now we prove that the following inclusion $V_1(\gamma, \varepsilon) \subseteq U(\gamma, \varepsilon)$ is true.
For if $r \in V_1(\gamma, \varepsilon)$ we have

$$\begin{aligned} 2\varepsilon \leq \frac{1}{h_r} \sum_{k, k_0 \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| &\leq \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| + \frac{1}{h_r} \sum_{k_0 \in J_r} \|\zeta_{k_0}(\gamma) - \zeta(\gamma)\| \\ &< \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| + \varepsilon \qquad [\text{by}(1)] \end{aligned}$$

This imples that

$$\frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| > \varepsilon \textit{ for every } \gamma \in \Lambda$$

and therefore $r \in U(\gamma, \varepsilon)$. Thus we conclude that $V_1(\gamma, \varepsilon) \in \mathcal{I}$,

Case 2. When k,k_0 belongs to different J_r.
We define the set $V_2(\gamma, \varepsilon) = \Big\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| \geq 2\varepsilon\Big\} \in \mathcal{I}$ for every

$\gamma \in \Lambda$. Now we prove that the following inclusion $V_2(\gamma, \varepsilon) \subseteq U(\gamma, \varepsilon)$ is true.
For if $r \in V_2(\gamma, \varepsilon)$ we have

$$2\varepsilon \leq \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| \leq \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| + \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_{k_0}(\gamma) - \zeta(\gamma)\|$$
$$< \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| + \varepsilon \qquad [\text{by}(2)]$$

This imples that

$$\frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| > \varepsilon \text{ for every } \gamma \in \Lambda$$

and therefore $r \in U(\gamma, \varepsilon)$. Thus we conclude that $V_2(\gamma, \varepsilon) \in \mathcal{I}$,

Hence from the above two cases, we see that for every $\varepsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ satisfying

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| \geq \varepsilon \right\} \in \mathcal{I}, \text{ for every } \gamma \in \Lambda.$$

i.e., (ζ_k) is a lacunary $\mathcal{I}$-Cauchy sequence almost surely. □

Definition 3.3. Let $\zeta, \zeta_1, \zeta_2, \cdots$ be complex uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $\theta = (k_r)$ be a lacunary sequence. A complex uncertain sequence (ζ_k) is said to be lacunary $\mathcal{I}^*$-convergent almost surely to ζ if there exists a set $X = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $X' = \{r \in \mathbb{N} : m_k \in J_r\} \in \mathcal{F}(\mathcal{I})$ and for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_{m_k}(\gamma) - \zeta(\gamma)\| = 0.$$

Symbolically we write $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta^*)} \zeta$.

Example 3.3. Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_k \in \Lambda} \frac{1}{2^k}$$

Let $\theta = (2^r)$, $r = 1, 2, 3, \cdots$ be a lacunary sequence and take $\mathcal{I}$=$\mathcal{I}_d$.
Also, the complex uncertain variables defined by

$$\zeta_k(\gamma) = i\beta_k \;\; if \;\gamma \in \{\gamma_1, \gamma_2, \cdots\},$$

where $\beta_k = \begin{cases} k, \; if \; k \in J_{r^2} \\ 0, \;\; otherwise \end{cases} \quad for \; k = 1, 2, 3, \cdots$ and $\zeta \equiv 0$.

Then there exists a set $X = \{m_1 < m_2 < \cdots < m_k < \cdots\} = \mathbb{N} \cap \left(\bigcup_{r=1}^{\infty} J_r \setminus \bigcup_{r=1}^{\infty} J_{r^2} \right)$ such that $X' = (\mathbb{N} \setminus Y) \in \mathcal{F}(\mathcal{I})$, where $Y = \{1, 4, 9, \cdots\} \in \mathcal{I}$ for which

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_{m_k}(\gamma) - \zeta(\gamma)\| = 0$$

for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$.
Thus the sequence (ζ_k) is lacunary $\mathcal{I}^*$-convergent almost surely to $\zeta \equiv 0$.

Theorem 3.6. *If* $\zeta_k \xrightarrow{A_s(\mathcal{I}^*_\theta)} \zeta$, *then* $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta$.

Proof. Let us assume that $\zeta_k \xrightarrow{A_s(\mathcal{I}^*_\theta)} \zeta$. Then there exists a set $X = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $X' = \{r \in \mathbb{N} : m_k \in J_r\} \in \mathcal{F}(\mathcal{I})$ and for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$,

$$\frac{1}{h_r} \sum_{k \in J_r} \|\zeta_{m_k}(\gamma) - \zeta(\gamma)\| < \varepsilon$$

for each $\varepsilon > 0$ and for all $k \geq k_0$.
Let $Y = \mathbb{N} \setminus X'$. It is clear that $Y \in \mathcal{I}$. Then for any $\varepsilon > 0$,

$$U(\gamma, \varepsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon \right\} \subseteq Y \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}.$$

Hence $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta$. □

Remark 3.2. But the converse of Theorem 3.6 is not true in general.

Example 3.4. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, where $D_j = \{2^{j-1}t : 2 \textit{ does not divide } t, t \in \mathbb{N}\}$ be the decomposition of $\mathbb{N}$ such that each D_j is infinite and $D_l \cap D_q = \phi$, for $l \neq q$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ that can intersect only finite number of $D_j's$. Then $\mathcal{I}$ is a non-trivial admissible ideal of $\mathbb{N}$ (Kostyrko et al. [16]).
Now we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_k \in \Lambda} \frac{k}{(2k+1)}, & if \sup\limits_{\gamma_k \in \Lambda} \frac{k}{(2k+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_k \in \Lambda^c} \frac{k}{(2k+1)}, & if \sup\limits_{\gamma_k \in \Lambda^c} \frac{k}{(2k+1)} < \frac{1}{2} \quad for\ k = 1, 2, 3, \cdots. \\ \frac{1}{2}, \ otherwise \end{cases}$$

Let $\theta = (2^r)$, $r = 1, 2, 3, \cdots$ be a lacunary sequence.
Also, the complex uncertain variables are defined by

$$\zeta_k(\gamma) = i\beta_r \text{ if } \gamma \in \{\gamma_1, \gamma_2, \cdots\} \textit{ and } k \in J_r,$$

where $\beta_r = \frac{1}{j}$, $if\ r \in D_j\ for\ r = 1, 2, 3, \cdots$ and $\zeta \equiv 0$.
It is clear that the sequence (ζ_k) is lacunary $\mathcal{I}$-convergent almost surely to $\zeta \equiv 0$. But this sequence is not lacunary $\mathcal{I}^*$-convergent almost surely to $\zeta \equiv 0$. Because for any set $H \in \mathcal{I}$ there exists $p \in \mathbb{N}$ such that $H \subseteq \bigcup_{j=1}^{p} D_j$ and as a consequence $D_{p+1} \subseteq \mathbb{N} \setminus H$. Let $A = \mathbb{N} \setminus H$, then $A \in \mathcal{F}(\mathcal{I})$ for which we can define a subsequence (ζ_{m_k}) which is not lacunary convergent almost surely to $\zeta \equiv 0$.
Hence the sequence (ζ_k) is not lacunary $\mathcal{I}^*$-convergent almost surely to $\zeta \equiv 0$.

Theorem 3.7. *Let* (ζ_k) *be a complex uncertain sequence in an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *such that* $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta$, *then* $\zeta_k \xrightarrow{A_s(\mathcal{I}^*_\theta)} \zeta$ *if* $\mathcal{I}$ *satisfies the condition (AP).*

Proof. Let us assume that $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta)} \zeta$. Then there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ and for any $\varepsilon > 0$, the set

$$U(\gamma, \varepsilon) = \Big\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \varepsilon \Big\} \in \mathcal{I} \textit{ for every } \gamma \in \Lambda.$$

Now we construct a countable family of mutually disjoint sets $\{U_p(\gamma)\}_{p\in\mathbb{N}}$ in $\mathcal{I}$ by considering

$$U_1(\gamma) = \Big\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq 1 \Big\}$$

and

$$U_p(\gamma) = \Big\{ r \in \mathbb{N} : \frac{1}{p} \leq \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| < \frac{1}{p-1} \Big\} = U(\gamma, \frac{1}{p}) \setminus U(\gamma, \frac{1}{p-1})$$

Since $\mathcal{I}$ satisfies the condition (AP), so for the above countable collection $\{U_p(\gamma)\}_{p\in\mathbb{N}}$ there exists another countable family of subsets $\{V_p(\gamma)\}_{p\in\mathbb{N}}$ of $\mathbb{N}$ satisfying $U_j(\gamma) \triangle V_j(\gamma)$ is finite for all $j \in \mathbb{N}$ and $V(\gamma) = \bigcup\limits_{j=1}^{\infty} V_j(\gamma) \in \mathcal{I}$.

Let $\varepsilon > 0$ be arbitrary. By Archimedean property, we can choose $p \in \mathbb{N}$ such that $\frac{1}{p+1} < \varepsilon$. Then

$$\Big\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \delta \Big\} \subseteq \Big\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \|\zeta_k(\gamma) - \zeta(\gamma)\| \geq \frac{1}{p+1} \Big\}$$
$$= \bigcup_{j=1}^{p+1} U_j(\gamma) \in \mathcal{I}.$$

Since $U_j(\gamma) \triangle V_j(\gamma)$ is finite $(j = 1, 2, \cdots, p+1)$ there exists a $r_0 \in \mathbb{N}$, such that

$$\bigcup_{j=1}^{p+1} V_j(\gamma) \cap (r_0, \infty) = \bigcup_{j=1}^{p+1} U_j(\gamma) \cap (r_0, \infty).$$

Thus for every $r \in \mathbb{N} \setminus V(\gamma) \in \mathcal{F}(\mathcal{I})$ and there exists $k_0 \in \mathbb{N}$, we choose $m_k \in J_r$ such that $k > k_0$. Therefore we must have $r \notin \bigcup\limits_{j=1}^{p+1} V_j(\gamma)$ and so $r \notin \bigcup\limits_{j=1}^{p+1} U_j(\gamma)$. Then there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ satisfying

$$\frac{1}{h_r} \sum_{k \in J_r} \|\zeta_{m_k}(\gamma) - \zeta(\gamma)\| < \frac{1}{p+1} < \varepsilon \text{ for every } \gamma \in \Lambda.$$

Hence $\zeta_k \xrightarrow{A_s(\mathcal{I}_\theta^*)} \zeta$. □

Definition 3.4. Let $\zeta, \zeta_1, \zeta_2, \cdots$ be complex uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. A complex uncertain sequence (ζ_k) is said to be a lacunary $\mathcal{I}^*$-Cauchy sequence almost surely if there exists a set $X = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $X' = \{r \in \mathbb{N} : m_k \in J_r\} \in \mathcal{F}(\mathcal{I})$ and for every $j \in \mathbb{N}$ satisfying

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in J_r} \left\| \zeta_{m_k}(\gamma) - \zeta_{m_j}(\gamma) \right\| = 0.$$

for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$.

Theorem 3.8. *If a complex uncertain sequence (ζ_k) is a lacunary $\mathcal{I}^*$-Cauchy sequence almost surely then it is a lacunary $\mathcal{I}$-Cauchy sequence almost surely.*

Proof. Let the complex uncertain sequence (ζ_k) is a lacunary $\mathcal{I}^*$-Cauchy sequence almost surely. Then for every $\varepsilon > 0$ there exists a set $X = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $X' = \{r \in \mathbb{N} : m_k \in J_r\} \in \mathcal{F}(\mathcal{I})$ and $k_0 \in \mathbb{N}$ satisfying

$$\frac{1}{h_r}\sum_{k\in J_r}\left\|\zeta_{m_k}(\gamma) - \zeta_{m_j}(\gamma)\right\| < \varepsilon$$

for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$ and for all $k, j \geq k_0$.
Let $Y = \mathbb{N} \setminus X'$. It is clear that $Y \in \mathcal{I}$. Then for any $\varepsilon > 0$,

$$U(\gamma,\varepsilon) = \left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in J_r}\|\zeta_k(\gamma) - \zeta_{k_0}(\gamma)\| \geq \varepsilon\right\} \subseteq Y \cup \{m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}.$$

Hence the sequence (ζ_k) is a lacunary $\mathcal{I}$-Cauchy sequence almost surely and the proof is complete. □

Remark 3.3. But the converse of Theorem 3.8 is not true in general.

Example 3.5. Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, where $D_j = \{2^{j-1}t : 2$ does not divide $t, t \in \mathbb{N}\}$ be the decomposition of $\mathbb{N}$ such that each D_j is infinite and $D_l \cap D_q = \phi$, for $l \neq q$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ that can intersect only finite number of $D_j{}'s$. Then $\mathcal{I}$ is a non-trivial admissible ideal of $\mathbb{N}$ (Kostyrko et al. [16]).
Now we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_k\in\Lambda}\frac{1}{2^k} \ for\ k = 1, 2, 3, \cdots .$$

Let $\theta = (2^r)$, $r = 1, 2, 3, \cdots$ be a lacunary sequence.
Also, the complex uncertain variables are defined by

$$\zeta_k(\gamma) = i\beta_r \text{ if } \gamma \in \{\gamma_1, \gamma_2, \cdots\} \ and\ k \in J_r,$$

where $\beta_r = \frac{1}{j+1}$, *if* $r \in D_j$ *for* $r = 1, 2, 3, \cdots$ and $\zeta \equiv 0$.
It is clear that the sequence (ζ_k) is lacunary $\mathcal{I}$-convergent almost surely to $\zeta \equiv 0$.
By theorem 3.5 the sequence (ζ_k) is lacunary $\mathcal{I}$-Cauchy sequence almost surely.

Next, we shall show that the complex uncertain sequence (ζ_k) is not a lacunary $\mathcal{I}^*$-Cauchy sequence almost surely. For this, if possible assume that the sequence (ζ_k) is lacunary $\mathcal{I}^*$-Cauchy sequence almost surely. Then for every $\varepsilon > 0$ there exists a set $X = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $X' = \{r \in \mathbb{N} : m_k \in J_r\} \in \mathcal{F}(\mathcal{I})$ and $k_0 \in \mathbb{N}$ satisfying

$$\frac{1}{h_r}\sum_{k,j\in J_r}\left\|\zeta_{m_k}(\gamma) - \zeta_{m_j}(\gamma)\right\| < \varepsilon \tag{3}$$

for every $\gamma \in \Lambda$ with $\mathcal{M}\{\Lambda\} = 1$ and for all $k, j \geq k_0$.
Since $(\mathbb{N}\setminus X') \in \mathcal{I}$, so there exists a 'p'$\in \mathbb{N}$ such that $(\mathbb{N}\setminus X') \subset D_1 \cup D_2 \cup \cdots \cup D_p$. But $D_i \subset X'$ $\forall i > p$. In particular D_{p+1}, $D_{p+2} \subset X'$. We see that from the construction

of $D_j' s$, for given any $k_0 \in \mathbb{N}$ there are $m_k \in J_r, r \in D_{p+1}$ and $m_j \in J_r, r \in D_{p+2}$ such that $m_k, m_j \geq k_0$. But then the following equality

$$\frac{1}{h_r}\sum_{k\in J_r}\left\|\zeta_{m_k}(\gamma)-\zeta_{m_j}(\gamma)\right\| = \left\|\frac{i}{p+2}-\frac{i}{p+3}\right\| = \frac{1}{(p+2)(p+3)}$$

contradicts (3) for any particular choice of ε with $\varepsilon = \frac{1}{3(p+2)(p+3)}$. This is a contradiction so our assumption was wrong and hence (ζ_k) is not $\mathcal{I}^*$-Cauchy sequence almost surely.

4. Conclusion

In this paper, the idea of convergence of lacunary $\mathcal{I}$ and $\mathcal{I}^*$-convergent almost surely of complex uncertain sequence have been introduced and some results related to this convergence have been studied. Also, we define lacunary $\mathcal{I}$ and $\mathcal{I}^*$-Cauchy sequence almost surely and study the relationship between them. These ideas and results are expected to be enough scope for researchers in the area of convergence of complex uncertain sequences. Also, these concepts can be generalized and applied for further studies.

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