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Existence and Asymptotic Behavior of a Nonlinear Axially Moving String with Variable Tension and Subject to Disturbances

Abdelkarim Kelleche and Athmane Abdallaoui

Abstract. In this paper, we consider the stabilization question for a nonlinear model of an axially moving string. The model is assumed to undergo the variable tension and variable disturbances. The Hamilton principle is used to describe the dynamic of transverse vibrations. We establish the well-posedness by means of the Faedo–Galerkin method. A boundary control with a time-varying delay is designed to stabilize uniformly the string. Then, we derive a decay rate of the solution assuming that the retarded term be dominated by the damping one. Some examples are given to clarify when the rate is exponential or polynomial.

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1. Introduction

For a variety of reasons, the vibration problem for axially moving systems is considered to be a major factor that menaces their good functioning and their life duration. The importance of these systems is due to their employment in different engineering applications. Their use has increased rapidly these last decades. This development has forced the researchers to explore more efficient techniques to attenuate or to reduce these effects. Several methods and approaches have been adopted to reach a typical model that describes the vibration behavior and dynamic characteristics of an axially moving string. Many researchers have investigated in this area to control these devices. This led to the emergence of several studies in this direction, Fung and Tseng in [4] showed the exponential stability of a linear model of an axially moving string using a feedback comprising the displacement, the velocity and the slope of the string at one of the endpoints. Fung et *al.* in [5] proved that the system is exponentially stable by employing a nonlinear feedback boundary controller including an MDS controller. Several others results concerned with nonlinear models have been investigated, see in [22, 23, 24]. The stabilisation using a boundary control of memory type has been investigated recently in [7] where the control leads to the same result compared to other types of control.

Model and mathematical formulation

Consider an axially moving string with a constant velocity c in the direction of its axis, as illustrated in Fig. 1, with length l and density ρ. Let t be the time, x be the spatial coordinate along the longitude of motion. The transverse displacement is

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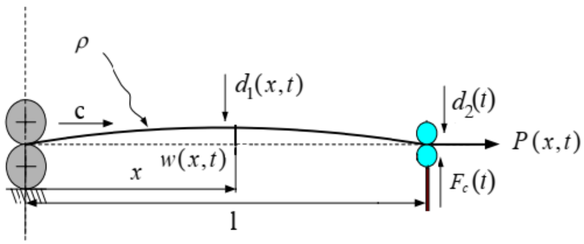

FIGURE 1. An axially moving string under boundary control force

indicated by $w(x,t)$. The string at the left boundary is assumed fixed, i.e., fixed in the sense that there is no vertical movement, but it allows the string to move in the horizontal direction. The tension P in the string is assumed to be a spatiotemporally variable function. This is resulted from external disturbances and/or gravity, etc. The string is subject to an external disturbance force $d_1(x,t)$ along the string and an external disturbance force $d_2(t)$ at the right boundary. A control force denoted by $F_c(t)$ is located at the right boundary to suppress the transverse vibrations of the string. We denote by $w_0(x)$, $w_1(x)$, respectively the initial displacement and the initial velocity of the string.

The governing equation of the transverse displacement and the boundary conditions of the controlled string can be derived applying the Hamilton's principle. We recall the generalized Hamilton's principle formula (see [19])

$$\delta \int_{t_0}^{t_1} (E_c - E_p + W) = 0 \tag{1}$$

where δ denotes the variational operator, t_0, t_1 are two time instants, $t_0 < t < t_1$ is the operating interval. E_c and E_p are the kinetic energy and the potential energy of the system, respectively. Their expressions are given by

$$E_c = \frac{1}{2} \int_0^l \rho \left(w_t + c w_x\right)^2 dx, \ t \geq 0 \tag{2}$$

and

$$E_p = \frac{1}{2} \int_0^l P(x,t) w_x^2 dx, \ t \geq 0. \tag{3}$$

The term W is the virtual work performed by the external forces

$$\delta W = \left[F_c(t) - d_2(t)\right] \delta w(l,t) + \int_0^l d_1(x,t) \delta w dx, \ t \geq 0. \tag{4}$$

The variation of the expressions (2) and (3) is given by

$$\delta E_c = \int_0^l \rho \left(w_t + c w_x\right) \delta \left(w_t + c w_x\right) dx, \ t \geq 0 \tag{5}$$

and

$$\delta E_p = \int_0^l P(x,t) w_x \delta w_x dx, \ t \geq 0. \tag{6}$$

Before proceeding to evaluate the Hamilton's principle formula (1), we define as in [28] and [21] the total derivative operator with respect to time by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} = (.)_t + c(.)_x. \tag{7}$$

The substitution of (4)–(6) into (1) gives

$$\begin{aligned} &\int_{t_0}^{t_1}\int_0^l \left[\rho\left(w_t + cw_x\right)\delta\left(w_t + cw_x\right)dx - P(x,t)w_x\delta w_x\right]dxdt \\ &\quad + \int_0^l d_1(x,t)\delta w dx + \int_{t_0}^{t_1}\left[F_c(t) - d_2(t)\right]\delta w(l,t)dt = 0. \end{aligned} \tag{8}$$

An integration by parts yields

$$\begin{aligned} &\rho\int_0^l \left[\left(w_t + cw_x\right)\delta w\right]_{t_0}^{t_1} dx - \rho\int_{t_0}^{t_1}\int_0^l \left(w_{tt} + 2cw_{xt} + c^2 w_{xx}\right)\delta w dx \\ &- \int_{t_0}^{t_1}\left[P(x,t)w_x\delta w\right]_0^l dt + \int_{t_0}^{t_1}\int_0^l \left(P(x,t)w_x\right)_x \delta w dxdt \\ &+ \int_0^l d_1(x,t)\delta w dx - \int_{t_0}^{t_1}\left(-F_c(t) + d_2(t)\right)\delta w(l,t)dt = 0 \end{aligned} \tag{9}$$

or

$$\begin{aligned} &-\rho\int_{t_0}^{t_1}\int_0^l \left(w_{tt} + 2cw_{xt} + c^2 w_{xx}\right)\delta w dx + \int_{t_0}^{t_1}\int_0^l \left(P(x,t)w_x\right)_x \delta w dxdt \\ &+ \int_{t_0}^{t_1} P(l,t)w_x(l,t)\delta w(l,t)dt + \int_0^l d_1(x,t)\delta w dx \\ &- \int_{t_0}^{t_1}\left(-F_c(t) + d_2(t)\right)\delta w(l,t)dt = 0 \end{aligned} \tag{10}$$

where we have used the fact $\delta w(0,t) = 0$ and the boundary conditions $\delta w(t_0) = \delta w(t_1) = 0$. Since the displacement δw is arbitrary, the following system is obtained

$$\begin{cases} \rho\left(w_{tt} + 2cw_{xt} + c^2 w_{xx}\right) - \left(P(x,t)w_x\right)_x = d_1(x,t), \;\; x \in (0,l)\,, \;\; t > 0, \\ w(0,t) = 0, \\ P(l,t)w_x(l,t) = -F_c(t) + d_2(t), \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), \;\; x \in (0,l)\,, \;\; t > 0, \end{cases} \tag{11}$$

Introducing the following non-dimensional parameters

$$\begin{aligned} x^* &= x/l, \; \alpha = \sqrt{1/\rho}, \; t^* = \alpha t/l, \; y\left(x^*, t^*\right) = w\left(x,t\right), \; T(x^*,t^*) = P(x,t), \\ v &= c/\alpha, \; f_1 = l^2 d_1, \; f = l d_2, \; f_c = l F_c. \end{aligned}$$

Inserting them into the system (11) and dropping all the stars, we are lead to

$$\begin{cases} y_{tt} + 2vy_{xt} + v^2 y_{xx} - \left(T(x,t)y_x\right)_x = f_1(x,t), \;\; x \in (0,1)\,, \;\; t > 0, \\ y(0,t) = 0, \quad t \geq 0, \\ T(1,t)y_x(1,t) = f_c\left(t\right) + f_2(t), \;\; t \geq 0, \\ y(x,0) = y_0(x), \;\; y_t(x,0) = y_1(x), \;\; x \in (0,1)\,, \end{cases} \tag{12}$$

Control formulation and closed loop system. The control objective is to design a bounded control force $f_c\left(t\right)$ that derives the string displacement $y(x,t)$ to zero with a fast manner as t goes to infinity. The energy associated to (12) is defined by

$$E(t) = \frac{1}{2}\left(\int_0^1 y_t^2 dx + \int_0^1 T(x,t) y_x^2 dx\right),\ t \geq 0.$$

By using (7), the derivative of $E(t)$ in the case of constant tension is equal to

$$\frac{d}{dt}E(t) = (y_t + vy_x)(1,t) f_c(t) - vy_x^2(0,t) + (y_t + vy_x)(1,t) f_2(t) + \int_0^1 (y_t + vy_x) f_1(x,t) dx,\ t \geq 0.$$

Our proposed control force $f_c(t)$ for this system is the following

$$f_c(t) = -\mu_1 (y_t + vy_x)(1,t) \ - \mu_2 (y_t + vy_x)(1, t - \tau(t)),\ t \geq 0. \tag{13}$$

The control input $f_c(t)$ acts here with delay, as in most practical situations the delays can not be neglected, due to delayed measurements, an intrinsic property of the system, feedback control action, etc. The positive real numbers μ_1 and μ_2 are the coefficients of the damping term and the retarded term, respectively. The function $\tau(t) > 0$, $t \geq 0$ indicates that the delay is a varying function with time.

As well known, the delay term, which occurs in different applications, is a factor of instability. It was shown that for the one dimensional wave equation when the control acts without delays, a simple frictional boundary damping capable to stabilize exponentially the system (see, e.g. [18, 29, 25]). However, when the delay acts with a weak delay, the instability of the system is provoked as shown in [1]. For that reason, the control $f_c(t)$ may be written as (see [30])

$$\begin{cases} f_c(t) = k\mu y_t(1,t) + k(1-\mu) y_t(1, t-\tau),\ k > 0,\ \mu \in (0,1),\ t > 0, \\ y_t(1, t-\tau) = f(1, t-\tau),\ t \in (0, \tau). \end{cases} \tag{14}$$

The authors in [30], showed that when $\mu > \frac{1}{2}$, the system is exponentially stable and it is unstable when $\mu < \frac{1}{2}$. In the case $\mu = \frac{1}{2}$, it was shown that if τ is rational the system is unstable and it is asymptotically stable if τ is irrational. The multi-dimensional case has also been treated by Nicaise and Pignotti in [20]. The assumption that $\mu_2 < \mu_1$ is requested to stabilize exponentially the system. In addition, the authors showed that if $\mu_2 \geq \mu_1$ the system is unstable. The systems with variable tension and delay term have been investigated in[2] where the author considered a plate equation with a memory term and a time delay term in the internal feedback. The global well-posedness has been established by using the Faedo-Galerkin approximations and some energy estimates. A general decay result of the energy was obtained provided that the weight of the delay is less than the weight of the damping. This result was generalized lately in[3] for two classes of plate equations with past history and strong time-dependent delay in the internal feedback.

The closed loop system associated to (12) by considering the control force (13) is given by

$$\begin{cases} y_{tt}+2vy_{xt}+v^2y_{xx}-(T(x,t)y_x)_x=f_1(x,t), \ \ x\in(0,1)\,,\ t>0, \\ y(0,t)=0, \quad t\geq 0, \\ T(1,t)y_x(1,t)=-\mu_1\left(y_t+vy_x\right)(1,t)\ -\mu_2\left(y_t+vy_x\right)(1,t-\tau(t))+f_2(t),\ t\geq 0, \\ y(x,0)=y_0(x),\ y_t(x,0)=y_1(x),\ \ x\in(0,1)\,, \\ \left(y_t+vy_x\right)(1,t-\tau)=f_0(1,t-\tau),\ t\in(-\tau(0),0) \end{cases} \tag{15}$$

where f_0 stands for the measure of the observation of the system in $(-\tau(0),0)$. Some stability results related to the system (15) without delay have been obtained, see [27, 10, 7, 8]. For Similar investigations in this regard, see [11, 12, 13, 14]. The question of the stabilization of an axially moving structure with delay, like systems (15) has not been investigated previously. Recently, the present authors investigated in [9] the stability of a delayed Kirchhof moving string where the delay acts in the boundary or in the inetrnal feedbacks. Their study was restricted on the case of homogeneous system: *i.e.* $f_1(x,t)=f_2(t)=0$ with $\tau(t)=\tau>0$. It was proven that if the delay coefficient satisfies $\mu_2<\mu_1$, the solution decays exponentially to zero.

Our main contributions throughout this work are summarized as follows:

(i) Proposing a more practical model that takes into account the influence of both internal and external factors depending on space such that the varying of tension, delay measurements and disturbance effects.

(ii) Proving the well-posedness of the system by means of the Faedo-Galerkin method.

(iii) Studying the asymptotic behaviour of solutions and deriving a decay rate of the system.

Our analysis is confronted by some difficulties: The axial movement of the string, the variation of the tension and the disturbance functions. The first difficulty requires the use of the Leibniz rule or the Reynolds Transport Theorem (for more details, see [28] and [21]). The second point is related to the dissipativity of the system and needs some extra assumption to deal with (see the total derivative of the energy below). For the third issue, we shall make use of an inequality which is new in this theory.

We have organised the content of this paper as follows. The second section is reserved to introduce our assumptions on the delay function, its coefficients and the disturbance functions. We need also to introduce some lemmas which will help in the proof of the results. In the third section, we give the statement of the well-posedness and present the proof using the Faedo-Galerkin approximations. The fourth section is concerned with the asymptotic behavior of the system. We prove under the condition that the delay coefficient μ_2 is dominated by the damping one μ_1, the system can be uniformly stabilized. The result obtained is illustrated by some examples clarifying when the system is exponentially stable or polynomially stable. The fifth section discusses some generalizations to other problems. In the last section, we conclude and suggest some possible future investigations.

2. Preliminaries

This section is concerned with introducing of our assumptions and some lemmas used in the proof of our result.

Let $L^2(0,1)$ be the usual Hilbert space with the inner product $(\cdot,\cdot)$ and the induced norm $\|.\|$. In order to state our existence result, we introduce

$$V = \left\{ w \in H^1(0,1) : w(0) = 0 \right\}.$$

The following inequalities will be utilized in this paper

Young inequality: Let $(a,b) \in \mathbb{R}^2$, for any $\eta > 0$, we have

$$ab \leq \eta a^2 + \frac{b^2}{4\eta} .$$

Poincaré inequality: Let $w \in V$, then the following inequalities hold

$$w(x) \leq \|w_x\|^2 , \ \forall x \in [0,1]$$

and

$$\|w\|^2 \leq \|w_x\|^2 .$$

Next, we introduce our assumptions. For the time-varying delay, we assume that

A1: : $\tau(t) \in W^{2,\infty}([0,T])$, $T > 0$ and there exist positive constants τ_0 and $\bar{\tau}$ such that

$$0 < \tau_0 \leq \tau(t) \leq \bar{\tau}, \ t > 0 \tag{16}$$

the delay derivative verifies

$$\tau'(t) \leq d < 1, \ t > 0 \tag{17}$$

and the coefficients μ_1 and μ_2 are related by

$$\mu_2 < \sqrt{1-d}\mu_1. \tag{18}$$

For the tension T, we suppose that

A2: $T(x,t)$ is continuously differentiable a.e. and satisfies

$$0 < T_{\min} \leq T_s(x,t) \leq T_{\max}, \tag{19}$$

$$|(T(x,t))_t| \leq (T_t)_{\max} \tag{20}$$

and

$$|(T(x,t))_t| \leq (T_x)_{\max} \tag{21}$$

for all $x \in [0,l]$ and $t \geq 0$ and for some known constants $T_{\min}$, $T_{\max}$, $(T_t)_{\max}$ and $(T_x)_{\max}$. To conserve the hyperbolicity of the system, we assume that

$$T_{\min} > v^2. \tag{22}$$

A3: The functions $f_1(x,.)$ and f_2 are continuous such that $f_1(.,t) \in L^2(0,1)$ for all $t \geq 0$. For simplicity, we denote $F(t) = \int_0^1 f_1^2(x,t)dx + f_2^2(t)$.

Due to the presence of the disturbance functions f_1 and f_2, the system is not necessarily dissipative (see 46). This does not allow to profit from the dissipativity of the system. To resolve this problem we introduce the following lemma, which was proven in [**?**].

Lemma 2.1. *Let $\chi(t)$, $\beta(t) \in C[0,+\infty)$ and let $v(t)$ be a nonnegative solution of the following inequality*

$$v'(t) \leq -\chi(t)v(t) + \beta(t), \ t \geq 0$$

such that there exists a positive function $\varphi(t) \in C^1[0,+\infty)$

$$\varphi(t)\chi(t) - \varphi'(t) \geq 0, \ t \geq 0$$

and

$$\beta(t) \leq \frac{1}{2\varphi(t)}\left(\chi(t) - \frac{\varphi'(t)}{\varphi(t)}\right), \ t \geq 0,$$

then

$$v(t) < \frac{1}{\varphi(t)}, \ t \geq 0$$

provided that $\varphi(0)v(0) < 1$.

3. Existence result

In this section, we present an existence and uniqueness result for the problem (15). In order to deal with the delay feedback term, we introduce the following new dependent variable

$$z(\rho, t) = y_t(1, t - \rho\tau(t)) + vy_x(1, t - \rho\tau(t)), \ \rho \in (0,1), \ t > 0.$$

Then, it is easy to check that

$$\tau(t)z_t(\rho, t) + (1 - \tau'(t)\rho)z_\rho(\rho, t) = 0, \ \ \rho \in (0,1), \ t > 0.$$

So, the problem (15) may be rewritten in the form

$$\begin{cases} y_{tt} + 2vy_{xt} + v^2 y_{xx} - (T(x,t)y_x)_x = f_1(x,t), \ \ x \in (0,1), \ t > 0, \\ \tau(t)z_t(\rho, t) + (1 - \tau'(t)\rho)z_\rho(\rho, t) = 0, \ \ \rho \in (0,1), \ t > 0, \\ y(0,t) = 0, \quad t \geq 0, \\ T(1,t)y_x(1,t) = -\mu_1 z(0,t) \ - \mu_2 z(1,t) + f_2(t), \ t \geq 0, \\ z(0,t) = (y_t + vy_x)(1,t), \ t > 0, \\ y(x,0) = y_0(x), \ y_t(x,0) = y_1(x), \ \ x \in (0,1), \\ z(\rho, 0) = f_0(1, -\tau\rho), \ \rho \in (0,1). \end{cases} \tag{23}$$

Definition 3.1. A pair of functions (y, z) is said to be a weak solution of (23) on $[0, T]$ if

$$y \in C([0,T), V) \cap C^1\left([0,T), L^2(0,1)\right), \ z \in C\left([0,T), L^2(0,1)\right).$$

In addition, (y, z) satisfies for any $(w, u) \in V \times L^2(0,1)$ and for all $t \in [0, T]$

$$\begin{cases} \left(\frac{d^2y}{dt^2}, w\right) = -(T(x,t)y_x, w_x) - w(1)\left[\mu_1 z(0,t) + \mu_2 z(1,t)\right] + (f_1(x,t), w) + w(1)f_2(t), \\ \tau(t)\left(z_t, u\right) + ((1 - \tau'(t)\rho)z_\rho, u) = 0 \end{cases} \tag{24}$$

and

$$y(0) = y_0, \ y_t(0) = y_1, \ z(0) = f_0.$$

For the existence of a local solution, we need only to have $f_1 \in L^2_{loc}\left(0, \infty, L^2(0,1)\right)$ and $f_2 \in L^2_{loc}(0,\infty)$. The third assumption is needed for the asymptotic behavior.

Theorem 3.1. *Let* $(y_0, y_1, f_0) \in V \times L^2(0,1) \times L^2(0,1)$. *Assume that* ***(A1)****-****(A2)*** *hold. Then, there exists a unique global weak solution of (23) such that*

$$y \in C([0,T), V), \ \frac{dy}{dt} \in C\left([0,T), L^2(0,1)\right), \ z \in C\left([0,T), L^2(0,1)\right) \ \textit{for any } T > 0. \tag{25}$$

Proof. In order to establish the existence of a weak solution to the system (23) we shall use a standard Galerkin approximation scheme. For this we consider a complete orthogonal system $\{w^i\}_{i=1}^{\infty}$ of $V \cap H^2(0,1)$. Then $\{w^i\}_{i=1}^{\infty}$ is an orthogonal basis of V and an orthonormal basis in $L^2(\Omega)$. Next, we define for $1 \leq i \leq m$ the sequence $u^i(x,\rho)$ by $u^i(x,0) = w^i(x)$. Then, we may extend $u^i(x,0)$ to $u^i(x,\rho)$ in $L^2((0,1),(0,1))$. Let $W_m = span\{w^1, w^2, ..., w^m\}$, $U_m = span\{u^1, u^2, ..., u^m\}$. The projection of the initial data on the finite dimensional subspace V_m and U_m is given by

$$y_0^m = \sum_{i=1}^{m} a_i w^i(x),\ \ y_1^m = \sum_{i=1}^{m} b_i w^i(x),\ \ z_0^m = \sum_{i=1}^{m} c_i u^i \tag{26}$$

satisfying

$$\begin{cases} y_0^m \to y_0 \text{ strongly in } V \cap H^2(0,1), \\ y_1^m \to y_1 \text{ strongly in } L^2(0,1), \\ z_0^m \to f_0 \text{ strongly in } L^2(0,1). \end{cases} \tag{27}$$

For each $m \in \mathbb{N}$, we seek the approximate solutions of the form

$$\begin{cases} y^m(x,t) = \sum_{i=1}^{m} c_m^i(t) w^i(x),\ \ x \in (0,1),\ \ t \geq 0, \\ z^m(\rho,t) = \sum_{i=1}^{m} d_m^i(t) u^i(1,\rho),\ \ \rho \in (0,1),\ \ t \geq 0 \end{cases} \tag{28}$$

for the following approximate problem in V_m

$$\begin{cases} \left(\frac{d^2}{dt^2} y^m, w\right) = -(T(x,t) y_x^m, w_x) - w(1)\left[\mu_1 z^m(0,t) \ + \mu_2 z^m(1,t)\right] \\ \qquad\qquad +(f_1(x,t), w) + w(1) f_2(t), \\ \tau(t)\left(z_t^m, u\right) + \left((1-\tau'(t)\rho) z_\rho^m, u\right) = 0, \\ z^m(0,t) = \left(y_t^m + v y_x^m\right)(1,t), \\ y^m(0) = y_0^m,\ \ y_t^m(0) = y_1^m,\ \ z^m(\rho,0) = z_0^m. \end{cases} \tag{29}$$

We deduce the existence of a solution (u^m, v^m) of (29) on a maximal time interval $[0, T_m)$, for each $m \in \mathbb{N}$. The system (29) leads to a system of ODEs for the unknown functions $\left(c_m^i(t), d_m^i(t)\right)_{i=1,m}$. Based on Cauchy-Peano theorem, we deduce the existence of a solution (y^m, z^m) of (29) on a maximal time interval $[0, t_m)$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independently of m and t.

A priori estimate: Taking $w = (y_t^m + v y_x^m)$ in the first equation of (29), it results that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|y_t^m + v y_x^m\|^2 + \int_0^1 T(x,t)\left(y_x^m\right)^2 dx\right) &= \frac{1}{2}\int_0^1 \left[(T(x,t))_t + v\left(T(x,t)\right)_x\right]\left(y_x^m\right)^2 dx \\ - z^m(0,t)\left[\mu_1 z^m(0,t) + \mu_2 z^m(1,t)\right] &+ \left(f_1(x,t), y_t^m + v y_x^m\right) + z^m(0,t) f_2(t) \end{aligned} \tag{30}$$

for all $t \in [0,T]$ with arbitrary fixed T. Applying Young inequality to the last two terms in (30) and considering the assumption (**A2**), we get for $\eta > 0$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|y_t^m + v y_x^m\|^2 + \int_0^1 T(x,t)\left(y_x^m\right)^2 dx\right) \\ &\leq \frac{1}{2}\left[(T_t)_{\max} + v\left(T_x\right)_{\max}\right]\|y_x^m\|^2 + \eta\,\|y_t^m + v y_x^m\|^2 - \left(\mu_1 - \frac{\mu_2}{2\sqrt{1-d}} + \eta\right)\left[z^m(0,t)\right]^2 \\ &\quad + \frac{\mu_2}{2}\sqrt{1-d}\left[z^m(1,t)\right]^2 + \frac{1}{4\eta}\left(\int_0^1 f_1^2(x,t) dx + f_2^2(t)\right). \end{aligned} \tag{31}$$

for all $t \in [0,T]$ with arbitrary fixed T. Let $\xi > 0$, by taking $u = z^m$ in the second equation of (29) followed by an integrating by parts, we get

$$\begin{aligned}
&\frac{\xi}{2}\tau(t)\frac{d}{dt}\|z^m\|^2 + \frac{\xi}{2}\tau'(t)\|z^m\|^2 = \xi\left((1-\tau'(t)\rho)z_\rho^m, z^m\right) \\
&= -\frac{\xi}{2}\int_0^1 (1-\tau'(t)\rho)\frac{\partial}{\partial\rho}\left(z_\rho^m\right)^2 d\rho \\
&= -\frac{\xi}{2}(1-\tau'(t))\left[z^m(1,t)\right]^2 + \frac{\xi}{2}\left[z^m(0,t)\right]^2,\ t\in[0,T).
\end{aligned} \tag{32}$$

Adding the resulting relations (31) and (32), we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|y_t^m + v y_x^m\|^2 + \int_0^1 T(x,t)\left(y_x^m\right)^2 dx + \xi\tau(t)\|z^m\|^2\right) \\
&\leq \frac{1}{2}\left[(T_t)_{\max} + v\,(T_x)_{\max}\right]\|y_x^m\|^2 + \eta\,\|y_t^m + v y_x^m\|^2 - \left(\mu_1 - \frac{\mu_2}{2\sqrt{1-d}} - \frac{\xi}{2} + \eta\right)\left[z^m(0,t)\right]^2 \\
&\quad - \left(\frac{\xi}{2}(1-\tau'(t)) - \frac{\mu_2}{2}\sqrt{1-d}\right)\left[z^m(1,t)\right]^2 + \frac{1}{4\eta}\left(\int_0^1 f_1^2(x,t)dx + f_2^2(t)\right)
\end{aligned} \tag{33}$$

for all $t \in [0,T]$ with arbitrary fixed T. Conisdering the assumption (**A2**) and choosing $\frac{\mu_2}{\sqrt{1-d}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}$ and η small enough, the relation (33) becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|y_t^m + v y_x^m\|^2 + \int_0^1 T(x,t)\left(y_x^m\right)^2 dx + \xi\tau(t)\|z^m\|^2\right) \leq \frac{1}{2}\left[(T_t)_{\max} + v\,(T_x)_{\max}\right] \\
&\quad \times \|y_x^m\|^2 + \eta\,\|y_t^m + v y_x^m\|^2 + \frac{1}{4\eta}\left(\int_0^1 \left(f_1^2(x,t)dx + f_2^2(t)\right)dx\right)
\end{aligned} \tag{34}$$

for all $t \in [0,T]$ with arbitrary fixed T. Letting

$$w_n(t) = \|y_t^m + v y_x^m\|^2 + \int_0^1 T(x,t)\left(y_x^m\right)^2 dx + \xi\tau(t)\|z^m\|^2.$$

The assumptions (**A1**) and (**A2**) and (34) lead to

$$\frac{d}{dt}w_n(t) \leq C_1\left(w_n(t) + F(t)\right) \tag{35}$$

where C_1 is a positive constant independant of m and t. Gronwall's inequality implies

$$w_n(t) \leq e^{C_1 t}\left(w_n(0) + \int_0^t F(s)ds\right) \tag{36}$$

Since $f_1 \in L^2_{loc}\left(0,\infty, L^2(0,1)\right)$ and $f_2 \in L^2_{loc}(0,\infty)$ then, it follows from (36) that

$$w_n(t) \leq e^{C_1 T}\left(w_n(0) + C_2\right) \tag{37}$$

for all $t \in [0,T]$ with arbitrary fixed T, where

$$w_n(0) = \|y_1^m + v\left(y_0^m\right)_x\|^2 + \int_0^1 T(x,0)\left(y_x^m\right)^2 dx + \xi\tau(0)\|z_0^m\|^2.$$

By the strong convergence of the initial data (27), it follows from (37) that

$$w_n(t) \leq e^{C_1 T}\left(w_n(0) + C_2\right) \leq C_3 \tag{38}$$

for all $t \in [0,T]$ with arbitrary fixed T and C_3 is a positive constant independent of t and m. From this, we conclude that

$$\begin{cases} y^m \text{ is uniformly bounded in } L^\infty(0,T;V), \\ y_t^m + vy_x^m \text{ is uniformly bounded in } L^\infty(0,T;L^2(0,1)), \\ z^m \text{ is uniformly bounded in } L^\infty(0,T;L^2(0,1)). \end{cases} \tag{39}$$

Passage to the limit

It follows from (39) that there exist subsequences still denoted by y^m, z^m and y, z such that

$$y \in L^\infty(0,T;L^2(0,1)),\ y_t + vy_x \in L^\infty(0,T;L^2(0,1)),\ z \in L^\infty(0,T;L^2(0,1)) \tag{40}$$

and

$$\begin{cases} y^\mu \rightharpoonup y \text{ weakly star in } L^\infty(0,T;V) \text{ and weakly in } L^2(0,T;V), \\ y_t^\mu + vy_x^\mu \rightharpoonup y_t + vy_x \text{ weakly star in } L^\infty(0,T;L^2(0,1)) \text{ and weakly in } L^2(0,T;L^2(0,1)), \\ z^\mu \rightharpoonup z \text{ weakly star in } L^\infty(0,T;L^2(0,1)) \text{ and weakly in } L^2(0,T;L^2(0,1)). \end{cases} \tag{41}$$

We apply Lions-Aubin theorem (see [17]), for any $T > 0$ to obtain the required compactness. The Passage to the limit in (29) permits to see that y is a solution of (23) satisfying (23). To finish the proof, there remains check that

$$y \in C([0,T),V),\ y_t + vy_x \in C\left([0,T),L^2(0,1)\right),\ z \in C\left([0,T),L^2(0,1)\right)$$

for any $T > 0$. It follows from (38) and the result in Reference [[26], Chapter II, Lemma 3.3] that y is weakly continuous from $[0,T]$ in V. Similarly, we deduce from (23) that

$$y_{tt} + 2vy_{xt} + v^2 y_{xx} = (T(x,t)y_x)_x + f_1(x,t).$$

Since $f_1 \in L^2(0,T;L^2(0,1)), y \in L^\infty(0,T;V)$ which implies that $y_x \in L^\infty(0,T;L^2(0,1))$ and $y_{xx} \in L^\infty(0,T;V')$, then $\frac{d^2y}{dt^2} \in L^\infty(0,T;V')$. This implies that y is weakly continuous from $[0,T]$ in V and $\frac{dy}{dt}$ is weakly continuous from $[0,T]$ in $L^2(0,1)$. Moreover y and z satisfy an equation (35), namely

$$\frac{d}{dt}\left(\|y_t + vy_x\|^2 + \int_0^1 T(x,t)\,(y_x)^2\,dx + \xi\tau(t)\,\|z\|^2\right) \le C_1\,(w(t) + F(t)) \tag{42}$$

This shows that the function

$$t \mapsto \|y_t + vy_x\|^2 + \int_0^1 T(x,t)y_x^2 dx + \xi\tau(t)\,\|z\|^2$$

is continuous on $[0,T]$. Gathering this with the properties of weak continuity, we deduce that

$$y \in C([0,T),V),\ y_t + vy_x \in C\left([0,T),L^2(0,1)\right),\ z \in C\left([0,T),L^2(0,1)\right).$$

Uniqueness: Let (y_1,z_1) and (y_2,z_2) be two solutions of (23) satisfying (25), and let $(y,z) = (y_1 - y_2, z_1 - z_2)$. Then (y,z) satisfy

$$\begin{cases} \left(\frac{d^2}{dt^2}y, w\right) = -(T(x,t)y_x, w_x) - w(1)\left[\mu_1 z(0,t) + \mu_2 z(1,t)\right], \\ \tau(t)\,(z_t,u) + (1 - \tau'(t)\rho\,(z_\rho,u) = 0, \\ z(0,t) = (y_t + vy_x)\,(1,t), \\ y(0) = y_0,\ y_t(0) = y_1,\ z(\rho,0) = z_0. \end{cases}$$

Taking $w = y_t + vy_x$, the following estimate is obtained (see (34))

$$\frac{1}{2}\frac{d}{dt}\left(\|y_t + vy_x\|^2 + \int_0^1 T(x,t)\,(y_x)^2\,dx + \xi\tau(t)\,\|z\|^2\right)$$
$$\leq \frac{1}{2}\left[(T_t)_{\max} + v\,(T_x)_{\max}\right]\|y_x\|^2 + \eta\,\|y_t + vy_x^m\|^2$$

for some constant $\eta > 0$. Letting

$$\varphi(t) = \|y_t + vy_x\|^2 + \int_0^1 T(x,t)y_x^2dx + \xi\tau(t)\,\|z\|^2\,.$$

The assumptions (**A1**) and (**A2**) lead to

$$\frac{d}{dt}\varphi(t) \leq K\varphi(t)$$

for some positive constant K. Gronwall's inequality implies

$$\varphi(t) \leq e^{Kt}\varphi(0) = 0$$

This termines the proof of the theorem. □

4. Asymptotic behavior

We define the energy associated to system (23) by

$$E(t) = \frac{1}{2}\|y_t + vy_x\|^2 + \frac{1}{2}\int_0^1 T(x,t)y_x^2dx + \frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t + vy_x\right)^2(1,s)ds,\ t \geq 0 \tag{43}$$

where ξ and λ are positive constant satisfying

$$\frac{\mu_2}{\sqrt{1-d}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{44}$$

and

$$\lambda < \frac{1}{\bar{\tau}}\left|\log\frac{\mu_2}{\xi\sqrt{1-d}}\right|. \tag{45}$$

Lemma 4.1. *The energy functional $E(t)$ defined by (43) satisfies along solution of (23)*

$$\begin{aligned}\frac{d}{dt}E(t) \quad &\leq \quad -\left(k_1 - \eta\right)z^2(0,t) - k_2z^2(1,t) + \frac{1}{2}\int_0^1 \left(T_t(x,t) + vT_x(x,t)\right)y_x^2dx \\ &\quad -\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t + vy_x\right)^2(1,s)ds + \frac{1}{4\eta}F(t),\ t \geq 0\end{aligned} \tag{46}$$

where $k_1 = \mu_1 - \frac{\mu_2}{2\sqrt{1-d}} - \frac{\xi}{2}$, $k_2 = \frac{\xi}{2}(1-d)e^{-\lambda\bar{\tau}} - \frac{\mu_2}{2}\sqrt{1-d}$ and η is a small positive constant to be determined later.

Proof. It follows by applying (7) that

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_0^1 (y_t+vy_x)\left(y_{tt}+2vy_{xt}+v^2y_{xx}\right)dx+\int_0^1 T(x,t)y_x\left(y_{xt}+vy_{xx}\right)dx\\ &+\frac{1}{2}\int_0^1 \left(T_t(x,t)+vT_x(x,t)\right)y_x^2dx+\int_0^1 (y_t+vy_x)f_1(x,t)dx\\ &+\frac{\xi}{2}\frac{d}{dt}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t+vy_x\right)^2(1,s)ds,\ t\geq 0. \end{aligned}\tag{47}$$

The substitution of the second derivative of y from (23) in (47) followed by an integration by parts produces

$$\begin{aligned}\frac{d}{dt}E(t) \leq& T(1,t)\left(y_t+vy_x\right)y_x(1,t)+\frac{1}{2}\int_0^1 \left(T_t(x,t)+vT_x(x,t)\right)y_x^2dx \\ &+\int_0^1 (y_t+vy_x)f_1(x,t)dx+\frac{\xi}{2}\frac{d}{dt}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t+vy_x\right)^2(1,s)ds,\ t\geq 0.\end{aligned}\tag{48}$$

The evaluation of last term gives

$$\begin{aligned}&\frac{d}{dt}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t+vy_x\right)^2(1,s)ds \\ &= z^2(0,t)-e^{-\lambda\tau(t)}(1-\tau'(t))z^2(1,t)-\lambda\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t+vy_x\right)^2(1,s)ds,\ t\geq 0.\end{aligned}\tag{49}$$

By inserting (49) and then using the boundary conditions, the relation (48) becomes

$$\begin{aligned}\frac{d}{dt}E(t) \leq& -z(0,t)\left[\mu_1 z(0,t)\ +\mu_2 z(1,t)-f_2(t)\right]+\int_0^1 (y_t+vy_x)f_1(x,t)dx+\frac{\xi}{2}z^2(0,t)\\ &-\frac{\xi}{2}e^{-\lambda\tau(t)}(1-\tau'(t))z^2(1,t)+\frac{1}{2}\int_0^1 \left(T_t(x,t)+vT_x(x,t)\right)y_x^2dx\\ &-\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t+vy_x\right)^2(1,s)ds,\ t\geq 0.\end{aligned}\tag{50}$$

We now apply Young inequality and consider the assumptions (**A2**). It holds for $\eta>0$ that

$$\begin{aligned}\frac{d}{dt}E(t) \leq& -\left(\mu_1-\frac{\mu_2}{2\sqrt{1-d}}-\frac{\xi}{2}+\eta\right)z^2(0,t)-\left[\frac{\xi}{2}(1-d)e^{-\lambda\bar{\tau}}-\frac{\mu_2}{2}\sqrt{1-d}\right]z^2(1,t)\\ &-\lambda\frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t+vy_x\right)^2(1,s)ds+\frac{1}{2}\int_0^1 \left(T_t(x,t)+vT_x(x,t)\right)y_x^2dx\\ &+\eta\left\|y_t+vy_x\right\|^2+\frac{1}{4\eta}F(t),\ t\geq 0.\end{aligned}$$

This proves the assertion (46). □

Our main result is summarized in the following

Theorem 4.2. *Assume that the hypotheses (**A1**)-(**A3**) hold. If the lower bound of the tension $T_{\min}$ is larger than its time and space derivatives $(T)_{t,\max}$ and $(T)_{x,\max}$*

and if there exists a positive function $\varphi(t) \in C^1[0,\infty)$ such that

$$\chi(t)\varphi(t) - \varphi'(t) \geq 0, \ t \geq 0$$

and

$$F(t) \leq \frac{B}{\varphi(t)} \left(\chi(t) - \frac{\varphi'(t)}{\varphi(t)} \right), \ t \geq 0$$

where B and $\chi(t)$ are given in 64 and 65 below, then

$$E(t) \leq \frac{K}{\varphi(t)}, \ t \geq 0.$$

such that $\varphi(0)L(0) < 1$, $t \geq 0$.

Proof. In order to prove the main result, we shall construct a lyamunov functional $L(t)$ which will play the role of an equivalent energy. The candidate functional $L(t)$ is defined by

$$L(t) = E(t) + \epsilon\Phi(t), \ t \geq 0$$

where

$$\Phi(t) = \int_0^1 x y_x (y_t + v y_x) \, dx, \ t \geq 0$$

and ϵ is a positive constant to be determined later. The first step consists to establish an equivalence result between $L(t)$ and $E(t)$: There exist β_i, $i = 1, 2$ such that for small ϵ

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \ t \geq 0. \tag{51}$$

The result follows immediately by applying Young inequality to the functional $\Phi(t)$ and by considering ϵ small enough.

The second step consists to establish a relation of the form $\frac{d}{dt}L(t) \leq -A_1 E(t) + A_2 F(t)$ for some positive constants A_1 and A_2 so that we can apply Lemma 2.1. A differentiation of the functional $\Phi(t)$ gives

$$\begin{aligned} \frac{d}{dt}\Phi(t) &= \int_0^1 x (y_{xt} + v y_{xx}) (y_t + v y_x) \, dx + v \int_0^1 y_x (y_t + v y_x) \, dx \\ &\quad + \int_0^1 x y_x \left(y_{tt} + 2v y_{xt} + v^2 y_{xx} \right) dx, \ t \geq 0. \end{aligned} \tag{52}$$

Replacing the expression of the second derivative of y from 23 into (52). It results that

$$\begin{aligned} \frac{d}{dt}\Phi(t) &= \int_0^1 x (y_{xt} + v y_{xx}) (y_t + v y_x) \, dx + v \int_0^1 y_x (y_t + v y_x) \, dx \\ &\quad + \int_0^1 x y_x \left(T(x,t) y_x \right)_x dx + \int_0^1 x y_x f_1(x,t) \, dx, \ t \geq 0. \end{aligned} \tag{53}$$

Next, we would like to estimate the last three terms in the right hand side of (53). Exploiting the expression of z, the boundary conditions, integration by parts and Young inequality to estimate

$$\int_0^1 x (y_{xt} + v y_{xx}) (y_t + v y_x) \, dx = \frac{1}{2} z^2(0,t) - \frac{1}{2} \|y_t + v y_x\|^2, \ t \geq 0, \tag{54}$$

$$v \int_0^1 y_x (y_t + v y_x) \, dx \leq \frac{v\sqrt{T_{\min}}}{2} \|y_x\|^2 + \frac{v}{2\sqrt{T_{\min}}} \|y_t + v y_x\|^2, \ t \geq 0 \tag{55}$$

and

$$\int_0^1 xy_x \left(T(x,t)y_x\right)_x dx = \frac{1}{2}T(1,t)y_x^2(1,t) - \frac{1}{2}\int_0^1 T(x,t)y_x^2 dx + \frac{1}{2}\int_0^1 T_x(x,t)y_x^2 dx,\ t \geq 0. \tag{56}$$

It results from the second boundary condition that

$$T(1,t)y_x^2(1,t) \leq \frac{3}{T(1,t)}\left[\mu_1^2 z^2(0,t) + \mu_2^2 z^2(1,t) + f_2^2(t)\right],\ t \geq 0. \tag{57}$$

Next, the insertion of (56) in (57) gives

$$\int_0^1 xy_x \left(T(x,t)y_x\right)_x dx \leq \frac{3}{2T(1,t)}\left[\mu_1^2 z^2(0,t) + \mu_2^2 z^2(1,t) + f_2^2(t)\right] - \frac{1}{2}\int_0^1 T(x,t)y_x^2 dx + \frac{1}{2}\int_0^1 T_x(x,t)y_x^2 dx,\ t \geq 0. \tag{58}$$

Using Young inequality, we estimate

$$\int_0^1 xy_x f_1(x,t)\, dx \leq \eta \left\|y_x\right\|^2 + \frac{1}{4\eta}\int_0^1 f_1^2(x,t)\, dx,\ t \geq 0. \tag{59}$$

Combining the estimates (53), (54), (55), (59) and (58) to obtain

$$\begin{aligned}\frac{d}{dt}\Phi(t) \leq{}& \frac{1}{2}\left(1 + \frac{3}{T(1,t)}\mu_1^2\right) z^2(0,t) + \frac{3\mu_2^2}{2T(1,t)}z^2(1,t) - \frac{1}{2}\left(1 - \frac{v}{\sqrt{T_{\min}}}\right)\left\|y_t + vy_x\right\|^2 \\ &- \frac{1}{2}\int_0^1 T(x,t)y_x^2 dx + \frac{1}{2}\int_0^1 T_x(x,t)y_x^2 dx + \left(\frac{vT_{\min}}{2} + \eta\right)\left\|y_x\right\|^2 + k_1 F(t), \\ &t \geq 0.\end{aligned} \tag{60}$$

where $k_1 = \max\left\{\frac{3}{2T(1,t)}, \frac{1}{4\eta}\right\}$. Exploiting the previous estimates (46) and (60), we entail that

$$\begin{aligned}\frac{d}{dt}L(t) \leq{}& -\left[(k_1 - \eta) - \frac{\epsilon}{2}\left(1 + \frac{3\mu_1^2}{T(1,t)}\right)\right] z^2(0,t) - \left(k_2 - \epsilon C\mu_2^2\right) z^2(1,t) \\ &+ \left[\eta - \frac{\epsilon}{2}\left(1 - \frac{v}{T_{\min}}\right)\right]\left\|y_t + vy_x\right\|^2 - \frac{\epsilon}{2}\int_0^1 T(x,t)y_x^2 dx + \frac{\epsilon}{2}\int_0^1 T_x(x,t)y_x^2 dx \\ &+ \frac{1}{2}\int_0^1 \left(T_t(x,t) + vT_x(x,t)\right) y_x^2 dx + \epsilon\left(\frac{v\sqrt{T_{\min}}}{2} + \eta\right)\left\|y_x\right\|^2 \\ &- \lambda\frac{\xi}{2}\int_{t-\tau(t)}^t e^{\lambda(s-t)}\left(y_t + vy_x\right)^2(1,s)ds + k_2 F(t),\ t \geq 0\end{aligned} \tag{61}$$

where $k_2 = \frac{1}{4\eta} + \epsilon k_1$. We now pick ϵ and η sufficiently small such that the coefficients of the first two terms in (61) are negative. It results that

$$\begin{aligned}\frac{d}{dt}L(t) \quad &\leq \quad \left[\eta - \frac{\epsilon}{2}\left(1 - \frac{v}{T_{\min}}\right)\right]\|y_t + vy_x\|^2 - \frac{\epsilon}{2}\int_0^1 T(x,t)y_x^2 dx \\ &\quad + \frac{1}{2T_{\min}}\int_0^1 \left(T_t(x,t) + vT_x(x,t) + \epsilon T_x(x,t)\right)T(x,t)y_x^2 dx \\ &\quad + \epsilon\left(\frac{v}{2\sqrt{T_{\min}}} + \frac{\eta}{T_{\min}}\right)\int_0^1 T(x,t)y_x^2 dx + k_2 F(t), \ t \geq 0\end{aligned} \tag{62}$$

where we have used the fact that $\|y_x\|^2 \leq \frac{1}{T_{\min}}\int_0^1 T(x,t)y_x^2 dx$. Our assumptions on the tension $T(x,t)$ permit to simplify (62) as follows

$$\begin{aligned}\frac{d}{dt}L(t) \leq \left[\eta - \frac{\epsilon}{2}\left(1 - \frac{v}{\sqrt{T_{\min}}}\right)\right]\|y_t + vy_x\|^2 - \frac{1}{2}\left\{\epsilon\left(1 - \frac{v}{\sqrt{T_{\min}}} - \frac{\eta}{T_{\min}}\right)\right. \\ \left. - \frac{1}{T_{\min}}\left[(T_t)_{\max} + v(T_x)_{\max} + \epsilon(T_x)_{\max}\right]\right\}\int_0^1 T(x,t)y_x^2 dx + k_2 F(t), \ t \geq 0.\end{aligned} \tag{63}$$

To eliminate the first term in the right hand side of (63), it suffices to consider again ϵ and η so small. The negativity of the second term is guaranteed by considering the lower bound of the tension $T_{\min}$ is larger than its time and space derivatives $(T_t)_{\max}$ and $(T_x)_{\max}$. This leads to

$$\frac{d}{dt}L(t) \leq -k_3 E(t) + k_2 F(t), \ t \geq 0 \tag{64}$$

where is k_3 a positive constant. Lemma 2.1 with: $\chi(t) = k_3$ and $\beta(t) = k_2 F(t)$ such that

$$F(t) \leq \frac{B}{\varphi(t)}\left(C_2 - \frac{\varphi'(t)}{\varphi(t)}\right), \ t \geq 0 \tag{65}$$

where $B = 1/(2k_2)$ and

$$C_2\varphi(t) - \varphi'(t) \geq 0 \tag{66}$$

allows us to conclude from 64 that

$$E(t) \leq \frac{K}{\varphi(t)}, \ t \geq 0$$

for some positive constant K such that $\varphi(0)L(0) < 1$, $t \geq 0$ where $L(0) = E(0) + \epsilon\Phi(0)$. □

Examples

Next, we illustrate the main result in Theorem 4.2 by two examples

Example 4.1. Assume that f_1 and f_2 are such that $f_1(x,t) = g(x)e^{-\frac{\alpha_1}{2}t}$ and $f_2(t) = f_0 e^{-\frac{\alpha_2}{2}t}$ for some positive constants α_1 and α_2.Clearly $F(t) = \|g\|_2^2 e^{-\alpha_1 t} + f_0^2 e^{-\alpha_2 t}$. Take $\varphi(t) = \varphi_0 e^{\beta t}$ with $0 < \beta < \lambda_4$ as this choice corresponds to the fact that

$$F(t) \leq \left(\|g\|_2^2 + f_0^2\right)e^{-\alpha t} \leq \frac{1}{2\lambda_2\varphi(t)}\left(\lambda_4 - \frac{\varphi'(t)}{\varphi(t)}\right) = \frac{(\lambda_4 - \beta)}{2\varphi_0\lambda_2}e^{-\beta t}$$

is fulfilled provided that $\beta \leq \alpha = \min\{\alpha_1, \alpha_2\}$ and $\|g\|_2^2 + f_0^2 \leq \frac{(\lambda_4 - \beta)}{2\varphi_0\lambda_2}$. We conclude that $E(t) \leq Ke^{-\beta t}$, $t \geq 0$ for some positive constant K.

Example 4.2. Assume that $f_1(x,t) = g(x)(1+t)^{-\frac{\alpha_1}{2}}$ and $f_2(t) = f_0(1+t)^{-\frac{\alpha_2}{2}}$ for some positive constants α_1 and α_2. Then $F(t) = \|g\|_2^2(1+t)^{-\alpha_1} + f_0^2(1+t)^{-\alpha_2}$. One can take $\varphi(t) = \varphi_0(1+t)^{\beta}$ with $\beta < \lambda_4$, we have

$$F(t) \leq \left(\|g\|_2^2 + f_0^2\right)(1+t)^{-\alpha} \leq \frac{\left[C_2 - \beta(1+t)^{-1}\right]}{2C_3\varphi_0}(1+t)^{-\beta}$$

provided that $\beta \leq \alpha = \min\{\alpha_1, \alpha_2\}$ and $\|g\|_2^2 + f_0^2 \leq \frac{\lambda_4 - \beta}{2\varphi_0\lambda_2}$. We conclude that $E(t) \leq K(1+t)^{-\beta}$, $t \geq 0$ for some positive constant K.

5. Some other systems

5.1. System with more than one delay. A similar result is obtained when the system acts with more than one delay term, that is, the second boundary condition in (15) is rewritten as

$$T(1,t)y_x(1,t) = -\mu_0(y_t + vy_x)(1,t) - \sum_{i=1}^{n}\mu_i(y_t + vy_x)(1, t - \tau_i(t)) + f_2(t),\ t \geq 0,$$

where the positive constants μ_0, $\mu_i, i = 1, ..., n$, are the coefficients of the damping and the delay terms, respectively. The delay functions satisfy: $\tau_i(t) \in W^{2,\infty}([0,T])$, $i = 1, ..., n$, $T > 0$ and there exist positive constants $\tilde{\tau}_i$ and $\bar{\tau}_i$ such that

$$0 < \tilde{\tau}_i \leq \tau_i(t) \leq \bar{\tau}_i,\ i = 1, ..., n,\ t > 0, \tag{67}$$

$$\tau_i'(t) \leq d_i < 1,\ i = 1, ..., n,\ t > 0 \tag{68}$$

and the coefficients μ_0 and $\mu_i, i = 1, ..., n$, satisfy

$$\mu_0 > \sum_{i=1}^{n}\frac{\mu_i}{\sqrt{1-d_i}}. \tag{69}$$

The energy that corresponds to the system (15) is defined by

$$E(t) = \frac{1}{2}\|y_t + vy_x\|^2 + \frac{1}{2}\int_0^1 T(x,t)\,y_x^2 dx + \sum_{i=1}^{n}\frac{\xi_i}{2}\int_{t-\tau_i(t)}^{t} e^{\lambda_i(s-t)}(y_t + vy_x)^2(1,s)ds,\ t \geq 0$$

where ξ_i , $i = 1, ..., n$, are positive constant satisfying

$$\sum_{i=1}^{n}\frac{\mu_i}{\sqrt{1-d_i}} < \sum_{i=1}^{n}\xi_i < 2\mu_0 - \sum_{i=1}^{n}\frac{\mu_i}{\sqrt{1-d_i}}.$$

and

$$\lambda_i < \frac{1}{\bar{\tau}_i}\left|\log\frac{\mu_i}{\xi_i\sqrt{1-d_i}}\right|.$$

6. Conclusion

Throughout this study, we have dealt with the stabilization of an axially moving string subject to a variable tension and variable disturbances. Local stability but of arbitrary rate is proved under some reasonable conditions. Two example were provided illustrating the exponential case and the polynomial case. For precisely these examples one may prove in fact a global stability result. This is established in the presence of a varying delay term which is known to be of destructive nature in other circumstances. Our future concerns are to examine the impact of the above considerations on other axially moving systems, namely moving beams, such as Euler-Bernoulli beams and Timoshenko beams.

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(A. Kelleche) Department of Mathematics, Faculty of Sciences of matter and Computer sciences, University Djilali Bounaama of Khemis Miliana, Algeria

(A. Abdallaoui) Laboratoire de mathématiques et physique appliqués, École normale supérieure de Bou Saada, Algeria