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Large radial solutions of an overdetermined eigenvalue problems for the polyharmonic operator

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Abstract. We consider an eigenvalue problem for the polyharmonic operator, with overdetermined boundary conditions. We give radial solutions on balls and those solutions are expressed by the mean of Bessel functions.

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1. Introduction

In this paper, we are interested in the overdetermined eigenvalue problems for the polyharmonic operator given by:

$$(P_m) \qquad \begin{cases} (-1)^m \Delta^m u = \lambda u + \mu & \text{in } B(0,R), \\ u = a, \ \dfrac{\partial u}{\partial \nu} = b & \text{on } \partial B(0,R), \\ \Delta u = c_1, \ \dots , \Delta^{m-1} u = c_{m-1} \quad \text{on } \partial B(0,R) & \text{if } m \geq 2, \end{cases}$$

where m is a positive integer, $\lambda > 0$, $\mu, a, b \in \mathbb{R}$, $c_p \in \mathbb{R}$, $\forall p \in \{1, ..., m-1\}$, $B(0,R)$ is an N-ball of raduis R in $\mathbb{R}^N$ with $N \geq 2$ and $\dfrac{\partial}{\partial \nu}$ is the outward normal derivative.

When $m = 1$, $\lambda = 0$, $\mu = 1$ and $a = 0$, problem (P_1) writes

$$\begin{cases} \Delta u = -1 & \text{in } B(0,R), \\ u = 0, \qquad \dfrac{\partial u}{\partial \nu} = b & \text{on } \partial B(0,R). \end{cases} \tag{1}$$

Serrin [7] showed that (1) admits a solution u on the ball of radius $R = N|b|$ and u is radially symmetric given by

$$u = \frac{R^2 - r^2}{2N}.$$

When $m = 2$, $\lambda = 0$, $\mu = -1$ and $a = b = 0$, problem (P_2) writes

$$\begin{cases} \Delta^2 u = -1 & \text{in } B(0,R), \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad \Delta u = c_1 & \text{on } \partial B(0,R). \end{cases} \tag{2}$$

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Bennett [1] showed that (2) admits a solution u on the ball of radius $R = [|c_1|(N^2 + 2N)]^{1/2}$ and u is radially symmetric given by

$$u = -\frac{1}{2N}\left(\frac{N+2}{4}(Nc_1)^2 + \frac{Nc_1}{2}r^2 + \frac{1}{4(N+2)}r^4\right).$$

When $m = 2$, $\lambda = 0$, $\mu = 1$ and $a = b = 0$, problem (P_2) writes

$$\begin{cases} \Delta^2 u = 1 & \text{in } B(0,R), \\ \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad \Delta u = c_1 & \text{on } \partial B(0,R). \end{cases} \tag{3}$$

Dalmaso [2] showed that (3) admits a solution u on the ball of radius $R = [c_1(N^2 + 2N)]^{1/2}$ and u is radially symmetric given by

$$u = \frac{(R^2 - r^2)^2}{8N(N+2)}.$$

In the case m=2, we have considered in [4] the following problem :

$$(P_{2,\tau}) \qquad \begin{cases} L_\tau u = \Delta^2 u - \tau \Delta u = \lambda u + \mu & \text{in } B(0,1), \\ \\ u = a, \quad \dfrac{\partial u}{\partial \nu} = b, \quad \Delta u = c_1 & \text{on } \partial B(0,1), \end{cases}$$

with $\tau \geq 0$. Under some conditions, we prove:

Theorem 1.1. *Let $\beta := \frac{N-2}{2}$, $\theta := \left(\frac{\sqrt{\tau^2 + 4\lambda} - \tau}{2}\right)^{\frac{1}{2}}$ and consider the Bessel function of the first kind of order β denoted by J_β. Then, we have the following.*
(i) Suppose $\mu = 0$, $a = c_1 = 0$, $b \neq 0$ and θ satisfies $J_\beta(\theta) = 0$. Then, there exists a radial solution to problem $(P_{2,\tau})$, given by

$$u(x) = v(r) = \frac{b}{\theta J_\beta'(\theta)}\, r^{-\beta} J_\beta(\theta r), \qquad \forall r = |x| \in (0,1].$$

(ii) Suppose $\mu = 0$, $a \neq 0$, $b = 0$, $c_1 = -\theta^2 a$ and θ satisfies $\theta J_\beta'(\theta) - \beta J_\beta(\theta) = 0$. Then, there exists a radial solution to problem $(P_{2,\tau})$, given by

$$u(x) = v(r) = \frac{a}{J_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r), \qquad \forall r = |x| \in (0,1].$$

(iii) Suppose $\mu \neq 0$, $a = b = 0$, $c_1 = -\dfrac{\theta^2 \mu}{\lambda}$ and θ satisfies $\theta J_\beta'(\theta) - \beta J_\beta(\theta) = 0$. Then, there exists a radial solution to problem $(P_{2,\tau})$, given by

$$u(x) = v(r) = \frac{\mu}{\lambda}\left[\frac{1}{J_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r) - 1\right], \qquad \forall r = |x| \in (0,1].$$

Our result deals with radial solutions of problem (P_m) when $B(0,R)$ is an N-ball of raduis R. Let $\beta := \frac{N-2}{2}$ and consider the Bessel function of the first kind of order β denoted by J_β.

We have the following theorem.

Theorem 1.2. *Let $B(0,R)$ the ball in $\mathbb{R}^N$ with $N \geq 2$. Then, we have the following.*
1- Suppose $\mu = 0,\ a = c_p = 0,\ b \neq 0$ and R satisfies $J_\beta(\lambda^{\frac{1}{2m}} R) = 0$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{b}{\lambda^{\frac{1}{2m}} J'_\beta(\lambda^{\frac{1}{2m}} R)} \left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}} r), \qquad \forall r = |x| \in (0,R].$$

2- Suppose $\mu = 0,\ a \neq 0,\ b = 0,\ c_p = (-1)^p \lambda^{\frac{p}{m}} a,\ \ \forall p \in \{1,...,m-1\}$ and R satisfies $\lambda^{\frac{1}{2m}} R J'_\beta(\lambda^{\frac{1}{2m}} R) - \beta J_\beta(\lambda^{\frac{1}{2m}} R) = 0$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{a}{J_\beta(\lambda^{\frac{1}{2m}} R)} \left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}} r), \qquad \forall r = |x| \in (0,R].$$

3- Suppose $\mu \neq 0,\ a \in \mathbb{R},\ b = 0,\ c_p = (-1)^p \lambda^{\frac{p}{m}} \left[a + \frac{\mu}{\lambda}\right],\ \forall p \in \{1,...,m-1\}$ and R satisfies $\lambda^{\frac{1}{2m}} R J'_\beta(\lambda^{\frac{1}{2m}} R) - \beta J_\beta(\lambda^{\frac{1}{2m}} R) = 0$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{\mu}{\lambda}\left[\frac{(\frac{\lambda a}{\mu} + 1)}{J_\beta(\lambda^{\frac{1}{2m}} R)} \left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}} r) - 1\right], \qquad \forall r = |x| \in (0,R].$$

Remark 1.1. Let $B(0,R)$ the ball in $\mathbb{R}^3$. Then, we have the following.
1- Suppose $\mu = 0,\ a = c_p = 0,\ b \neq 0$ and $R = k\pi\lambda^{-\frac{1}{2m}}, \forall k \in \mathbb{N}^*$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{(-1)^k k\pi b}{\lambda^{\frac{1}{2m}}} \frac{\sin(\lambda^{\frac{1}{2m}} r)}{\lambda^{\frac{1}{2m}} r}, \qquad \forall r = |x| \in (0,R].$$

2- Suppose $\mu = 0,\ a \neq 0,\ b = 0,\ c_p = (-1)^p \lambda^{\frac{p}{m}} a,\ \ \forall p \in \{1,...,m-1\}$ and R satisfies $\tan(\lambda^{\frac{1}{2m}} R) = \lambda^{\frac{1}{2m}} R$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = a\ \frac{R}{r} \frac{\sin(\lambda^{\frac{1}{2m}} r)}{\sin(\lambda^{\frac{1}{2m}} R)}, \qquad \forall r = |x| \in (0,R].$$

3- Suppose $\mu \neq 0,\ a \in \mathbb{R},\ b = 0,\ c_p = (-1)^p \lambda^{\frac{p}{m}} \left[a + \frac{\mu}{\lambda}\right],\ \forall p \in \{1,...,m-1\}$ and R satisfies $\tan(\lambda^{\frac{1}{2m}} R) = \lambda^{\frac{1}{2m}} R$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{\mu}{\lambda}\left[\left((\frac{\lambda a}{\mu} + 1)\ \frac{R}{r} \frac{\sin(\lambda^{\frac{1}{2m}} r)}{\sin(\lambda^{\frac{1}{2m}} R)}\right) - 1\right], \qquad \forall r = |x| \in (0,R].$$

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We assume that the domain is a ball in $\mathbb{R}^N$ with $N \geq 2$ and for sake of simplicity, we consider the ball $B(0,R)$. As mentioned in the Introduction, radial solutions for problem (P_m) are constructed with the use of Bessel functions.

We recall that the Bessel functions , discovered by Bernoulli and used by Bessel as part of his studies of the movement of the planets induced by gravitational interaction, are canonical solutions $y(r)$ of the differential Bessel equation:

$$r^2y''(r)+ry'(r)+(r^2-\alpha^2)y(r)=0, \tag{4}$$

for any real or complex number α. Most often, α is a natural integer (then called the order of the function), or a half-integer.

There are two kinds of Bessel functions:

- The Bessel functions of the first kind, denoted by J_n, solutions of the above differential equation which are defined at 0.
- The Bessel functions of the second kind, denoted by Y_n, solutions which are not defined in 0 (but which have an infinite limit in 0).

We also recall that for integer values $\alpha=n$, the Bessel functions of the first kind can be represented by integrals given by

$$J_n(x)=\frac{1}{\pi}\int_0^\pi \cos(nt-x\sin t)\,\mathrm{d}t$$

or else by

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\,(nt-x\sin t)}\,\mathrm{d}t.$$

For $\alpha=n+\frac{1}{2}$, the Bessel functions of the first kind given by

$$J_{n+\frac{1}{2}}(x)=(-1)^n\sqrt{\frac{2x}{\pi}}\,x^n\Big(\frac{d}{xdx}\Big)^n\Big\{\frac{\sin x}{x}\Big\},\quad \forall x\in\mathbb{R}^+. \tag{5}$$

This definition can extend to the non-integer case of α (for $Re(x)>0$), by adding another term

$$J_\alpha(x)=\frac{1}{\pi}\int_0^\pi \cos(\alpha t-x\sin t)\,\mathrm{d}t-\frac{\sin(\alpha\pi)}{\pi}\int_0^\infty \mathrm{e}^{-x\sinh(t)-\alpha t}\,\mathrm{d}dt.$$

Here, we consider the Bessel functions of the first kind and we denote them by J_β, where $\beta=\frac{N-2}{2}$ $(N\geq 2)$ is an integer or half-integer.

Now, writing $y(r)=J_\beta(\lambda^{\frac{1}{2m}}r)$ with $r\in(0,R]$ and taking $\alpha=\beta$ in the equation (4), we are led to the equation

$$J''_\beta(\lambda^{\frac{1}{2m}}r)+\frac{1}{\lambda^{\frac{1}{2m}}r}J'_\beta(\lambda^{\frac{1}{2m}}r)+\Big(1-\frac{\beta^2}{\lambda^{\frac{1}{m}}r^2}\Big)J_\beta(\lambda^{\frac{1}{2m}}r)=0,\quad \forall r\in(0,R]. \tag{6}$$

Proof of Theorem 1.2. 1- Let $\mu=0$, $b\neq 0$ and suppose that R satisfies $J_\beta(\lambda^{\frac{1}{2m}}R)=0$. Consider the radial function

$$u(x)=v(r)=\frac{b}{\lambda^{\frac{1}{2m}}J'_\beta(\lambda^{\frac{1}{2m}}R)}\Big(\frac{R}{r}\Big)^\beta J_\beta(\lambda^{\frac{1}{2m}}r),\qquad \forall r=|x|\in(0,R]. \tag{7}$$

We have

$$v'(r)=-\frac{b\beta R^\beta}{\lambda^{\frac{1}{2m}}J'_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r)+\frac{bR^\beta}{J'_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J'_\beta(\lambda^{\frac{1}{2m}}r).$$

Using (6), we have

$$v^{''}(r) = -(2\beta+1)r^{-1}\left[-\frac{b\beta R^{\beta}}{\lambda^{\frac{1}{2m}}J_{\beta}^{'}(\lambda^{\frac{1}{2m}}R)}\ r^{-\beta-1}J_{\beta}(\lambda^{\frac{1}{2m}}r)+\frac{bR^{\beta}}{J_{\beta}^{'}(\lambda^{\frac{1}{2m}}R)}\ r^{-\beta}J_{\beta}^{'}(\lambda^{\frac{1}{2m}}r)\right]$$

$$-\ \frac{\lambda^{\frac{1}{m}}b}{\lambda^{\frac{1}{2m}}J_{\beta}^{'}(\lambda^{\frac{1}{2m}}R)}\left(\frac{R}{r}\right)^{\beta}J_{\beta}(\lambda^{\frac{1}{2m}}r) = -(2\beta+1)r^{-1}v'(r)-\lambda^{\frac{1}{m}}v(r)$$

$$= -(N-1)r^{-1}v'(r)-\lambda^{\frac{1}{m}}v(r).$$

Then

$$-\Delta u(x) = -\left[\frac{N-1}{r}\,v'(r)+v''(r)\right] = \lambda^{\frac{1}{m}}v(r) = \lambda^{\frac{1}{m}}u(x).$$

By induction, we obtain

$$(-1)^p\Delta^p u(x) = \lambda^{\frac{p}{m}}\,u(x),\quad \forall p\in\{1,...,m\}.$$

In particular for $p=m$, we have

$$(-1)^m\Delta^m u(x) = \lambda u(x),\quad \forall m\geq 1.$$

Then $u(x)=v(r)$ given by (7) is a solution of the equation

$$(-1)^m\Delta^m u = \lambda u \qquad \text{on}\ \ B(0,R).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=R} = \frac{1}{\lambda^{\frac{1}{2m}}J_{\beta}^{'}(\lambda^{\frac{1}{2m}}R)}J_{\beta}(\lambda^{\frac{1}{2m}}R) = 0,$$

$$\frac{\partial u}{\partial\nu}(x) = \nabla u(x)\cdot\vec{\nu}(x) = \frac{1}{r}\,v'(r)\ \vec{x}\cdot\vec{\nu}(x)\Big|_{r=R}$$

$$= -\frac{b\beta R^{\beta}}{\lambda^{\frac{1}{2m}}J_{\beta}^{'}(\lambda^{\frac{1}{2m}}R)}\ r^{-\beta-1}J_{\beta}(\lambda^{\frac{1}{2m}}r)+\frac{bR^{\beta}}{J_{\beta}^{'}(\lambda^{\frac{1}{2m}}R)}\ r^{-\beta}J_{\beta}^{'}(\lambda^{\frac{1}{2m}}r))\Big|_{r=R} = b$$

and

$$\Delta u(x) = (-1)^1\lambda^{\frac{1}{m}}\,u(x) = 0.$$

By induction, for $m\geq 2$ we obtain

$$\Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}\,u(x),\quad \forall p\in\{1,...,m-1\},$$

then

$$c_p = \Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}\,u(x) = 0,\quad \forall p\in\{1,...,m-1\}.$$

2- Let $\mu=0$, $a\neq 0$ and suppose that R satisfies $\lambda^{\frac{1}{2m}}RJ_{\beta}^{'}(\lambda^{\frac{1}{2m}}R)-\beta J_{\beta}(\lambda^{\frac{1}{2m}}R)=0$. Consider the radial function

$$u(x) := v(r) = \frac{a}{J_{\beta}(\lambda^{\frac{1}{2m}}R)}\ \left(\frac{R}{r}\right)^{\beta}J_{\beta}(\lambda^{\frac{1}{2m}}r),\qquad \forall r=|x|\in(0,R]. \tag{8}$$

We have

$$v'(r) = -\frac{a\beta R^{\beta}}{J_{\beta}(\lambda^{\frac{1}{2m}}R)}\ r^{-\beta-1}J_{\beta}(\lambda^{\frac{1}{2m}}r)+\frac{a\,\lambda^{\frac{1}{2m}}\,R^{\beta}}{J_{\beta}(\lambda^{\frac{1}{2m}}R)}\ r^{-\beta}J_{\beta}^{'}(\lambda^{\frac{1}{2m}}r).$$

Using (6), we have

$$v^{''}(r) = -(2\beta+1)r^{-1}\left[-\frac{a\beta R^{\beta}}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r) + \frac{a\,\lambda^{\frac{1}{2m}}\,R^{\beta}}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J^{'}_\beta(\lambda^{\frac{1}{2m}}r)\right]$$
$$-\lambda^{\frac{1}{m}}\frac{a}{J_\beta(\lambda^{\frac{1}{2m}}R)}\left(\frac{R}{r}\right)^{\beta}J_\beta(\lambda^{\frac{1}{2m}}r) = -(2\beta+1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}v(r)$$
$$= -(N-1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}v(r).$$

Then

$$-\Delta u(x) = -\left[\frac{N-1}{r}\,v'(r) + v''(r)\right] = \lambda^{\frac{1}{m}}v(r) = \lambda^{\frac{1}{m}}u(x).$$

By induction, we obtain

$$(-1)^p\Delta^p u(x) = \lambda^{\frac{p}{m}}\,u(x), \quad \forall p\in\{1,...,m\}.$$

In particular for $p=m$, we have

$$(-1)^m\Delta^m u(x) = \lambda u(x), \quad \forall m\geq 1.$$

Then $u(x)=v(r)$ given by (8) is a solution of the equation

$$(-1)^m\Delta^m u = \lambda u \quad \text{ on } \; B(0,R).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=R} = a,$$

$$\frac{\partial u}{\partial\nu}(x) = \nabla u(x)\cdot\vec{\nu}(x) = \frac{1}{r}\,v'(r)\;\vec{x}\cdot\vec{\nu}(x)\Big|_{r=R}$$
$$= -\frac{a}{R\,J_\beta(\lambda^{\frac{1}{2m}}R)}\Big(\lambda^{\frac{1}{2m}}R\,J^{'}_\beta(\lambda^{\frac{1}{2m}}R) - \;\beta\,J_\beta(\lambda^{\frac{1}{2m}}R)\Big) = 0$$

and

$$\Delta u(x) = (-1)^1\lambda^{\frac{1}{m}}\,u(x) = (-1)^1\lambda^{\frac{1}{m}}\,a.$$

By induction, for $m\geq 2$ we obtain

$$\Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}\,u(x), \quad \forall p\in\{1,...,m-1\},$$

then

$$c_p = \Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}\,u(x) = (-1)^p\lambda^{\frac{p}{m}}a, \quad \forall p\in\{1,...,m-1\}.$$

3 - Let $\mu\neq 0$, $a\in\mathbb{R}$ and suppose that R satisfies $\lambda^{\frac{1}{2m}}RJ^{'}_\beta(\lambda^{\frac{1}{2m}}R) - \beta J_\beta(\lambda^{\frac{1}{2m}}R) = 0$. Consider the radial function

$$u(x) = v(r) = \frac{\mu}{\lambda}\left[\frac{(\frac{\lambda a}{\mu}+1)}{J_\beta(\lambda^{\frac{1}{2m}}R)}\;\left(\frac{R}{r}\right)^{\beta}J_\beta(\lambda^{\frac{1}{2m}}r) - 1\right], \qquad \forall r=|x|\in(0,R]. \tag{9}$$

We have

$$v^{'}(r) = \frac{\mu}{\lambda}\left[\frac{-(\frac{\lambda a}{\mu}+1)\beta R^{\beta}}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r) + \frac{(\frac{\lambda a}{\mu}+1)\lambda^{\frac{1}{2m}}R^{\beta}}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J^{'}_\beta(\lambda^{\frac{1}{2m}}r)\right].$$

Using (6), we obtain

$$v^{''}(r) = -(2\beta+1)r^{-1}\,\frac{\mu}{\lambda}\left[\frac{-(\frac{\lambda a}{\mu}+1)\beta R^{\beta}}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r)+\frac{(\frac{\lambda a}{\mu}+1)\lambda^{\frac{1}{2m}}R^{\beta}}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J_\beta^{'}(\lambda^{\frac{1}{2m}}r)\right]$$

$$-\lambda^{\frac{1}{m}}\left[\frac{\mu}{\lambda}\,\frac{(\frac{\lambda a}{\mu}+1)}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,\left(\frac{R}{r}\right)^{\beta}J_\beta(\lambda^{\frac{1}{2m}}r)\right] = -(2\beta+1)r^{-1}v'(r)-\lambda^{\frac{1}{m}}\left[v(r)+\frac{\mu}{\lambda}\right]$$

$$= -(N-1)r^{-1}v'(r)-\lambda^{\frac{1}{m}}\left[v(r)+\frac{\mu}{\lambda}\right].$$

Thus

$$-\Delta u(x) = \left[\frac{N-1}{r}\,v'(r)+v''(r)\right] = \lambda^{\frac{1}{m}}\left[v(r)+\frac{\mu}{\lambda}\right] = \lambda^{\frac{1}{m}}\left[u(x)+\frac{\mu}{\lambda}\right].$$

By induction, we obtain

$$(-1)^p\Delta^p u(x) = \lambda^{\frac{p}{m}}\left[u(x)+\frac{\mu}{\lambda}\right],\quad \forall p\in\{1,...,m\}.$$

In particular for $p=m$, we have

$$(-1)^m\Delta^m u(x) = \lambda u(x)+\mu,\quad \forall m\geq 1.$$

Then $u(x)=v(r)$ given by (9) is a solution of the equation

$$(-1)^m\Delta^m u = \lambda u+\mu \quad \text{on}\;\; B(0,R).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=R} = a,$$

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\cdot\vec{\nu}(x) = \frac{1}{r}\,v'(r)\;\vec{x}\cdot\vec{\nu}(x)\Big|_{r=R}$$

$$= \frac{\mu(\frac{\lambda a}{\mu}+1)}{\lambda J_\beta(\lambda^{\frac{1}{2m}}R)}\Big(\lambda^{\frac{1}{2m}}R\,J_\beta^{'}(\lambda^{\frac{1}{2m}}R)-\beta\,J_\beta(\lambda^{\frac{1}{2m}}R)\Big) = 0$$

and

$$\Delta u(x) = -\lambda^{\frac{1}{m}}\left[u(x)+\frac{\mu}{\lambda}\right] = -\lambda^{\frac{1}{m}}\left[a+\frac{\mu}{\lambda}\right].$$

By induction, for $m\geq 2$ we obtain

$$\Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}\left[u(x)+\frac{\mu}{\lambda}\right],\quad \forall p\in\{1,...,m-1\},$$

and so

$$c_p = \Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}\left[a+\frac{\mu}{\lambda}\right],\quad \forall p\in\{1,...,m-1\}.$$

$\square$

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