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Existence and localization results for semi-linear problems

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Abstract. This survey paper presents the new method worked out in [14] and [15] for the existence and localization of solutions to evolution operator equations, which is based on Krasnoselskii's compression-expansion fixed point theorem in cones. The main idea is to handle two equivalent operator forms of the equation, one of fixed point type giving the operator to which Krasnoselskii's theorem applies and an other one of coincidence type which is used to localize a positive solution in a shell. Applications are presented for a boundary value problem associated to a fourth order partial differential equation on a rectangular domain and for nonlinear wave equations.

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1. Introduction

Krasnoselskii's compression-expansion fixed point theorem in cones [6] is one of the most useful results of nonlinear functional analysis, for the investigation of the existence, localization and multiplicity of nonnegative solutions to two-point boundary value problems. Such applications can be found in [1-3], [5-8] and [10-13]. All these applications are based on upper and lower inequalities for the appropriate Green's functions. Similar inequalities for boundary value problems related to partial differential equations are not known and Krasnoselskii's Theorem has appeared quite unapplicable to this type of problems. Recently, in [14] and [15], we have proposed a method of application of Krasnoselskii's Theorem to some classes of nonlinear operator equations making Krasnoselskii's Theorem applicable to several problems involving partial differential equations.

Let us recall Krasnoselskii's compression-expansion fixed point theorem in the form given in [4, p 325].

Theorem 1.1 (Krasnoselskii)**.** *Let $(E, |.|)$ be a normed linear space, $C \subset E$ a proper wedge and $N : C \to C$ a completely continuous map. Assume that for some numbers ρ and R with $0 < \rho < R$, one of the following conditions is satisfied:*
(a) $|N(x)| \leq |x|$ *for* $|x| = \rho$ *and* $|N(x)| \geq |x|$ *for* $|x| = R$;
(b) $|N(x)| \geq |x|$ *for* $|x| = \rho$ *and* $|N(x)| \leq |x|$ *for* $|x| = R$.
Then N has a fixed point x with $\rho \leq |x| \leq R$.

In this paper we shall work in the space $E = C([0,T]; X)$. Here $0 < T < \infty$ and X is a Banach space with norm $|.|_X$. We denote by $|.|_{\infty,X}$ the norm on $C([0,T]; X)$ defined by $|u|_{\infty,X} = \max_{t\in[0,T]} |u(t)|_X$.

We also use the notation $|.|_p$, or $|.|_{L^p[a,b]}$, for the norm of $L^p[a,b]$ $(1 \leq p \leq \infty)$.

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2. Existence and Localization Results

Let V, W be Banach spaces, $A : D(A) \subset C([0,T];V) \to C([0,T];W)$ be a linear map, $B : V \to W$ a linear continuous map and $F : C([0,T];V) \to C([0,T];W)$ be a nonlinear map. We discuss the operator equation

$$\begin{cases} (Au)(t) - Bu(t) = F(u)(t), & t \in [0,T] \\ u \in D(A) \subset C([0,T];V) \end{cases} \tag{2.1}$$

which is seen as

$$(A - B)u = F(u), \quad u \in D(A)$$

for short. Here for a function $u \in C([0,T];V)$, by Bu we shall mean the function $(Bu)(t) = Bu(t)$, $t \in [0,T]$. Obviously, if the operator $A - B$ from $D(A)$ to $C([0,T];W)$ is invertible, then (2.1) is equivalent to the fixed point problem

$$u = (A - B)^{-1} F(u), \quad u \in C([0,T];V). \tag{2.2}$$

Assume in addition that $V \subset W$ with continuous injection, A and B are invertible and $A^{-1}B^{-1} = B^{-1}A^{-1}$ on $C([0,T];W)$. Here again, for $u \in C([0,T];W)$, by $B^{-1}u$ we mean the function in $C([0,T];V)$ defined as $\left(B^{-1}u\right)(t) = B^{-1}u(t)$ for $t \in [0,T]$. Notice that since $V \subset W$ with continuous injection, any function in $C([0,T];V)$ also belongs to $C([0,T];W)$ making possible the compositions $A^{-1}B^{-1}$ and $B^{-1}A^{-1}$. Now (2.1) is equivalent to the coincidence equation

$$\left(B^{-1} - A^{-1}\right) u = A^{-1}B^{-1}F(u), \quad u \in C([0,T];V) \tag{2.3}$$

since

$$(A - B)^{-1} = \left(B^{-1} - A^{-1}\right)^{-1} A^{-1}B^{-1} \quad \text{on } C([0,T];W).$$

Indeed, if $v \in C([0,T];W)$ and we denote $u = (A - B)^{-1} v$, then $Au - Bu = v$. It follows successively that $B^{-1}Au - u = B^{-1}v$ and $A^{-1}B^{-1}Au - A^{-1}u = A^{-1}B^{-1}v$. Hence $B^{-1}A^{-1}Au - A^{-1}u = A^{-1}B^{-1}v$, that is $\left(B^{-1} - A^{-1}\right) u = A^{-1}B^{-1}v$. Therefore $u = \left(B^{-1} - A^{-1}\right)^{-1} A^{-1}B^{-1}v$.

Equation (2.2) gives us the operator $(A - B)^{-1} F$ to which Krasnoselskii's Theorem applies, while (2.3) is used to localize a positive solution in a shell.

Our assumptions are as follows:

(h1) there exist two Banach spaces $(X, |.|_X)$, $(Y, |.|_Y)$ such that $V \subset X \subset W$, $V \subset Y$, the injection of V in X is compact and the injections of X in W and of V in Y are continuous. Also W is ordered by the closed positive cone W_+, $V \cap W_+ \neq \{0\}$ and $|.|_Y$ is monotone on V with respect to the order relation induced in V by the cone $V \cap W_+$.

(h2) A is invertible and its inverse A^{-1} has the representation

$$\left(A^{-1}v\right)(t) = \int_0^T k(t,s)\, v(s)\, ds, \quad v \in C([0,T];W)$$

where $k : [0,T]^2 \to \mathbf{R}_+$ is such that $k(t,.) \in L^1[0,T]$ for each $t \in [0,T]$ and the map $t \mapsto k(t,.)$ is continuous from $[0,T]$ to $L^1[0,T]$.

(h3) B is continuous from V to W, invertible and the linear map $B^{-1} : W \to V$ is positive (i.e. $B^{-1}(W_+) \subset W_+$) or negative (i.e. $-B^{-1}$ is positive). Also $A^{-1}B^{-1} = B^{-1}A^{-1}$ on $C([0,T];W)$.

(h4) the linear map $(A - B)^{-1}$ is continuous from $C([0,T];W)$ in $C([0,T];V)$, and compact from $C([0,T];W)$ in $C([0,T];W)$.

(h5) the map $F : C \to C([0,T];W_+)$ is continuous and sends bounded sets into bounded sets, when C is endowed with the topology of $C([0,T];X)$. Here C is the cone of $C([0,T];V)$

$$C = (A-B)^{-1} C([0,T];W_+).$$

(h6) there exists $\alpha > 0$ such that

$$\left|B^{-1}F(u)\right|_{\infty,Y} \le \frac{\alpha}{\max_{t\in[0,T]} |\kappa(t,.)|_{L^1[0,T]}}$$

for every $u \in C$ with $\|u\| = \left|\left(B^{-1} - A^{-1}\right)u\right|_{\infty,Y} = \alpha$.

(h7) there exists an interval $[a,b] \subseteq [0,T]$, a map $\phi : C \to W_+$, a number $\beta > 0$, $\beta \ne \alpha$ and a point $t^* \in [0,T]$ such that

$$\phi(u) \le F(u)(t), \quad t \in [a,b] \quad \text{and}$$

$$\left|B^{-1}\phi(u)\right|_Y \ge \frac{\beta}{|k(t^*,.)|_{L^1[a,b]}}$$

for all $u \in C$ with $\|u\| = \beta$.

Theorem 2.1. *If the conditions* (h1)-(h7) *are satisfied, then* (2.1) *has at least one solution* u *with* $u \in C$ *and*

$$\min\{\alpha,\beta\} \le \|u\| \le \max\{\alpha,\beta\}.$$

Theorem 2.1 yields the following existence and localization result for the problem

$$\begin{cases} (Au)(t) - Bu(t) = f\left(\sigma\left(B^{-1} - A^{-1}\right)u(t)\right), & t \in [0,T] \\ u \in D(A) \subset C([0,T];V). \end{cases} \tag{2.4}$$

Theorem 2.2. *Assume that conditions* (h1)-(h4) *hold. In addition assume that there exists* $0 < M < 1$, $\kappa \in L^1[0,T]$ *and an interval* $[a,b] \subseteq [0,T]$, $a < b$, *such that*

$$\begin{aligned} k(t,s) &\le \kappa(s), \quad && t \in [0,T], \quad a.e.\ s \in [0,T] \\ M\kappa(s) &\le k(t,s), \quad && t \in [a,b], \quad a.e.\ s \in [0,T]. \end{aligned}$$

Let $f : V \cap W_+ \to W_+$ *be continuous from its domain with the topology of* V *to* W, *nondecreasing with respect to the order induced by* W_+, *and let* f *send bounded sets into bounded sets. In addition assume that the following conditions are satisfied:*

(i) *there exists* $\alpha > 0$ *such that*

$$\frac{\alpha}{\sup_{w\in B^{-1}(W_+),\ |w|_Y=\alpha} |B^{-1}f(\sigma w)|_Y} \ge \max_{t\in[0,T]} |\kappa(t,.)|_{L^1[0,T]};$$

(ii) *there exists* $\beta > 0$, $\beta \ne \alpha$ *and* $t^* \in [0,T]$ *such that*

$$\frac{\beta}{\inf_{w\in B^{-1}(W_+),\ |w|_Y=\beta} |B^{-1}f(\sigma M v)|_Y} \le |k(t^*,.)|_{L^1[a,b]}.$$

Then (2.4) *has at least one solution* u *with*

$$\min\{\alpha,\beta\} \le \|u\| \le \max\{\alpha,\beta\}$$

and

$$0 \le \sigma M\left(B^{-1} - A^{-1}\right)u(t) \le \sigma\left(B^{-1} - A^{-1}\right)u(t') \tag{2.5}$$

for all $t \in [0,T]$ *and* $t' \in [a,b]$. (*Inequalities* (2.5) *are understand with respect to the order induced by* W_+)

We note that the use of different spaces V, W, X and Y is convenient when treating partial differential equations in terms of weak solutions.

3. Applications

1. Consider the nonlinear problem associated to a fourth order partial differential equation:

$$\begin{cases} v_{ttxx} + g(v) = 0, & (t,x) \in [0,T] \times [0,h] \\ v = 0 \text{ for } t = 0, \ t = T, \ x = 0 \text{ and } x = h. \end{cases} \tag{3.1}$$

We use the convention that $v(t,x) = v(t)(x)$.

Let $C_0^2[0,h]$ be the space of all functions $u \in C^2[0,h]$ with $u(0) = u(h) = 0$, endowed with the C^2-norm. If we let A and B be the operators $Au = -\frac{d^2u}{dt^2}$ ($u \in D(A)$, $D(A) = \{u \in C^2([0,T]; C_0^2[0,h]) : u(0) = u(T) = 0\}$) and $Bu = -\frac{d^2u}{dx^2}$ $\left(u \in C_0^2[0,h]\right)$ and we make the substitution $v = \left(B^{-1} - A^{-1}\right)u$, we can write (3.1) under the form

$$\begin{cases} u_{tt} - u_{xx} = g\left(\left(B^{-1} - A^{-1}\right)u(t)\right) & \text{on } [0,T] \\ u(0) = u(T) = 0 \\ u \in C^2\left([0,T]; C_0^2[0,h]\right). \end{cases} \tag{3.2}$$

In this case B^{-1} is positive. Problem (3.2) is of type (2.4) with $\sigma = 1$ and $f(u)(t) = g(u(t))$. This choice of B is suitable to guarantee the existence of a positive solution to (3.1).

Also, if A is the above operator while $Bu = +\frac{d^2u}{dx^2}$, then (3.1) is equivalent to

$$\begin{cases} u_{tt} + u_{xx} = -g\left(\left(B^{-1} - A^{-1}\right)u(t)\right) & \text{on } [0,T] \\ u(0) = u(T) = 0 \\ u \in C^2\left([0,T]; C_0^2[0,h]\right). \end{cases} \tag{3.3}$$

This time, B^{-1} is negative. Problem (3.3) is of type (2.4) with $\sigma = -1$ and $f(u)(t) = -g(-u(t))$. In this approach we shall be able to guarantee the existence of negative solutions for (3.1).

Before we state two existence and localization results for positive and respectively negative solutions to (3.1), we introduce some notations.

Here the kernel k of A^{-1} is the Green function corresponding to the operator $-\frac{d^2}{dt^2}$, the interval $[0,T]$ and the boundary condition $u(0) = u(T) = 0$, i.e.

$$k(t,s) = \begin{cases} \frac{s(T-t)}{T}, & 0 \le s \le t \le T \\ \frac{t(T-s)}{T}, & 0 \le t \le s \le T. \end{cases}$$

To be more precise we shall denote this function by k^T. Notice that for every subinterval $[a,b]$ of $[0,T]$, $0 < a < b < T$, k^T satisfies the following upper and lower inequalities

$$\begin{aligned} k^T(t,s) &\le k^T(s,s) \quad \text{for } t \in [0,T] \text{ and } s \in [0,T] \\ M_{a,b}^T k^T(s,s) &\le k^T(t,s) \quad \text{for } t \in [a,b] \text{ and } s \in [0,T]. \end{aligned}$$

Here

$$M_{a,b}^T = \min\left\{\frac{a}{T}, \frac{T-b}{T}\right\}.$$

We shall use the notation $|u|_\infty$ to denote the norm of $C[0,h]$ and $\|u\|$ for the norm $\max_{t\in[0,T]} |u(t)|_\infty$ on $C([0,T]; C[0,h])$.

Positive solutions to (3.1) are guaranteed by the following result.

Theorem 3.1. *Let $g : \mathbf{R}_+ \to \mathbf{R}_+$ be continuous and nondecreasing. Assume that there exists $\alpha > 0$, $\beta > 0$ with $\beta \neq \alpha$, two intervals $[a, b]$, $[a_0, b_0]$ with $0 < a < b < T$, $0 < a_0 < b_0 < h$ and two points $t^* \in [0, T]$, $x^* \in [0, h]$ such that*

$$\frac{(T\,h)^2}{48} \leq \frac{\alpha}{g(\alpha)}$$

and

$$\frac{\beta}{g\left(M^h_{a_0,b_0}\,M^T_{a,b}\,\beta\right)} \leq \left|k^h(x^*, .)\right|_{L^1[a_0,b_0]} \left|k^T(t^*, .)\right|_{L^1[a,b]}.$$

Then problem (3.1) *has at least one solution v with*

$$0 \leq M^T_{a,b} v(t) \leq v(t') \quad \textit{for all } t \in [0, T],\ t' \in [a, b]; \quad \textit{and}$$

$$\min\{\alpha, \beta\} \leq \|v\| \leq \max\{\alpha, \beta\}.$$

Proof. Theorem 2.2 is used with $V = C^2_0[0, h]$, $W = X = Y = C[0, h]$, $W_+ = C([0, h]; \mathbf{R}_+)$ and $f(u)(x) = g(u(x))$. □

Similarly, negative solutions to (3.1) are guaranteed by the following result.

Theorem 3.2. *Let $g : \mathbf{R}_- \to \mathbf{R}_-$ be continuous and nondecreasing. Assume that there exists $\alpha > 0$, $\beta > 0$ with $\beta \neq \alpha$, two intervals $[a, b]$, $[a_0, b_0]$ with $0 < a < b < T$, $0 < a_0 < b_0 < h$ and two points $t^* \in [0, T]$, $x^* \in [0, h]$ such that*

$$\frac{(T\,h)^2}{48} \leq \frac{-\alpha}{g(-\alpha)}$$

and

$$\frac{-\beta}{g\left(-M^h_{a_0,b_0}\,M^T_{a,b}\,\beta\right)} \leq \left|k^h(x^*, .)\right|_{L^1[a_0,b_0]} \left|k^T(t^*, .)\right|_{L^1[a,b]}.$$

Then problem (3.1) *has at least one solution v with*

$$v(t') \leq M^T_{a,b} v(t) \leq 0 \quad \textit{for all } t \in [0, T],\ t' \in [a, b]; \quad \textit{and}$$

$$\min\{\alpha, \beta\} \leq \|v\| \leq \max\{\alpha, \beta\}.$$

Notice that multiple (positive and negative) solutions to problem (3.1) are guaranteed by Theorems 3.1 and 3.2 if the assumptions are satisfied for several disjoint intervals $[\alpha, \beta]$ (or $[\beta, \alpha]$).

2. Consider the boundary value problem for the nonlinear wave equation

$$\begin{cases} -u''(t) + \Delta u(t) - mu(t) = F(u)(t), \quad t \in [0, T] \\ u(0) = u(T) = 0 \\ u \in C\left([0, T]; H^1_0(\Omega)\right) \cap C^2\left([0, T]; H^{-1}(\Omega)\right). \end{cases} \tag{3.4}$$

Here $0 < T < \infty$, $\Omega \subset \mathbf{R}^n$ is a bounded open subset, $m > -\lambda_1$ (λ_1 is the first eigenvalue corresponding to $-\Delta$ and to the homogenous Dirichlet boundary condition) and F is a map from $C\left([0, T]; H^1_0(\Omega)\right)$ to $C\left([0, T]; H^{-1}(\Omega)\right)$.

Let $V = H^1_0(\Omega)$, $W = H^{-1}(\Omega)$, $A : D(A) \to C\left([0, T]; H^{-1}(\Omega)\right)$ be given by

$$(Au)(t) = -u''(t)$$

where $D(A) = \left\{u \in C^2\left([0, T]; H^{-1}(\Omega)\right) : u(0) = u(T) = 0\right\}$,and let $B : H^1_0(\Omega) \to H^{-1}(\Omega)$ be defined by

$$Bu = -\Delta u + m\,u, \quad u \in H^1_0(\Omega).$$

Since $m > -\lambda_1$, B is invertible and its inverse B^{-1} is a linear continuous and positive (by the maximum principle) operator.

Basic theory on the non-homogenous linear wave equation (see [9]) guarantees that the operator $A - B$ from $C([0,T];\, H_0^1(\Omega)) \cap D(A)$ to $C([0,T];\, H^{-1}(\Omega))$ is invertible and its inverse $(A-B)^{-1}$ is a linear operator, completely continuous from $C\left([0,T];H^{-1}(\Omega)\right)$ to $C([0,T];\, L^p(\Omega))$ for $(2^*)' \le p < 2^*$ if $n \ge 3$ and any $p \ge 1$ if $n = 1$ or $n = 2$. Also $A^{-1}B^{-1} = B^{-1}A^{-1}$ on $C\left([0,T];H^{-1}(\Omega)\right)$.

Under suitable conditions on F, the complete continuity of $(A-B)^{-1}$ implies that the nonlinear operator $N := (A-B)^{-1} F$ is completely continuous.

We have the following result on the existence and the localization of a solution of (3.4).

Theorem 3.3. *Let* $(2^*)' \le p < 2^*$, $1 \le q \le 2^*$ *if* $n \ge 3$ *and* $p \ge 1$, $q \ge 1$ *if* $n = 1$ *or* $n = 2$. *Let* C *be the cone of* $C([0,T];L^p(\Omega))$ *given by*

$$C = \left\{ u \in C([0,T];L^p(\Omega)) : u = (A-B)^{-1} v,\ v \in C\left([0,T];H^{-1}(\Omega;\mathbf{R}_+)\right) \right\}.$$

1^0 *Assume that the following two conditions are satisfied:*

(h1) $F : C \to C\left([0,T];H^{-1}(\Omega;\mathbf{R}_+)\right)$ *is continuous and sends bounded sets into bounded sets,*

(h2) *there exists* $\alpha > 0$ *such that*

$$\left|B^{-1}F(u)(t)\right|_q \le \frac{6\alpha}{T^2}, \ \ t \in [0,T]$$

for every $u \in C$ *with* $\|u\| := \left|\left(B^{-1} - A^{-1}\right)u\right|_{\infty,q} = \alpha$.

Then (3.4) *has at least one solution* $u \in C$ *with* $\|u\| \le \alpha$.

2^0 *Assume that* (h1), (h2) *and the following additional condition are satisfied:*

(h3) *there exists an interval* $[a,b]$ *with* $0 < a < b < T$, *a map* $\phi : C \to H^{-1}(\Omega;\mathbf{R}_+)$ *and a number* $\beta > 0$, $\beta \ne \alpha$, *such that*

$$\phi(u) \le F(u)(t), \ \ t \in [a,b]$$

and

$$\left|B^{-1}\phi(u)\right|_q \ge \frac{\beta}{g^*_{a,b}}$$

for all $u \in C$ *with* $\|u\| = \beta$.

Then (3.4) *has at least one solution* $u \in C$ *with*

$$\min\{\alpha,\beta\} \le \|u\| \le \max\{\alpha,\beta\}.$$

Proof. The result follows from Theorem 2.1 where $X = L^p(\Omega)$, $Y = L^q(\Omega)$ and $N = (A-B)^{-1} F$. □

We can specialize Theorem 3.3 to discuss the existence and the localization of solutions for nonlinear problems of the form

$$\begin{cases} -u''(t) + \Delta u(t) - mu(t) = h\left(\left|\left(B^{-1} - A^{-1}\right)u(t)\right|_q\right) f(t,u(t)) \ \text{ on } [0,T] \\ u(0) = u(T) = 0 \\ u \in C\left([0,T];H_0^1(\Omega)\right) \cap C^2\left([0,T];H^{-1}(\Omega)\right). \end{cases} \tag{3.5}$$

Theorem 3.4. *Let* $h : \mathbf{R}_+ \to \mathbf{R}_+$ *be continuous and nondecreasing and* $f : [0,T] \times \mathbf{R} \to \mathbf{R}_+$ *be continuous and*

$$f(t,u) \le c + d|u|^\gamma, \ \ t \in [0,T], \ u \in \mathbf{R}$$

for some $c, d > 0$, $1 \leq \gamma < 2^* - 1 = \frac{n+2}{n-2}$ *if* $n \geq 3$ *and* $1 \leq \gamma < \infty$ *if* $n = 1$ *or* $n = 2$. *Let* $\left|B^{-1}\right|$ *denotes the norm of operator* B^{-1} *from* $L^r(\Omega)$ *to* $L^q(\Omega)$ *where* $r = (2^*)'$, $q = 2^*$ *if* $n \geq 3$ *and* $r \geq 1$, $q \geq 1$ *in case that* $n = 1$ *or* $n = 2$. *Denote* $c^* = c\,|1|_r$ *and let* $p = \gamma r$. *Assume that there exists* $\alpha > 0$ *such that*

$$\left|B^{-1}\right| \left(c^* + d c_0^{-\gamma} \alpha^\gamma\right) \frac{T^2}{6} \leq \frac{\alpha}{h(\alpha)}.$$

Then problem (3.5) *has at least one solution* u *with*

$$\|u\| = \left|\left(B^{-1} - A^{-1}\right) u\right|_{\infty,q} \leq \alpha.$$

If in addition there exists an interval $[a,b]$ *with* $0 < a < b < T$, *a number* σ *with*

$$0 < \sigma \leq f(t,u), \quad t \in [a,b], \; u \in \mathbf{R}$$

and a number $\beta > 0$, $\beta \neq \alpha$, *such that*

$$\frac{\sigma\, g_{a,b}^* \left|\varphi_1\right|_q}{(\lambda_1 + m)\left|\varphi_1\right|_\infty} \geq \frac{\beta}{h(k_{a,b}\beta)}$$

then (3.5) *has at least one solution* u *with*

$$\min\{\alpha,\beta\} \leq \|u\| \leq \max\{\alpha,\beta\}.$$

Proof. The result follows from Theorem 3.3 with $p = \gamma (2^*)'$ and

$$F(u)(t) = h\left(\left|\left(B^{-1} - A^{-1}\right) u(t)\right|_q\right) f(t, u(t)).$$

□

Notice that multiple solutions to problem (3.5) are guaranteed by Theorem 3.4 if f and h satisfy all the assumptions for several disjoint intervals $[\alpha,\beta]$ (or $[\beta,\alpha]$).

In particular, for $h \equiv 1$, Theorem 3.4 yields the following existence and localization result for the problem

$$\begin{cases} -u''(t) + \Delta u(t) - m u(t) = f(t, u(t)), & t \in [0,T] \\ u(0) = u(T) = 0 \\ u \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^2\left([0,T]; H^{-1}(\Omega)\right). \end{cases} \tag{3.6}$$

Theorem 3.5. *Let* $f : [0,T] \times \mathbf{R} \to \mathbf{R}_+$ *be continuous and*

$$f(t,u) \leq c + d\,|u|^\gamma, \quad t \in [0,T], \; u \in \mathbf{R}$$

for some $c, d > 0$, $1 \leq \gamma < 2^* - 1$ *if* $n \geq 3$ *and* $1 \leq \gamma < \infty$ *if* $n = 1$ *or* $n = 2$. *Let* $\left|B^{-1}\right|$ *denotes the norm of operator* B^{-1} *from* $L^r(\Omega)$ *to* $L^q(\Omega)$ *where* $r = (2^*)'$, $q = 2^*$ *if* $n \geq 3$ *and* $r \geq 1$, $q \geq 1$ *in case that* $n = 1$ *or* $n = 2$. *Let* $c^* = c\,|1|_r$ *and* $p = \gamma r$. *Assume that there exists* $\alpha > 0$ *such that*

$$\left|B^{-1}\right| \left(c^* + d c_0^{-\gamma} \alpha^\gamma\right) \frac{T^2}{6} \leq \alpha.$$

Then problem (3.6) *has at least one solution* u *with*

$$\|u\| = \left|\left(B^{-1} - A^{-1}\right) u\right|_{\infty,q} \leq \alpha.$$

If in addition there exists an interval $[a,b]$ *with* $0 < a < b < T$ *and a number* σ *with*

$$0 < \sigma \leq f(t,u), \quad t \in [a,b], \; u \in \mathbf{R},$$

then $\|u\| \geq \beta$, *where*

$$\beta := \frac{\sigma\, g_{a,b}^* \left|\varphi_1\right|_q}{(\lambda_1 + m)\left|\varphi_1\right|_\infty}.$$

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