Large radial solutions of an overdetermined eigenvalue problems for the polyharmonic operator

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ABSTRACT. We consider an eigenvalue problem for the polyharmonic operator, with overdetermined boundary conditions. We give radial solutions on balls and those solutions are expressed by the mean of Bessel functions.

1. Introduction

In this paper, we are interested in the overdetermined eigenvalue problems for the polyharmonic operator given by:

$$(P_m) \quad \begin{cases} (-1)^m \Delta^m u = \lambda u + \mu & \text{in } B(0,R), \\ u = a, \ \dfrac{\partial u}{\partial \nu} = b & \text{on } \partial B(0,R), \\ \Delta u = c_1, \ \dots, \Delta^{m-1} u = c_{m-1} & \text{on } \partial B(0,R) \quad \text{if } m \geq 2, \end{cases}$$

where m is a positive integer, $\lambda > 0$, $\mu, a, b \in \mathbb{R}$, $c_p \in \mathbb{R}$, $\forall p \in \{1, ..., m-1\}$, $B(0,R)$ is an N-ball of raduis R in $\mathbb{R}^N$ with $N \geq 2$ and $\dfrac{\partial}{\partial \nu}$ is the outward normal derivative.

When $m = 1$, $\lambda = 0$, $\mu = 1$ and $a = 0$, problem (P_1) writes

$$\begin{cases} \Delta u = -1 & \text{in } B(0,R), \\ \\ u = 0, \qquad \dfrac{\partial u}{\partial \nu} = b & \text{on } \partial B(0,R). \end{cases} \tag{1}$$

Serrin [7] showed that (1) admits a solution u on the ball of radius $R = N|b|$ and u is radially symmetric given by

$$u = \frac{R^2 - r^2}{2N}.$$

When $m = 2$, $\lambda = 0$, $\mu = -1$ and $a = b = 0$, problem (P_2) writes

$$\begin{cases} \Delta^2 u = -1 & \text{in } B(0,R), \\ \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad \Delta u = c_1 & \text{on } \partial B(0,R). \end{cases} \tag{2}$$

Bennett [1] showed that (2) admits a solution u on the ball of radius $R = [|c_1|(N^2 + 2N)]^{1/2}$ and u is radially symmetric given by

$$u = -\frac{1}{2N}\left(\frac{N+2}{4}(Nc_1)^2 + \frac{Nc_1}{2}r^2 + \frac{1}{4(N+2)}r^4\right).$$

When $m = 2$, $\lambda = 0$, $\mu = 1$ and $a = b = 0$, problem (P_2) writes

$$\begin{cases} \Delta^2 u = 1 & \text{in } B(0,R), \\ \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad \Delta u = c_1 & \text{on } \partial B(0,R). \end{cases} \tag{3}$$

Dalmaso [2] showed that (3) admits a solution u on the ball of radius $R = [c_1(N^2 + 2N)]^{1/2}$ and u is radially symmetric given by

$$u = \frac{(R^2 - r^2)^2}{8N(N+2)}.$$

In the case m=2, we have considered in [4] the following problem :

$(P_{2,\tau})$
$$\begin{cases} L_\tau u = \Delta^2 u - \tau \Delta u = \lambda u + \mu & \text{in } B(0,1), \\ \\ u = a, \quad \dfrac{\partial u}{\partial \nu} = b, \quad \Delta u = c_1 & \text{on } \partial B(0,1), \end{cases}$$

with $\tau \geq 0$. Under some conditions, we prove:

Theorem 1.1. Let $\beta := \frac{N-2}{2}$, $\theta := \left(\frac{\sqrt{\tau^2 + 4\lambda} - \tau}{2}\right)^{\frac{1}{2}}$ and consider the Bessel function of the first kind of order β denoted by J_β. Then, we have the following.

(i) Suppose $\mu = 0$, $a = c_1 = 0$, $b \neq 0$ and θ satisfies $J_\beta(\theta) = 0$. Then, there exists a radial solution to problem $(P_{2,\tau})$, given by

$$u(x) = v(r) = \frac{b}{\theta J'_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r), \qquad \forall r = |x| \in (0,1].$$

(ii) Suppose $\mu = 0$, $a \neq 0$, $b = 0$, $c_1 = -\theta^2 a$ and θ satisfies $\theta J'_\beta(\theta) - \beta J_\beta(\theta) = 0$. Then, there exists a radial solution to problem $(P_{2,\tau})$, given by

$$u(x) = v(r) = \frac{a}{J_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r), \qquad \forall r = |x| \in (0,1].$$

(iii) Suppose $\mu \neq 0$, $a = b = 0$, $c_1 = -\dfrac{\theta^2 \mu}{\lambda}$ and θ satisfies $\theta J'_\beta(\theta) - \beta J_\beta(\theta) = 0$. Then, there exists a radial solution to problem $(P_{2,\tau})$, given by

$$u(x) = v(r) = \frac{\mu}{\lambda}\left[\frac{1}{J_\beta(\theta)}\, r^{-\beta} J_\beta(\theta r) - 1\right], \qquad \forall r = |x| \in (0,1].$$

Our result deals with radial solutions of problem (P_m) when $B(0,R)$ is an N-ball of raduis R. Let $\beta := \frac{N-2}{2}$ and consider the Bessel function of the first kind of order β denoted by J_β.

We have the following theorem.

Theorem 1.2. *Let $B(0, R)$ the ball in $\mathbb{R}^N$ with $N \geq 2$. Then, we have the following.*

1- Suppose $\mu = 0$, $a = c_p = 0$, $b \neq 0$ and R satisfies $J_\beta(\lambda^{\frac{1}{2m}} R) = 0$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{b}{\lambda^{\frac{1}{2m}} J'_\beta(\lambda^{\frac{1}{2m}} R)} \left(\frac{R}{r} \right)^\beta J_\beta(\lambda^{\frac{1}{2m}} r), \qquad \forall r = |x| \in (0, R].$$

2- Suppose $\mu = 0$, $a \neq 0$, $b = 0$, $c_p = (-1)^p \lambda^{\frac{p}{m}} a$, $\forall p \in \{1, ..., m-1\}$ and R satisfies $\lambda^{\frac{1}{2m}} R J'_\beta(\lambda^{\frac{1}{2m}} R) - \beta J_\beta(\lambda^{\frac{1}{2m}} R) = 0$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{a}{J_\beta(\lambda^{\frac{1}{2m}} R)} \left(\frac{R}{r} \right)^\beta J_\beta(\lambda^{\frac{1}{2m}} r), \qquad \forall r = |x| \in (0, R].$$

3- Suppose $\mu \neq 0$, $a \in \mathbb{R}$, $b = 0$, $c_p = (-1)^p \lambda^{\frac{p}{m}} \left[a + \frac{\mu}{\lambda} \right]$, $\forall p \in \{1, ..., m-1\}$ and R satisfies $\lambda^{\frac{1}{2m}} R J'_\beta(\lambda^{\frac{1}{2m}} R) - \beta J_\beta(\lambda^{\frac{1}{2m}} R) = 0$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{\mu}{\lambda} \left[\frac{(\frac{\lambda a}{\mu} + 1)}{J_\beta(\lambda^{\frac{1}{2m}} R)} \left(\frac{R}{r} \right)^\beta J_\beta(\lambda^{\frac{1}{2m}} r) - 1 \right], \qquad \forall r = |x| \in (0, R].$$

Remark 1.1. Let $B(0, R)$ the ball in $\mathbb{R}^3$. Then, we have the following.

1- Suppose $\mu = 0$, $a = c_p = 0$, $b \neq 0$ and $R = k\pi\lambda^{-\frac{1}{2m}}, \forall k \in \mathbb{N}^*$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{(-1)^k k\pi b}{\lambda^{\frac{1}{2m}}} \frac{\sin(\lambda^{\frac{1}{2m}} r)}{\lambda^{\frac{1}{2m}} r}, \qquad \forall r = |x| \in (0, R].$$

2- Suppose $\mu = 0$, $a \neq 0$, $b = 0$, $c_p = (-1)^p \lambda^{\frac{p}{m}} a$, $\forall p \in \{1, ..., m-1\}$ and R satisfies $\tan(\lambda^{\frac{1}{2m}} R) = \lambda^{\frac{1}{2m}} R$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = a \frac{R}{r} \frac{\sin(\lambda^{\frac{1}{2m}} r)}{\sin(\lambda^{\frac{1}{2m}} R)}, \qquad \forall r = |x| \in (0, R].$$

3- Suppose $\mu \neq 0$, $a \in \mathbb{R}$, $b = 0$, $c_p = (-1)^p \lambda^{\frac{p}{m}} \left[a + \frac{\mu}{\lambda} \right]$, $\forall p \in \{1, ..., m-1\}$ and R satisfies $\tan(\lambda^{\frac{1}{2m}} R) = \lambda^{\frac{1}{2m}} R$. Then, the problem (P_m) admits a radial solution, given by :

$$u(x) = v(r) = \frac{\mu}{\lambda} \left[\left((\frac{\lambda a}{\mu} + 1) \frac{R}{r} \frac{\sin(\lambda^{\frac{1}{2m}} r)}{\sin(\lambda^{\frac{1}{2m}} R)} \right) - 1 \right], \qquad \forall r = |x| \in (0, R].$$

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We assume that the domain is a ball in $\mathbb{R}^N$ with $N \geq 2$ and for sake of simplicity, we consider the ball $B(0, R)$. As mentioned in the Introduction, radial solutions for problem (P_m) are constructed with the use of Bessel functions.

We recall that the Bessel functions , discovered by Bernoulli and used by Bessel as part of his studies of the movement of the planets induced by gravitational interaction, are canonical solutions $y(r)$ of the differential Bessel equation:

$$r^2 y''(r) + r y'(r) + (r^2 - \alpha^2) y(r) = 0, \tag{4}$$

for any real or complex number α. Most often, α is a natural integer (then called the order of the function), or a half-integer.

There are two kinds of Bessel functions:

- The Bessel functions of the first kind, denoted by J_n, solutions of the above differential equation which are defined at 0.
- The Bessel functions of the second kind, denoted by Y_n, solutions which are not defined in 0 (but which have an infinite limit in 0).

We also recall that for integer values $\alpha = n$, the Bessel functions of the first kind can be represented by integrals given by

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos(nt - x \sin t) \, \mathrm{d}t$$

or else by

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^\pi \mathrm{e}^{-\mathrm{i}\,(nt - x \sin t)} \, \mathrm{d}t.$$

For $\alpha = n + \frac{1}{2}$, the Bessel functions of the first kind given by

$$J_{n+\frac{1}{2}}(x) = (-1)^n \sqrt{\frac{2x}{\pi}} \, x^n \left(\frac{d}{x dx}\right)^n \left\{\frac{\sin x}{x}\right\}, \quad \forall x \in \mathbb{R}^+. \tag{5}$$

This definition can extend to the non-integer case of α (for $Re(x) > 0$), by adding another term

$$J_\alpha(x) = \frac{1}{\pi} \int_0^\pi \cos(\alpha t - x \sin t) \, \mathrm{d}t - \frac{\sin(\alpha \pi)}{\pi} \int_0^\infty \mathrm{e}^{-x \sinh(t) - \alpha t} \, \mathrm{d}dt.$$

Here, we consider the Bessel functions of the first kind and we denote them by J_β, where $\beta = \frac{N-2}{2}$ ($N \geq 2$) is an integer or half-integer.

Now, writing $y(r) = J_\beta(\lambda^{\frac{1}{2m}} r)$ with $r \in (0, R]$ and taking $\alpha = \beta$ in the equation (4), we are led to the equation

$$J_\beta''(\lambda^{\frac{1}{2m}} r) + \frac{1}{\lambda^{\frac{1}{2m}} r} J_\beta'(\lambda^{\frac{1}{2m}} r) + \left(1 - \frac{\beta^2}{\lambda^{\frac{1}{m}} r^2}\right) J_\beta(\lambda^{\frac{1}{2m}} r) = 0, \quad \forall r \in (0, R]. \tag{6}$$

Proof of Theorem 1.2. 1- Let $\mu = 0, b \neq 0$ and suppose that R satisfies $J_\beta(\lambda^{\frac{1}{2m}} R) = 0$. Consider the radial function

$$u(x) = v(r) = \frac{b}{\lambda^{\frac{1}{2m}} J_\beta'(\lambda^{\frac{1}{2m}} R)} \left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}} r), \qquad \forall r = |x| \in (0, R]. \tag{7}$$

We have

$$v'(r) = -\frac{b\beta R^\beta}{\lambda^{\frac{1}{2m}} J_\beta'(\lambda^{\frac{1}{2m}} R)} \, r^{-\beta - 1} J_\beta(\lambda^{\frac{1}{2m}} r) + \frac{bR^\beta}{J_\beta'(\lambda^{\frac{1}{2m}} R)} \, r^{-\beta} J_\beta'(\lambda^{\frac{1}{2m}} r).$$

Using (6), we have

$$v''(r) = -(2\beta+1)r^{-1}\left[-\frac{b\beta R^\beta}{\lambda^{\frac{1}{2m}}J'_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r) + \frac{bR^\beta}{J'_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J'_\beta(\lambda^{\frac{1}{2m}}r)\right]$$

$$-\frac{\lambda^{\frac{1}{m}}b}{\lambda^{\frac{1}{2m}}J'_\beta(\lambda^{\frac{1}{2m}}R)}\left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}}r) = -(2\beta+1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}v(r)$$

$$= -(N-1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}v(r).$$

Then

$$-\Delta u(x) = -\left[\frac{N-1}{r}\,v'(r) + v''(r)\right] = \lambda^{\frac{1}{m}}v(r) = \lambda^{\frac{1}{m}}u(x).$$

By induction, we obtain

$$(-1)^p\Delta^p u(x) = \lambda^{\frac{p}{m}}u(x), \quad \forall p \in \{1, ..., m\}.$$

In particular for $p = m$, we have

$$(-1)^m\Delta^m u(x) = \lambda u(x), \quad \forall m \geq 1.$$

Then $u(x) = v(r)$ given by (7) is a solution of the equation

$$(-1)^m\Delta^m u = \lambda u \quad \text{on } B(0, R).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=R} = \frac{1}{\lambda^{\frac{1}{2m}}J'_\beta(\lambda^{\frac{1}{2m}}R)}J_\beta(\lambda^{\frac{1}{2m}}R) = 0,$$

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\cdot\vec{\nu}(x) = \frac{1}{r}v'(r)\,\vec{x}\cdot\vec{\nu}(x)\Big|_{r=R}$$

$$= -\frac{b\beta R^\beta}{\lambda^{\frac{1}{2m}}J'_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r) + \frac{bR^\beta}{J'_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J'_\beta(\lambda^{\frac{1}{2m}}r))\Big|_{r=R} = b$$

and

$$\Delta u(x) = (-1)^1\lambda^{\frac{1}{m}}u(x) = 0.$$

By induction, for $m \geq 2$ we obtain

$$\Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}u(x), \quad \forall p \in \{1, ..., m-1\},$$

then

$$c_p = \Delta^p u(x) = (-1)^p\lambda^{\frac{p}{m}}u(x) = 0, \quad \forall p \in \{1, ..., m-1\}.$$

2- Let $\mu = 0$, $a \neq 0$ and suppose that R satisfies $\lambda^{\frac{1}{2m}}RJ'_\beta(\lambda^{\frac{1}{2m}}R) - \beta J_\beta(\lambda^{\frac{1}{2m}}R) = 0$. Consider the radial function

$$u(x) := v(r) = \frac{a}{J_\beta(\lambda^{\frac{1}{2m}}R)}\left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}}r), \qquad \forall r = |x| \in (0, R]. \qquad (8)$$

We have

$$v'(r) = -\frac{a\beta R^\beta}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r) + \frac{a\lambda^{\frac{1}{2m}}R^\beta}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J'_\beta(\lambda^{\frac{1}{2m}}r).$$

Using (6), we have

$$v^{''}(r) = -(2\beta+1)r^{-1}\left[-\frac{a\beta R^\beta}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r) + \frac{a\,\lambda^{\frac{1}{2m}}\,R^\beta}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J_\beta^{'}(\lambda^{\frac{1}{2m}}r)\right]$$

$$-\lambda^{\frac{1}{m}}\frac{a}{J_\beta(\lambda^{\frac{1}{2m}}R)}\left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}}r) = -(2\beta+1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}v(r)$$

$$= -(N-1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}v(r).$$

Then

$$-\Delta u(x) = -\left[\frac{N-1}{r}\,v'(r) + v''(r)\right] = \lambda^{\frac{1}{m}}v(r) = \lambda^{\frac{1}{m}}u(x).$$

By induction, we obtain

$$(-1)^p\Delta^p u(x) = \lambda^{\frac{p}{m}}u(x), \quad \forall p \in \{1,...,m\}.$$

In particular for $p = m$, we have

$$(-1)^m\Delta^m u(x) = \lambda u(x), \quad \forall m \geq 1.$$

Then $u(x) = v(r)$ given by (8) is a solution of the equation

$$(-1)^m\Delta^m u = \lambda u \quad \text{on } B(0,R).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=R} = a,$$

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\cdot\vec{\nu}(x) = \frac{1}{r}\,v'(r)\,\vec{x}\cdot\vec{\nu}(x)\Big|_{r=R}$$

$$= -\frac{a}{R\,J_\beta(\lambda^{\frac{1}{2m}}R)}\left(\lambda^{\frac{1}{2m}}R\,J_\beta^{'}(\lambda^{\frac{1}{2m}}R) - \beta\,J_\beta(\lambda^{\frac{1}{2m}}R)\right) = 0$$

and

$$\Delta u(x) = (-1)^1\lambda^{\frac{1}{m}}u(x) = (-1)^1\lambda^{\frac{1}{m}}a.$$

By induction, for $m \geq 2$ we obtain

$$\Delta^P u(x) = (-1)^P\lambda^{\frac{P}{m}}u(x), \quad \forall p \in \{1,...,m-1\},$$

then

$$c_p = \Delta^P u(x) = (-1)^P\lambda^{\frac{P}{m}}u(x) = (-1)^P\lambda^{\frac{P}{m}}a, \quad \forall p \in \{1,...,m-1\}.$$

3 - Let $\mu \neq 0$, $a \in \mathbb{R}$ and suppose that R satisfies $\lambda^{\frac{1}{2m}}RJ_\beta^{'}(\lambda^{\frac{1}{2m}}R) - \beta J_\beta(\lambda^{\frac{1}{2m}}R) = 0$. Consider the radial function

$$u(x) = v(r) = \frac{\mu}{\lambda}\left[\frac{(\frac{\lambda a}{\mu}+1)}{J_\beta(\lambda^{\frac{1}{2m}}R)}\left(\frac{R}{r}\right)^\beta J_\beta(\lambda^{\frac{1}{2m}}r) - 1\right], \qquad \forall r = |x| \in (0,R]. \quad (9)$$

We have

$$v^{'}(r) = \frac{\mu}{\lambda}\left[\frac{-(\frac{\lambda a}{\mu}+1)\beta R^\beta}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta-1}J_\beta(\lambda^{\frac{1}{2m}}r) + \frac{(\frac{\lambda a}{\mu}+1)\lambda^{\frac{1}{2m}}R^\beta}{J_\beta(\lambda^{\frac{1}{2m}}R)}\,r^{-\beta}J_\beta^{'}(\lambda^{\frac{1}{2m}}r)\right].$$

Using (6), we obtain

$$v''(r) = -(2\beta+1)r^{-1}\frac{\mu}{\lambda}\left[\frac{-(\frac{\lambda a}{\mu}+1)\beta R^{\beta}}{J_{\beta}(\lambda^{\frac{1}{2m}}R)}\, r^{-\beta-1}J_{\beta}(\lambda^{\frac{1}{2m}}r) + \frac{(\frac{\lambda a}{\mu}+1)\lambda^{\frac{1}{2m}}R^{\beta}}{J_{\beta}(\lambda^{\frac{1}{2m}}R)}\, r^{-\beta}J'_{\beta}(\lambda^{\frac{1}{2m}}r)\right]$$

$$-\lambda^{\frac{1}{m}}\left[\frac{\mu}{\lambda}\frac{(\frac{\lambda a}{\mu}+1)}{J_{\beta}(\lambda^{\frac{1}{2m}}R)}\left(\frac{R}{r}\right)^{\beta}J_{\beta}(\lambda^{\frac{1}{2m}}r)\right] = -(2\beta+1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}\left[v(r)+\frac{\mu}{\lambda}\right]$$

$$= -(N-1)r^{-1}v'(r) - \lambda^{\frac{1}{m}}\left[v(r)+\frac{\mu}{\lambda}\right].$$

Thus

$$-\Delta u(x) = \left[\frac{N-1}{r}v'(r)+v''(r)\right] = \lambda^{\frac{1}{m}}\left[v(r)+\frac{\mu}{\lambda}\right] = \lambda^{\frac{1}{m}}\left[u(x)+\frac{\mu}{\lambda}\right].$$

By induction, we obtain

$$(-1)^{p}\Delta^{p}u(x) = \lambda^{\frac{p}{m}}\left[u(x)+\frac{\mu}{\lambda}\right], \quad \forall p \in \{1,...,m\}.$$

In particular for $p=m$, we have

$$(-1)^{m}\Delta^{m}u(x) = \lambda u(x) + \mu, \quad \forall m \geq 1.$$

Then $u(x) = v(r)$ given by (9) is a solution of the equation

$$(-1)^{m}\Delta^{m}u = \lambda u + \mu \quad \text{on } B(0,R).$$

Moreover, on ∂B, we have

$$u(x) = v(r)\Big|_{r=R} = a,$$

$$\frac{\partial u}{\partial \nu}(x) = \nabla u(x)\cdot\vec{\nu}(x) = \frac{1}{r}v'(r)\,\vec{x}\cdot\vec{\nu}(x)\Big|_{r=R}$$

$$= \frac{\mu(\frac{\lambda a}{\mu}+1)}{\lambda J_{\beta}(\lambda^{\frac{1}{2m}}R)}\left(\lambda^{\frac{1}{2m}}R\,J'_{\beta}(\lambda^{\frac{1}{2m}}R) - \beta\,J_{\beta}(\lambda^{\frac{1}{2m}}R)\right) = 0$$

and

$$\Delta u(x) = -\lambda^{\frac{1}{m}}\left[u(x)+\frac{\mu}{\lambda}\right] = -\lambda^{\frac{1}{m}}\left[a+\frac{\mu}{\lambda}\right].$$

By induction, for $m \geq 2$ we obtain

$$\Delta^{p}u(x) = (-1)^{p}\lambda^{\frac{p}{m}}\left[u(x)+\frac{\mu}{\lambda}\right], \quad \forall p \in \{1,...,m-1\},$$

and so

$$c_{p} = \Delta^{p}u(x) = (-1)^{p}\lambda^{\frac{p}{m}}\left[a+\frac{\mu}{\lambda}\right], \quad \forall p \in \{1,...,m-1\}.$$

$\square$

Acknowledgment

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