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Hermite-Hadamard Inequalities for Generalized $(\mathfrak{m}-\mathrm{F})$-Convex Function in the Framework of Local Fractional Integrals

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Abstract. This work presents new versions of the Hermite-Hadamard Inequality, for $(\mathfrak{m}-\mathrm{F})$-convex functions, defined on fractal sets $\mathbb{R}^{\varsigma}$ $(0<\varsigma\leq 1)$. So, we show some new results for twice differentiable functions using local fractional calculus, as well as some new definitions. We will construct these new integral inequality using the generalized Hölder-integral inequality and the power mean integral inequality. Furthermore, we present some new inequalities for the midpoint and trapezoid formulas in a novel type of fractal calculus. The conclusions in this paper are substantial advancements and generalizations of prior research reported in the literature.

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1. Introduction

In the last few years, the study of generalized Hermite-Hadamard type inequalities has become an area of significant interest in mathematical analysis. This research area involves investigating the properties of generalized $(\mathfrak{m}-\mathrm{F})$-convex functions via local fractional integrals. These functions are a generalization of convex functions and have wide-ranging applications in several fields of mathematics, including optimization, geometry, and mathematical physics. Using local fractional integrals, we will discuss recent developments in the study of generalized Hermite-Hadamard type inequalities for generalized $(\mathfrak{m}-\mathrm{F})$-convex functions. We will draw from the work of Dragomir and Pearce [1] to provide a comprehensive understanding of this topic and highlight its applications in various fields of mathematics. This review will showcase the ongoing significance and impact of the research area and its potential to address several fundamental problems in mathematical analysis.

We will need the following terms and works of writing before we start.

Definition 1.1. A function $\Upsilon:[\ell_1,\ell_2]\to\mathbb{R}$ is said to be convex function, if we have

$$\Upsilon\left(\hbar \mathrm{x}+(1-\hbar)\,\mathrm{y}\right)\leq \hbar\Upsilon\left(\mathrm{x}\right)+(1-\hbar)\,\Upsilon\left(\mathrm{y}\right)$$

for all $\mathrm{x},\mathrm{y}\in[\ell_1,\ell_2]$ and $\hbar\in[0,1]$. A function Υ is said to be concave if $-\Upsilon$ is convex.

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Readers interested in going through many of these extensions and generalizations of the classical notion of convexity can consult [1, 2].

For convex functions, we have the following inequality, undoubtedly one of the most famous in Mathematics, due to its multiple connections and applications:

$$\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right) \leq \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(\mathrm{x})dx \leq \frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2} \tag{1}$$

this is called the Hermite–Hadamard inequality. Both inequality hold if Υ is concave in the opposite direction. For more detail see [1, 4].

Definition 1.2. We call the function $\Upsilon : [\ell_1, \ell_2] \to \mathbb{R}$, $\mathfrak{m}$-convex, if we have

$$\Upsilon\left(\hbar u + \mathfrak{m}(1-\hbar)v\right) \leq \hbar\Upsilon\left(u\right) + \mathfrak{m}\left(1-\hbar\right)\Upsilon\left(v\right) \tag{2}$$

with $\hbar \in [0, 1]$ and for all $u, v \in [\ell_1, \mathfrak{m}\ell_2]$.

Definition 1.3. The real function $\Upsilon : I \to \mathbb{R}$ is called F-convex if

$$\Upsilon(\hbar(\ell_1) + (1-\hbar)\ell_2) \leq \hbar\Upsilon(\ell_1) + (1-\hbar)\Upsilon(\ell_2) - \hbar(1-\hbar)\mathrm{F}(\ell_1-\ell_2) \tag{3}$$

for some fixed function $\mathrm{F} : \mathbb{R} \to \mathbb{R}^{\varsigma}$, for all $\ell_1, \ell_2 \in I$ and $\hbar \in [0, 1]$.

For more detail see [5].

In this Remark we will introduce other important concepts of convexity [6, 7, 8, 9].

Remark 1.1. Let us notice that a highly convex function served as the basis for the definition of the class of F-convex functions. However, they also combined a number of other significant convexity ideas:

(i) Putting $\mathrm{F}(\ell) = -\mathrm{c}\ell^2$, we recreate the c-convex functions;
(ii) Making $\mathrm{F}(\ell) = -\mathrm{c}|\ell|$ with $\mathrm{c} > 0$, we recreate essentially convex functions;
(iii) If $\mathrm{F}(\ell) = -\mathrm{c}|\ell|^{\mathrm{p}}$ with $\mathrm{c} > 0$ and $\mathrm{p} > 0$, we have the convex functions of essentially order p;
(iv) For $\mathrm{F}(\ell) = -|\ell|\omega(|x|)$ without diminishing, upper-semi continuous function $\omega : [0, \infty) \to (0, \infty]$ with $\omega(0) = 0$, we get the semi convex-functions.

Fractional calculus is widely used in various areas of mathematics, physics and engineering, [10]. In recent years, fractal sets have gained significant interest from scientists and engineers. Fractal Calculus is a relatively new field in Mathematical Sciences, this local Calculus is designed for the study and visualization of fractal sets, that is, it is a generalization of differentiation and integration of the functions defined on fractal sets. This Local Fractional Calculus has found various applications in pure and applied research, in the latter, in areas as dissimilar as music and soil mechanics, including cryptography, without forgetting applications in software engineering.

The origin of Fractal Calculation is in Yang's seminal paper [11] and has since become a widely used topic in pure and applied research [12, 13, 14]. Numerous studies have investigated the characteristics of functions on fractal spaces and, using these applications, have developed numerous fractional calculus concepts [15, 16]. The concept of a generalized convex function on the fractal space $\mathbb{R}^{\varsigma}$ $(0 < \varsigma \leq 1)$ was defined by the authors in paper [17], and they also obtained the generalized Hermite-Hadamard inequality for such functions within the framework of this Fractal Calculus.

The passage you provided discusses a number of studies that offer fresh approaches and improvements to the local fractional calculus discipline. These papers specifically address coupled nonlinear systems of partial differential equations with non-differentiable solutions using effective methods. Additionally, local fractional integral operators with Mittag-Leffler kernels are used in the papers to create new inequalities for generalized h-convex functions. These inequalities are used to derive generalized Fractal Jensen-Mercer and Hermite Mercer type inequalities via h-convex functions. We must point out that other additional results have been found in other works reported in the literature, for example, readers can consult [18, 19, 20].

Subsequently, we present the set $\mathbb{R}^{\varsigma}$ and on this basis, we classify the definitions of local fractional derivatives, local fractional integrals, and other related operators, with reference to the Gao-Yang-Kang notion. This is to properly determine the range of the fractional order parameter in these derivatives and integrals.

The Yang's theory of fractional sets [11] can be written as:

The element set of the ς-type is provided below for $0 < \varsigma \leq 1$.

(i) $\mathbb{Z}^{\varsigma}$: The element set is described as a collection of integers of the ς-type is

$$\{0^{\varsigma}, \pm 1^{\varsigma}, \pm 2^{\varsigma}, \ldots, \pm n^{\varsigma}, \ldots\};$$

(ii) $\mathbb{Q}^{\varsigma}$: A collection of rational numbers of the ς-type set is defined as

$$\{\mathfrak{m}^{\varsigma} = (\sigma_1/\sigma_2)^{\varsigma} : \sigma_1, \sigma_2 \in \mathbb{Z}, \sigma_2 \neq 0\};$$

(iii) $\mathbb{J}^{\varsigma}$: A collection of irrational numbers of the ς-type set is defined as

$$\{\mathfrak{m}^{\varsigma} \neq (\sigma_1/\sigma_2)^{\varsigma} : \sigma_1, \sigma_2 \in \mathbb{Z}, \sigma_2 \neq 0\};$$

(iv) $\mathbb{R}^{\varsigma}$: A collection of real numbers of the ς-type is used to define the set. $\mathbb{R}^{\varsigma}=\mathbb{Q}^{\varsigma} \cup \mathbb{J}^{\varsigma}$.

The operations listed below hold for the real line numbers $\sigma_1^{\varsigma}, \sigma_2^{\varsigma}$ and ρ^{ς} in the collection $\mathbb{R}^{\varsigma}$:

(i) $\sigma_1^{\varsigma} + \sigma_2^{\varsigma}$ and $\sigma_1^{\varsigma}\sigma_2^{\varsigma}$ belong to the set $\mathbb{R}^{\varsigma}$;
(ii) $\sigma_1^{\varsigma} + \sigma_2^{\varsigma} = \sigma_2^{\varsigma} + \sigma_1^{\varsigma} = (\sigma_1 + \sigma_2)^{\varsigma} = (\sigma_1 + \sigma_2)^{\varsigma}$;
(iii) $\sigma_1^{\varsigma} + (\sigma_2^{\varsigma} + \rho^{\varsigma}) = (\sigma_1 + \sigma_2)^{\varsigma} + \rho^{\varsigma}$;
(iv) $\sigma_1^{\varsigma}\sigma_2^{\varsigma} = \sigma_2^{\varsigma}\sigma_1^{\varsigma} = (\sigma_1\sigma_2)^{\varsigma} = (\sigma_2\sigma_1)^{\varsigma}$;
(v) $\sigma_1^{\varsigma}(\sigma_2^{\varsigma}\rho^{\varsigma}) = (\sigma_1^{\varsigma}\sigma_2^{\varsigma})\rho^{\varsigma}$;
(vi) $\sigma_1^{\varsigma}(\sigma_2^{\varsigma} + \rho^{\varsigma}) = \sigma_1^{\varsigma}\sigma_2^{\varsigma} + \sigma_1^{\varsigma}\rho^{\varsigma}$;
(vii) $\sigma_1^{\varsigma} + 0^{\varsigma} = 0^{\varsigma} + \sigma_1^{\varsigma} = \sigma_1^{\varsigma}$ and $\sigma_1^{\varsigma}1^{\varsigma} = 1^{\varsigma}\sigma_1^{\varsigma} = \sigma_1^{\varsigma}$.

Let's go over some basics of local fractional calculus $\mathbb{R}^{\varsigma}$:

In the realm of local fractional calculus, a function that is non-differentiable is considered to be continuous at x_0, and if the function Υ is locally continuous on the interval (ℓ_1, ℓ_2), it is denoted as $\Upsilon \in C_{\varsigma}(\ell_1, \ell_2)$ [21]. Various endeavors have been undertaken to establish definitions for local fractional derivative and integral [22]. Once more, we'll go over the formulas for local fractional calculus [23].

Definition 1.4. The local fractional derivative of order ς of $\Upsilon(\mathrm{x})$ at $\mathrm{x} = \mathrm{x}_0$ is defined by

$$\Upsilon^{(\varsigma)}(\mathrm{x}_0) = {}_{\mathrm{x}_0}D_{\mathrm{x}}^{\varsigma}\Upsilon(\mathrm{x}) = \left.\frac{d^{(\varsigma)}\Upsilon(\mathrm{x})}{d(\mathrm{x}^{\varsigma})}\right|_{\mathrm{x}=\mathrm{x}_0} = \lim_{\mathrm{x}\to\mathrm{x}_0} \frac{\Delta^{\varsigma}(\Upsilon(\mathrm{x}) - \Upsilon(\mathrm{x}_0))}{(\mathrm{x}_{-}\mathrm{x}_0)^{\varsigma}}, \tag{4}$$

where $\Delta^{\varsigma}(\Upsilon(\mathrm{x})-\Upsilon(\mathrm{x}_0)) \cong \Gamma(1+\varsigma)(\Upsilon(\mathrm{x})-\Upsilon(\mathrm{x}_0))$ and Γ which is the popular gamma function. Let $\Upsilon^{(\varsigma)}(\mathrm{x}) = D^{\varsigma}_{\mathrm{x}}\Upsilon(\mathrm{x})$. If there holds $\Upsilon^{(k+1)\varsigma}(\mathrm{x}) = \overbrace{D^{\varsigma}_{\mathrm{x}}\cdots D^{\varsigma}_{\mathrm{x}}\Upsilon(\mathrm{x})}$ for any $\mathrm{x} \in I \subseteq \mathbb{R}$, then we indicate $\Upsilon \in D_{(k+1)\varsigma}(I)$, where $k \in \mathbb{N}$.

For more detail see [17].

Definition 1.5. Let $\Upsilon \in C_{\varsigma}[\ell_1, \ell_2]$. Then again, let $P = \hbar_0 \cdots \hbar_N, (N \in \mathbb{N}$ be a division of the interval $[\ell_1, \ell_2]$ that satisfies $\ell_1 = \hbar_0 < \hbar_1 < \cdots < \hbar_{N-1} < \hbar_N = \ell_2$. For this partition P, as well, let $\Delta\hbar := \max_{0\le j\le N-1}\Delta\hbar_j$, where $\Delta\hbar_j := \hbar_{j+1} - \hbar_j$ and $j = 0, \ldots, N-1$. Then, Υ is a local fractional integral on the interval $[\ell_1, \ell_2]$ of order ς (denoted by ${}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon$) is defined by

$$ {}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon(\hbar) = \frac{1}{\Gamma(1+\varsigma)}\int_{\ell_1}^{\ell_2}\Upsilon(\hbar)(d\hbar)^{\varsigma} := \frac{1}{\Gamma(1+\varsigma)}\lim_{\Delta\hbar\to 0}\sum_{j=0}^{N-1}\Upsilon(\hbar_j)(\Delta\hbar_j)^{\varsigma} $$

considering that the limit is true (indeed, this limit exists if $\Upsilon \in C_{\varsigma}[\ell_1, \ell_2]$). Here, it follows that ${}_{\ell_1}I^{(\varsigma)}_{\ell_1}\Upsilon = 0$ if $\ell_1 = \ell_2$ and ${}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon = -{}_{\ell_2}I^{(\varsigma)}_{\ell_1}\Upsilon$ if $\ell_1 < \ell_2$. If ${}_{\ell_1}I^{(\varsigma)}_{\mathrm{x}}\Psi$ holds for any $\mathrm{x} \in [\ell_1, \ell_2]$ and a function $\Psi : [\ell_1, \ell_2] \to \mathbb{R}^{\varsigma}$, after that, we indicate $\Psi \in I^{(\varsigma)}_{\mathrm{x}}[\ell_1, \ell_2]$.

For more detail see [17].

Below we present some known results that will be useful in our work and that facilitate the reading of it (for convenience, local fractional derivative is l.f.d. and local fractional integration is l.f.i.).

Lemma 1.1. [17]

(i) *(The l.f.d. of* $\mathrm{x}^{k\varsigma}$*):*

$$ \frac{d^{\varsigma}\mathrm{x}^{k\varsigma}}{d\mathrm{x}^{\varsigma}} = \frac{\Gamma(1+k\varsigma)}{\Gamma(1+(k-1)\varsigma)}\mathrm{x}^{(k-1)\varsigma}. \tag{5} $$

(ii) *(The l.f.i. is the anti-differentiation):*
If $\Upsilon(\mathrm{x}) = \Psi^{(\varsigma)}(\mathrm{x}) \in C_{\varsigma}[\ell_1, \ell_2]$*, then, we have*

$$ {}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon(\mathrm{x}) = \Psi(\ell_2) - \Psi(\ell_1). \tag{6} $$

(iii) *(The formula of integration by parts):*
Assume that $\Upsilon(\mathrm{x}), \Psi(\mathrm{x}) \in D_{\varsigma}[\ell_1, \ell_2]$ *and* $\Upsilon^{(\varsigma)}(x), \Psi^{(\varsigma)}(\mathrm{x}) \in C_{\varsigma}[\ell_1, \ell_2]$*. Then, we have*

$$ {}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon(\mathrm{x})\Psi^{(\varsigma)}(\mathrm{x}) = \Upsilon(\mathrm{x})\Psi(\mathrm{x})\Big|_{\ell_1}^{\ell_2} - {}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon^{(\varsigma)}(\mathrm{x})\Psi(\mathrm{x}). \tag{7} $$

(iv) *(The l.f.i., in this case definite, of* $\mathrm{x}^{k\varsigma}$*):*

$$ \frac{1}{\Gamma(1+\varsigma)}\int_{\ell_1}^{\ell_2}\mathrm{x}^{k\varsigma}(d\mathrm{x})^{\varsigma} = \frac{\Gamma(1+k\varsigma)}{\Gamma(1+(k+1)\varsigma)}\left({\ell_2}^{(k+1)\varsigma} - {\ell_1}^{(k+1)\varsigma}\right) \tag{8} $$

where $k \in \mathbb{R}$*.*

Additional details can be found in [22].

Lemma 1.2. *Let* $\Upsilon, \Psi \in C_\varsigma[\ell_1, \ell_2]$ *and* $\mathrm{p}, \mathrm{q} > 1$ *with* $\frac{1}{\mathrm{p}} + \frac{1}{\mathrm{q}} = 1$, *then*

$$\frac{1}{\Gamma(1+\varsigma)} \int_{\ell_1}^{\ell_2} |\Upsilon(\mathrm{x})\Psi(\mathrm{x})|(d\mathrm{x})^\varsigma \leq \left(\frac{1}{\Gamma(1+\varsigma)} \int_{\ell_1}^{\ell_2} |\Upsilon(\mathrm{x})|(d\mathrm{x})^\varsigma\right)^{\frac{1}{\mathrm{p}}} \times \left(\frac{1}{\Gamma(1+\varsigma)} \int_{\ell_1}^{\ell_2} |\Upsilon(\mathrm{x})||\Psi(\mathrm{x})|^{\mathrm{q}}(d\mathrm{x})^\varsigma\right)^{\frac{1}{\mathrm{q}}}. \tag{9}$$

Definition 1.6. The function $\Upsilon : I \subseteq \mathbb{R} \to \mathbb{R}^\varsigma$ is called a generalized convex function on I, if the following inequality is fulfilled

$$\Upsilon(\hbar\ell_1 + (1-\hbar)\ell_2) \leq \hbar^\varsigma \Upsilon(\ell_1) + (1-\hbar)^\varsigma \Upsilon(\ell_2) \tag{10}$$

for any $\ell_1, \ell_2 \in I$ and $\hbar \in [0, 1]$.

For more detail see [17].

Definition 1.7. The function $\Upsilon : I \subseteq \mathbb{R} \to \mathbb{R}^\varsigma$ is called a generalized $\mathfrak{m}$-convex function on I if the following inequality is fulfilled

$$\Upsilon(\hbar\ell_1 + \mathfrak{m}(1-\hbar)\ell_2) \leq \hbar^\varsigma \Upsilon(\ell_1) + \mathfrak{m}^\varsigma(1-\hbar)^\varsigma \Upsilon(\ell_2) \tag{11}$$

for any $\ell_1, \mathfrak{m}\ell_2 \in I$ and $\hbar \in [0, 1]$.

Definition 1.8. The function $\Upsilon : I \to \mathbb{R}^\varsigma$ is called generalized F-convex if

$$\Upsilon(\hbar\ell_1 + (1-\hbar)\ell_2) \leq \hbar^\varsigma \Upsilon(\ell_1) + (1-\hbar)^\varsigma \Upsilon(\ell_2) - \hbar^\varsigma(1-\hbar)^\varsigma \mathrm{F}(\ell_1 - \ell_2) \tag{12}$$

for some fixed function $\mathrm{F} : \mathbb{R} \to \mathbb{R}^\varsigma$, all $\ell_1, \ell_2 \in I$ and $\hbar \in [0, 1]$.

For more detail see [24].

Theorem 1.3. [17] *Let* $\Upsilon : [\ell_1, \ell_2] \to \mathbb{R}$ *be a generalized convex function with* $\ell_1 < \ell_2$. *Then for all* $\mathrm{x} \in [\ell_1, \ell_2]$, *the following inequalities hold:*

$$\Upsilon\left(\frac{\ell_1 + \ell_2}{2}\right) \leq \frac{\Gamma(1+\varsigma)}{(\ell_2 - \ell_1)^\varsigma} {}_{\ell_1}I_{\ell_2}^{(\varsigma)} \Upsilon(\mathrm{x}) \leq \frac{\Upsilon(\ell_1) + \Upsilon(\ell_2)}{2^\varsigma}. \tag{13}$$

2. Generalized $(\mathfrak{m} - \mathrm{F})$-convexity

We then go over the class of convex functions we'll be using in our research: the idea of generalized $(\mathfrak{m} - \mathrm{F})$-convex function on a fractal set.

Definition 2.1. The function $\Upsilon : I \to \mathbb{R}^\varsigma$ is called a generalized $(\mathfrak{m} - \mathrm{F})$-convex if

$$\Upsilon(\hbar\ell_1 + (1-\hbar)\mathfrak{m}\ell_2) \leq \hbar^\varsigma \Upsilon(\ell_1) + \mathfrak{m}(1-\hbar)^\varsigma \Upsilon(\ell_2) - \mathfrak{m}\hbar^\varsigma(1-\hbar)^\varsigma \mathrm{F}(\ell_1 - \ell_2) \tag{14}$$

for some fixed function $\mathrm{F} : \mathbb{R} \to \mathbb{R}^\varsigma$, for all $\ell_1, \mathfrak{m}\ell_2 \in I$ and $\hbar \in [0, 1]$.

Remark 2.1. We can see the following results based on the Definition 2.1.

(i) Putting $\mathfrak{m} = 1$ in the above Definition, we obtain the class of Generalized F-convex function [24].
(ii) Considering $\mathrm{F}(\ell) = 0$, $\mathfrak{m} = 1$ and $\varsigma = 1$, the Definition 2.1 it boils down to the class of convex function [1].
(iii) In the case that $\mathrm{F}(\ell) = 0$ and $\mathfrak{m} = 1$, the Definition 2.1 becomes the the class of generalized convex function [17].

(iv) If we have $\varsigma = 1$ and $\mathfrak{m} = 1$ from Definition 2.1, we have the definition 1.3. This way any generalized $(\mathfrak{m} - \mathrm{F})$-convex functions is a F-convex [5].
(v) Making $\mathrm{F}(\ell) = c^{\varsigma}|\ell|^{2\varsigma}$ and $\mathfrak{m} = 1$ from Definition 2.1, we obtain the class of generalized strongly convex function; see in [25, Definition (2.1)].
(vi) Taking $\varsigma = 1$, $\mathfrak{m} = 1$ and $\mathrm{F}(\ell) = c\mathfrak{x}^2$ from Definition 2.1, we obtain the [25, Definition 1.1].
(vii) Finally, putting $\varsigma = 1$, $\mathfrak{m} = 1$ and $\mathrm{F}(\ell) = c\|\cdot\|^2$ from above Definition we obtain the [26, Definition 1].

3. Main results

3.1. Midpoint inequality. Next we will present various results related to inequalities of the midpoint type for generalized $(\mathfrak{m} - \mathrm{F})$-convex function.

Theorem 3.1. *Let be some fixed function $\Upsilon : \mathbb{R} \to \mathbb{R}^{\varsigma}$, locally fractional integrable on each compact subinterval of $(-\beta, \beta)$, where $\beta = supI - infI$. If $\Upsilon : I \to \mathbb{R}^{\varsigma}$ is an generalized $(\mathfrak{m} - \mathrm{F})$-convex function, then*

$$\Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right) + \frac{\mathfrak{m}\Gamma(1+\varsigma)}{4(\mathfrak{m}\ell_2 - \ell_1)^{\varsigma}} {}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2} F\left(\ell_1 + \ell_2 - 2\mathrm{x}\right) \leq \frac{\Gamma(1+\varsigma)}{(\mathfrak{m}\ell_2 - \ell_1)^{\varsigma}} {}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2} \Upsilon(\mathrm{x})$$

for all $\ell_1, \mathfrak{m}\ell_2 \in I$ and $\ell_1 \neq \mathfrak{m}\ell_2$.

Proof. Taking into account the $(\mathfrak{m} - \mathrm{F})$-convexity of a function Υ and equality

$$\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right) = \frac{\hbar\ell_1 + (1-\hbar)\mathfrak{m}\ell_2 + (1-\hbar)\ell_1 + \hbar\mathfrak{m}\ell_2}{2},$$

we get

$$\Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right) = \Upsilon\left(\frac{\hbar\ell_1 + (1-\hbar)\mathfrak{m}\ell_2 + (1-\hbar)\ell_1 + \hbar\mathfrak{m}\ell_2}{2}\right),$$

and

$$\Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right) \leq \frac{\Upsilon(\hbar\ell_1 + (1-\hbar)\mathfrak{m}\ell_2) + \Upsilon((1-\hbar)\ell_1 + \hbar\mathfrak{m}\ell_2)}{2} - \mathfrak{m}\frac{1}{4}F((2\hbar - 1)(\ell_1 - \ell_2))$$

for all $\ell_1, \mathfrak{m}\ell_2 \in I$ and $\hbar \in [0,1]$. Thus

$$\Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right) + \frac{\mathfrak{m}}{4}F(2\hbar - 1)(\ell_1 - \ell_2) \leq \frac{\Upsilon(\hbar\ell_1 + (1-\hbar)\mathfrak{m}\ell_2) + \Upsilon((1-\hbar)\ell_1 + \hbar\mathfrak{m}\ell_2)}{2}. \tag{15}$$

Fixing different $\ell_1, \mathfrak{m}\ell_2 \in I$ in (15) we obtain a generalized locally fractional integrable, so we have after integration with respect to '$\hbar$' on the interval $[0,1]$, the following

$$\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right)(d\hbar)^{\varsigma} + \frac{\mathfrak{m}}{4}\frac{1}{\Gamma(1+\varsigma)}\int_0^1 F((2\hbar - 1)(\ell_1 - \ell_2))(d\hbar)^{\varsigma}$$
$$\leq \frac{1}{2\Gamma(1+\varsigma)}\int_0^1 (\Upsilon(\hbar\ell_1 + (1-\hbar)\mathfrak{m}\ell_2) + \Upsilon(1-\hbar)\ell_1 + \hbar\mathfrak{m}\ell_2))(d\hbar)^{\varsigma}.$$

Substitutions $(1-\hbar)\ell_1+\hbar\mathfrak{m}\ell_2=\mathrm{x}, \mathfrak{m}(2\hbar-1)(\ell_1-\ell_2)=2\mathrm{x}$ and $\hbar\ell_1+(1-\hbar)\mathfrak{m}\ell_2=\mathrm{x}$ above the integrals, we have

$$\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)+\frac{\mathfrak{m}\Gamma(1+\varsigma)}{4(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}F\left(\ell_1+\ell_2-2\mathrm{x}\right)\leq\frac{\Gamma(1+\varsigma)}{(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(\mathrm{x}).$$

Thus, the proof is completed. □

Remark 3.1. From Theorem 3.1 we obtain, with $\mathfrak{m}=1$, the following inequality:

$$\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right)+\frac{\Gamma(1+\varsigma)}{4(\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{\ell_2}F\left(\ell_1+\ell_2-2\mathrm{x}\right)\leq\frac{\Gamma(1+\varsigma)}{(\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon(\mathrm{x}),$$

which appeared in [24, Theorem 3.1].

Remark 3.2. In the same way, putting $\varsigma=1$ and $\mathfrak{m}=1$, from Theorem 3.1, we have

$$\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right)+\frac{1}{4(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_2}\mathrm{F}\left(\ell_1+\ell_2-2\mathrm{x}\right)dx\leq\frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(\mathrm{x})dx,$$

inequality of [5, Lemma 3].

3.2. Trapezoid inequality.

Now, for generalized $(\mathfrak{m}-\mathrm{F})$-convex function, we will present various results related to integral inequalities of Trapezoidal type.

Theorem 3.2. *Let be some fixed function $\Upsilon:\mathbb{R}\to\mathbb{R}^{\varsigma}$, locally fractional integrable on each compact subinterval of $(-\beta,\beta)$, with $\beta=\frac{\sup I-\inf I}{2}$. If a generalized $(\mathfrak{m}-\mathrm{F})$ - convex function $\Upsilon:I\to\mathbb{R}^{\varsigma}$ is one-sided differentiable and $\Upsilon_-\leq\Upsilon_+$ for all $\hbar\in[\ell_1,\mathfrak{m}\ell_2]$ and $\ell_1\neq\mathfrak{m}\ell_2$, then we have the following inequality*

$$\frac{1}{(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(\mathrm{x})\leq\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\Upsilon(\ell_1)+\left(\frac{1}{\Gamma(1+\varsigma)}-\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)\mathfrak{m}\Upsilon(\ell_2)$$
$$-\left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\right)\mathfrak{m}\mathrm{F}(\ell_1-\ell_2).$$

Proof. From generalized $(\mathfrak{m}-\mathrm{F})$-convexity of function Υ we have

$$\Upsilon(\hbar\ell_1+(1-\hbar)\mathfrak{m}\ell_2)\leq\hbar^{\varsigma}\Upsilon(\ell_1)+\mathfrak{m}(1-\hbar)^{\varsigma}\Upsilon(\ell_2)-\mathfrak{m}\hbar^{\varsigma}(1-\hbar)^{\varsigma}\mathrm{F}(\ell_1-\ell_2)\tag{16}$$

for all $\ell_1,\mathfrak{m}\ell_2\in I$ and $\hbar\in[0,1]$.

Integrating member by member over the interval $[0,1]$, with respect to h, we derived

$$\frac{1}{\Gamma(1+\varsigma)}\int_0^1\Upsilon(\hbar\ell_1+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}$$
$$\leq\frac{1}{\Gamma(1+\varsigma)}\int_0^1\hbar^{\varsigma}\Upsilon(\ell_1)(d\hbar)^{\varsigma}+\frac{\mathfrak{m}}{\Gamma(1+\varsigma)}\int_0^1(1-\hbar)^{\varsigma}\Upsilon(\ell_2)(d\hbar)^{\varsigma}$$
$$-\frac{\mathfrak{m}}{\Gamma(1+\varsigma)}\int_0^1\hbar^{\varsigma}(1-\hbar)^{\varsigma}\mathrm{F}(\ell_1-\ell_2)(d\hbar)^{\varsigma}.$$

Substitution $\hbar\ell_1+(1-\hbar)\ell_2=\mathrm{x}$, allows us to get

$$\frac{1}{\Gamma(1+\varsigma)}\int_0^1\Upsilon(\hbar\ell_1+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}=\frac{\Gamma(1+\varsigma)}{(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{m\ell_2}\Upsilon(\mathrm{x}),$$

so, the following result is obtained

$$\frac{1}{(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(\mathrm{x}) \leq \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\Upsilon(\ell_1) + \left(\frac{1}{\Gamma(1+\varsigma)} - \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)\mathfrak{m}\Upsilon(\ell_2) - \left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)} - \frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\right)\mathfrak{m}\mathrm{F}(\ell_1-\ell_2).$$

Thus we complete the proof. □

Remark 3.3. From Theorem 3.2, making $\mathfrak{m}=1$, we have the following:

$$\frac{1}{(\ell_2-\ell_1)^{\varsigma}}{}_{\ell_1}I^{(\varsigma)}_{\ell_2}\Upsilon(\mathrm{x}) \leq \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\Upsilon(\ell_1) + \left(\frac{1}{\Gamma(1+\varsigma)} - \frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)\Upsilon(\ell_2) - \left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)} - \frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\right)\mathrm{F}(\ell_1-\ell_2) \tag{17}$$

inequality which appeared in [24, Theorem 4.1].

Remark 3.4. Analogously, putting $\varsigma=1$ and $\mathfrak{m}=1$ from Theorem 3.2, we have:

$$\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right) + \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\mathrm{F}\left(\mathrm{x}-\frac{\ell_1+\ell_2}{2}\right)d\mathrm{x} \leq \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(\mathrm{x})d\mathrm{x} \leq \frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2} - \frac{1}{6}\mathrm{F}(\ell_1-\ell_2),$$

inequality of [5, Lemma 1].

4. Results for Midpoint inequalities

Next, we present a result that establishes equality for twice (2ς) differentiable functions: $\Upsilon^{(2\varsigma)}$.

The following considerations will be necessary in the rest of the work:

(A_1) $I \subseteq \mathbb{R}$ is an interval, I^o is the interior of I;
(A_2) $\ell_1, \mathfrak{m}\ell_2 \in I^o$, such that $\Upsilon^{(\varsigma)} \in D_{\varsigma}(I^0)$ and $\Upsilon^{(2\varsigma)} \in C_{\varsigma}[\ell_1, \mathfrak{m}\ell_2]$;
(A_3) $\hbar \in [0,1]$ and $0<\varsigma<1$.

Lemma 4.1. *If we take into account the assumptions* $(A_1), (A_2), (A_3)$, *a function* $\Upsilon : I^0 \subset \mathbb{R} \to \mathbb{R}^{\varsigma}$ *twice locally fractional differentiable on* I°, *the we have*

$$\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)\right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) = \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}(I_1+I_2), \tag{18}$$

where

$$I_1 = \frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\hbar^{2\varsigma}\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}$$

and

$$I_2 = \frac{1}{\Gamma(1+\varsigma)}\int_{\frac{1}{2}}^{1}(1-\hbar)^{2\varsigma}\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}.$$

Proof. Take I_1 and I_2, integrating by parts in each of these integrals leads us

$$I_1 = -\frac{1}{(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}2^{2\varsigma}}\Upsilon^{(\varsigma)}\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) + \frac{\Gamma(1+2\varsigma)}{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\Upsilon(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma} \tag{19}$$

and

$$I_2 = \frac{1}{(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}2^{2\varsigma}}\Upsilon^{(\varsigma)}\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) + \frac{\Gamma(1+2\varsigma)}{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}\Gamma(1+\varsigma)}\int_{\frac{1}{2}}^{1}\Upsilon(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}. \tag{20}$$

If we make the change of variables $\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2 = z$ in each of these integrals, and adding the results obtained, we obtain easily

$$I_1+I_2 = -\frac{\Gamma(1+2\varsigma)(1^{\varsigma}+1^{\varsigma})}{2^{\varsigma}\Gamma(1+\varsigma)(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) + \frac{\Gamma(1+2\varsigma)}{(\mathfrak{m}\ell_2-\ell_1)^{3\varsigma}\Gamma(1+\varsigma)} \times\left(\int_{\ell_1}^{\frac{\ell_1+\mathfrak{m}\ell_2}{2}}\Upsilon(z)(dz)^{\varsigma} + \int_{\frac{\ell_1+\mathfrak{m}\ell_2}{2}}^{\mathfrak{m}\ell_2}\Upsilon(z)(dz)^{\varsigma}\right). \tag{21}$$

Multiplying the two members of (21) by $\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}$, we have the (22) what is the desired inequality. □

Remark 4.1. The following result is obtained from Lemma 4.1 putting $\mathfrak{m}=1$:

$$\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\ell_2}^{(\varsigma)}\Upsilon(z)\right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right) = \frac{(\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(I_1+I_2\right), \tag{22}$$

which appeared in [24, Lemma 5.1].

Remark 4.2. From Lemma 4.1 we obtain, making $\varsigma=1$:

$$\frac{1}{(\mathfrak{m}\ell_2-\ell_1)}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz - \Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) = \frac{(\mathfrak{m}\ell_2-\ell_1)}{2}\int_0^{\frac{1}{2}}\hbar^2\Upsilon''(\hbar\ell_1+(1-\hbar)\mathfrak{m}\ell_2)d\hbar + \frac{(\mathfrak{m}\ell_2-\ell_1)}{2}\int_{\frac{1}{2}}^{1}(1-\hbar)^2\Upsilon''(\hbar\ell_1+(1-\hbar)\mathfrak{m}\ell_2)d\hbar.$$

Remark 4.3. From Lemma 4.1 we have, with $\varsigma=1$ and $\mathfrak{m}=1$:

$$\frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz - \Upsilon\left(\frac{\ell_1+\ell_2}{2}\right) = \frac{(\ell_2-\ell_1)}{2}\int_0^{\frac{1}{2}}\hbar^2\Upsilon''(\hbar\ell_1+(1-\hbar)\ell_2)d\hbar + \frac{(\ell_2-\ell_1)}{2}\int_{\frac{1}{2}}^{1}(1-\hbar)^2\Upsilon''(\hbar\ell_1+(1-\hbar)\ell_2)d\hbar.$$

Theorem 4.2. *Considerations* $(A_1), (A_2), (A_3)$ *are valids. Let us consider a function* $\Upsilon : I^0 \subset \mathbb{R} \to \mathbb{R}^{\varsigma}$, *twice locally fractional differentiable on* I°. *If* $\left|\Upsilon^{(2\varsigma)}\right|$ *is a generalized* $(\mathfrak{m} - \mathrm{F})$*-convex function, so the following inequality:*

$$\begin{aligned}
&\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)\right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right| \\
&\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right| + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right] \\
&\quad - \mathfrak{m}(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}\mathrm{F}(\ell_1-\ell_2)\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]. \qquad (23)
\end{aligned}$$

holds, for all $\varsigma \geq 1$.

Proof. Since the $|\Upsilon^{(2\varsigma)}|$ is a generalized $(\mathfrak{m} - \mathrm{F})$-convex function, we have from $|I_1|$, the following

$$\begin{aligned}
|I_1| \leq{}& \left|\Upsilon^{(2\varsigma)}(\ell_1)\right|\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\hbar^{2\varsigma+\varsigma}(d\hbar)^{\varsigma} \\
&+ \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\hbar^{2\varsigma}(1-\hbar)^{\varsigma}(d\hbar)^{\varsigma} \\
&- \mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\hbar^{2\varsigma}\hbar^{\varsigma}(1-\hbar)^{\varsigma}(d\hbar)^{\varsigma}. \qquad (24)
\end{aligned}$$

Calculating both integrals in the previous inequality, we have

$$\begin{aligned}
|I_1| \leq{}& \left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|\right. \\
&\left.+\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma} - \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\right)\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right] \quad - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2) \\
&\times\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]. \qquad (25)
\end{aligned}$$

And likewise integrating I_2, we obtain

$$\begin{aligned}
|I_2| \leq{}& \left[\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma} - \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\right)\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|\right. \\
&\left.+\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right] - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2) \\
&\times\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]. \qquad (26)
\end{aligned}$$

By adding (25) and (26), we get

$$\begin{aligned}
|I_1| + |I_2| \leq{}& \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right| + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right] \\
&- 2^{\varsigma}\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]. \qquad (27)
\end{aligned}$$

If we multiply the previous inequality by $\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}$, we obtain the inequality sought. □

Remark 4.4. (i) From (23), putting $\varsigma = 1$, we derived the following result, not reported in the literature, for $(\mathfrak{m}-\mathrm{F})$-convex function

$$\left|\frac{1}{\mathfrak{m}\ell_2-\ell_1}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz-\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^2}{48}\left[|\Upsilon''(\ell_1)|+\mathfrak{m}\,|\Upsilon''(\ell_2)|\right]-(\mathfrak{m}\ell_2-\ell_1)^2\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left(\frac{3}{320}\right).$$

(ii) If we take $\mathrm{F}(\ell) = c^{\varsigma}|\ell|^{2\varsigma}$ in (23), that is, we consider generalized strongly $\mathfrak{m}$-convex function, we have the following

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\mathfrak{m}\ell_2}^{(\varsigma)}\Upsilon(z)\right]-\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|+\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right]$$
$$-(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}\mathfrak{m}c^{\varsigma}(\ell_1-\ell_2)^{2\varsigma}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma}\left(\frac{1}{16}\right)^{\varsigma}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right], \tag{28}$$

a new inequality in literature.

(iii) If we take $\mathfrak{m} = 1$ and $\mathrm{F}(\ell) = c^{\varsigma}|\ell|^{2\varsigma}$ from (23), we have:

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\ell_2}^{(\varsigma)}\Upsilon(z)\right]-\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right)\right|$$
$$\leq \frac{(\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|+\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right]-(\ell_2-\ell_1)^{4\varsigma}c^{\varsigma}$$
$$\times\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma}\left(\frac{1}{16}\right)^{\varsigma}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]. \tag{29}$$

inequality which appeared in [24, Remark 5.2].

(iv) Putting $\mathrm{F}(\ell) = 0$ in (23), we have this new result:

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\mathfrak{m}\ell_2}^{(\varsigma)}\Upsilon(z)\right]-\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right|\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}$$
$$\times\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|+\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right]. \tag{30}$$

(v) Considering $\varsigma = 1$, $\mathfrak{m} = 1$ and $\mathrm{F}(\ell) = 0$ from (23), we have:

$$\left|\frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz-\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right)\right|\leq\frac{(\ell_2-\ell_1)^2}{48}\left[|\Upsilon''(\ell_1)|+|\Upsilon''(\ell_2)|\right],$$

inequality of [27, Proposition 1].

Theorem 4.3. *From the considerations* $(A_1),(A_2),(A_3)$*, let* $\mathrm{p},\mathrm{q}\geq 1$ *with* $\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}}=1$*, and consider a function* $\Upsilon: I^0\subset\mathbb{R}\to\mathbb{R}^{\varsigma}$ *twice locally fractional differentiable on* I°*. If* $|\Upsilon^{(2\varsigma)}|^{\mathrm{q}}$ *is generalized* $(\mathfrak{m}-\mathrm{F})$*-convex on* $[\ell_1,\mathfrak{m}\ell_2]$*, then we have:*

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\mathfrak{m}\ell_2}^{(\varsigma)}\Upsilon(z)\right]-\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{\frac{1}{\mathrm{p}}}W \tag{31}$$

where

$$W = \left\{ \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)} \left(\frac{1}{8}\right)^{\varsigma} \right) \left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}} + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}} \right] \right.$$
$$\left. - 2^{\varsigma}\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left[\left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} \left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)} \left(\frac{1}{32}\right)^{\varsigma} \right) \right] \right\}^{\frac{1}{\mathrm{q}}}.$$

Proof. From Lemma 4.1, it is very easy to obtain the following result, by means of the Inequality of Hölder:

$$\left| \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}} \left[{}_{\ell_1}I^{(\varsigma)}_{(\mathfrak{m}\ell_2)}\Upsilon(z) \right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right) \right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}(|I_1|+|I_2|), \tag{32}$$

and

$$|I_1| \leq \left| \frac{1}{\Gamma(1+\varsigma)} \int_0^{\frac{1}{2}} \hbar^{\frac{2\varsigma}{\mathrm{p}}}\hbar^{\frac{2\varsigma}{\mathrm{q}}}\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma} \right|$$
$$\leq \left(\frac{1}{\Gamma(1+\varsigma)} \int_0^{\frac{1}{2}} \hbar^{2\varsigma}(d\hbar)^{\varsigma} \right)^{\frac{1}{\mathrm{p}}} \left(\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}} \frac{1}{\Gamma(1+\varsigma)} \int_0^{\frac{1}{2}} \hbar^{3\varsigma}(d\hbar)^{\varsigma} \right.\right.$$
$$\left. + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}} \frac{1}{\Gamma(1+\varsigma)} \int_0^{\frac{1}{2}} \hbar^{2\varsigma}(1-\hbar)^{\varsigma}(d\hbar)^{\varsigma} \right]$$
$$\left. - \frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)}{\Gamma(1+\varsigma)} \int_0^{\frac{1}{2}} \hbar^{3\varsigma}(1-\hbar)^{\varsigma}(d\hbar)^{\varsigma} \right)^{\frac{1}{\mathrm{q}}}, \tag{33}$$

or

$$|I_1| \leq \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)} \left(\frac{1}{8}\right)^{\varsigma} \right)^{\frac{1}{\mathrm{p}}} \left\{ \left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} \left(\frac{1}{16}\right)^{\varsigma} \left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}} \right.\right.$$
$$\left. + \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)} \left(\frac{1}{8}\right)^{\varsigma} - \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} \left(\frac{1}{16}\right)^{\varsigma} \right) \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}} \right]$$
$$\left. - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2) \left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} \left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)} \left(\frac{1}{32}\right)^{\varsigma} \right) \right\}^{\frac{1}{\mathrm{q}}}. \tag{34}$$

Similarly from $|I_2|$, we obtain the inequality

$$|I_2| \leq \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)} \left(\frac{1}{8}\right)^{\varsigma} \right)^{\frac{1}{\mathrm{p}}} \left\{ \left[\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)} \left(\frac{1}{8}\right)^{\varsigma} - \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} \left(\frac{1}{16}\right)^{\varsigma} \right) \right.\right.$$
$$\left. \times\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}} + \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} \left(\frac{1}{16}\right)^{\varsigma} \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}} \right]$$
$$\left. - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2) \left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} \left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)} \left(\frac{1}{32}\right)^{\varsigma} \right) \right\}^{\frac{1}{\mathrm{q}}}. \tag{35}$$

From (34) and (35), we obtain the following

$$|I_1|+|I_2| \leq \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)} \left(\frac{1}{8}\right)^{\varsigma} \right)^{\frac{1}{\mathrm{p}}} W. \tag{36}$$

Given this last inequalities (36) and (32), we derive (31), inequality desired. □

Remark 4.5. (i) Putting $\varsigma = 1$ in (31), we can get the following result, valids for $(\mathfrak{m} - \mathrm{F})$-convex function:

$$\left|\frac{1}{(\mathfrak{m}\ell_2 - \ell_1)}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz - \Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2 - \ell_1)^2}{48}\left[[|\Upsilon''(\ell_1)|^{\mathfrak{q}} + \mathfrak{m}|\Upsilon''(\ell_2)|^{\mathfrak{q}}] - \mathfrak{m}\mathrm{F}(\ell_1 - \ell_2)\left(\frac{3}{320}\right)\right]^{\frac{1}{\mathfrak{q}}},$$

a new inequality, not reported in the literature.

(ii) If we take $\mathrm{F}(\ell) = c^{\varsigma}|\ell|^{2\varsigma}$ in (31), for generalized strongly $\mathfrak{m}$-convex function, we have the following new inequality:

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2 - \ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\mathfrak{m}\ell_2}^{(\varsigma)}\Upsilon(z)\right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right)\right| \leq \frac{(\mathfrak{m}\ell_2 - \ell_1)^{2\varsigma}}{2^{\varsigma}}$$
$$\times\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{\frac{1}{\mathfrak{p}}}\left\{\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathfrak{q}} + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathfrak{q}}\right]\right.$$
$$\left. - 2^{\varsigma}\mathfrak{m}c^{\varsigma}(\ell_1 - \ell_2)^{2\varsigma}\left[\left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right)\right]\right\}^{\frac{1}{\mathfrak{q}}}.$$

(iii) The inequality [24, Remark (5.3)], can be obtained by doing $\mathfrak{m} = 1$ and $\mathrm{F}(\ell) = c^{\varsigma}|\ell|^{2\varsigma}$ in (31).

(iv) Similarly, the inequality what we present next, is reached from (31), if we consider $\mathrm{F}(\ell) = 0$:

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2 - \ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\mathfrak{m}\ell_2}^{(\varsigma)}\Upsilon(z)\right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right)\right| \leq \frac{(\mathfrak{m}\ell_2 - \ell_1)^{2\varsigma}}{2^{\varsigma}}$$
$$\times\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{\frac{1}{\mathfrak{p}}}\left[\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)\left(\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathfrak{q}} + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathfrak{q}}\right)\right]^{\frac{1}{\mathfrak{q}}}.$$

a new inequality not published.

(v) The following inequality, is easily obtained putting $\mathrm{F}(\ell) = 0$, $\mathfrak{m} = 1$ and $\varsigma = 1$ in (31):

$$\left|\frac{1}{\ell_2 - \ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz - \Upsilon\left(\frac{\ell_1 + \ell_2}{2}\right)\right| \leq \frac{(\ell_2 - \ell_1)^2}{48}[|\Upsilon''(\ell_1)|^{\mathfrak{q}} + |\Upsilon''(\ell_2)|^{\mathfrak{q}}]^{\frac{1}{\mathfrak{q}}}, \tag{37}$$

posed in [27, Theorem 2].

Theorem 4.4. *From the considerations* $(A_1), (A_2), (A_3)$, *let* $|\Upsilon^{(2\varsigma)}|^{\mathfrak{q}}$, *a generalized* $(\mathfrak{m} - \mathrm{F})$-*convex on* $[\ell_1, \mathfrak{m}\ell_2]$, *the following inequality:*

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2 - \ell_1)^{\varsigma}}\left[{}_{\ell_1}I_{\mathfrak{m}\ell_2}^{(\varsigma)}\Upsilon(z)\right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1 + \mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2 - \ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{1-\frac{1}{\mathfrak{q}}}E, \tag{38}$$

holds, for $\mathrm{q} \geq 1$ *and*

$$E = \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{\frac{1}{\mathrm{q}}}\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}} + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}} - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]\right]^{\frac{1}{\mathrm{q}}}.$$

Proof. The following result, can be obtained from Lemma 4.1

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)\right] - \frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right| \leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}(|I_1|+|I_2|), \tag{39}$$

using the generalized power mean integral inequality. Where from

$$\begin{aligned}|I_1| &= \left|\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}} \hbar^{2\varsigma}\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}\right| \\ &\leq \left(\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}} \hbar^{2\varsigma}(d\hbar)^{\varsigma}\right)^{1-\frac{1}{\mathrm{q}}}\left\{\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\hbar^{3\varsigma}(d\hbar)^{\varsigma}\right.\right. \\ &\quad + \mathfrak{m}\left.\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\hbar^{2\varsigma}(1-\hbar)^{\varsigma}(d\hbar)^{\varsigma}\right] \\ &\quad \left. - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left[\frac{1}{\Gamma(1+\varsigma)}\int_0^{\frac{1}{2}}\hbar^{2\varsigma}\hbar^{\varsigma}(1-\hbar)^{\varsigma}(d\hbar)^{\varsigma}\right]\right\}^{\frac{1}{\mathrm{q}}},\end{aligned}$$

or

$$\begin{aligned}|I_1| &\leq \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{1-\frac{1}{\mathrm{q}}}\left\{\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}\right.\right. \\ &\quad + \left.\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma} - \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\right)\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}\right] \\ &\quad \left. - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]\right\}^{\frac{1}{\mathrm{q}}}.\end{aligned} \tag{40}$$

Similarly from $|I_2|$, we obtain the inequalities

$$\begin{aligned}|I_2| &\leq \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{1-\frac{1}{\mathrm{q}}}\left\{\left[\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma} - \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\right)\right.\right. \\ &\quad \times \left.\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}} + \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}\right] \\ &\quad \left. - \mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]\right\}^{\frac{1}{\mathrm{q}}}.\end{aligned} \tag{41}$$

Considering the inequalities (40) and (41), is obtained

$$|I_1|+|I_2|\leq\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{1-\frac{1}{\mathfrak{q}}}E. \tag{42}$$

The inequality (38), is easily obtained by putting (42) in (39). □

Remark 4.6. (i) Putting $\varsigma=1$, from (38) we have the following

$$\left|\frac{1}{(\mathfrak{m}\ell_2-\ell_1)}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz-\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^2}{48}\left[[|\Upsilon''(\ell_1)|^{\mathfrak{q}}+\mathfrak{m}|\Upsilon''(\ell_2)|^{\mathfrak{q}}]-\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left(\frac{3}{320}\right)\right]^{\frac{1}{\mathfrak{q}}},$$

a new inequality for $(\mathfrak{m}-\mathrm{F})$-convex function.

(ii) If we take $\mathrm{F}(\ell)=c^{\varsigma}|\ell|^{2\varsigma}$ in (38), we have the following

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)\right]-\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{1-\frac{1}{\mathfrak{q}}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{\frac{1}{\mathfrak{q}}}$$
$$\times\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathfrak{q}}+\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathfrak{q}}-\mathfrak{m}c^{\varsigma}(\ell_1-\ell_2)^{2\varsigma}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\left(\frac{1}{16}\right)^{\varsigma}\right.\right.$$
$$\left.\left.-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\left(\frac{1}{32}\right)^{\varsigma}\right]\right]^{\frac{1}{\mathfrak{q}}}, \tag{43}$$

also a new inequality for generalized strongly $\mathfrak{m}$-convex function.

(iii) If we take $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=c^{\varsigma}|\ell|^{2\varsigma}$ in (38), we have the inequality stated in [24, Remark (5.4)].

(iv) The following inequality is obtained, taking $\mathrm{F}(\ell)=0$ in 38:

$$\left|\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)\right]-\frac{\Gamma(1+2\varsigma)}{2^{\varsigma}\Gamma(1+\varsigma)}\Upsilon\left(\frac{\ell_1+\mathfrak{m}\ell_2}{2}\right)\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{1-\frac{1}{\mathfrak{q}}}E,$$

where

$$E=\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}\left(\frac{1}{8}\right)^{\varsigma}\right)^{\frac{1}{\mathfrak{q}}}\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathfrak{q}}+\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathfrak{q}}\right]^{\frac{1}{\mathfrak{q}}}.$$

(v) Considering $\varsigma=1$, $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=0$ in (38), we have the following result:

$$\left|\frac{1}{(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz-\Upsilon\left(\frac{\ell_1+\ell_2}{2}\right)\right|\leq\frac{(\ell_2-\ell_1)^2}{48}[|\Upsilon''(\ell_1)|^{\mathfrak{q}}+|\Upsilon''(\ell_2)|^{\mathfrak{q}}]^{\frac{1}{\mathfrak{q}}},$$

inequality posed in [27, Theorem 2].

5. Results for trapezoid inequalities

The following equality, is basic in obtaining the results of this section for $\Upsilon^{(2\varsigma)}$.

Lemma 5.1. *From the considerations* $(A_1), (A_2), (A_3)$, *let* $|\Upsilon^{(2\varsigma)}|^{\mathfrak{q}}$, *a generalized* $(\mathfrak{m}-\mathrm{F})$*-convex on* $[\ell_1, \mathfrak{m}\ell_2]$, *twice locally fractional differentiable function on* I°, *the following inequality:*

$$\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}_2-\ell_1)^{\varsigma}}\left[2^{\varsigma}\Gamma(1+2\varsigma)\right.$$
$$\left.-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right]=\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(J_1+J_2\right), \tag{44}$$

holds, where

$$J_1=\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{\varsigma}(1-\hbar)^{2\varsigma}\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}$$

and

$$J_2=\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{\varsigma}(1-\hbar)^{2\varsigma}\Upsilon^{(2\varsigma)}((1-\hbar)\ell_1+\hbar\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}.$$

Proof. The integrals J_1 and J_2, can be calculated by making the change of variables $1-\hbar=\mathfrak{s}$, so we have, after integrating twice by parts:

$$\begin{aligned}J_1&=\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \mathfrak{s}^{2\varsigma}(1-\mathfrak{s})^{\varsigma}\Upsilon^{(2\varsigma)}((1-\mathfrak{s})\ell_1+\mathfrak{s}\mathfrak{m}\ell_2)(d\mathfrak{s})^{\varsigma}\\&=\frac{\mathfrak{m}\Upsilon(\ell_2)}{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]\\&\quad+\frac{\Gamma(1+2\varsigma)}{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\times\frac{1}{\Gamma(1+\varsigma)}\int_0^1\Upsilon((1-\mathfrak{s})\ell_1+\mathfrak{s}\mathfrak{m}\ell_2)(d\mathfrak{s})^{\varsigma}\\&\quad-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\frac{1}{\Gamma(1+\varsigma)}\int_0^1\mathfrak{s}^{\varsigma}\Upsilon((1-\mathfrak{s})\ell_1+\mathfrak{s}\mathfrak{m}\ell_2)(d\mathfrak{s})^{\varsigma}.\end{aligned}$$

Similarly for J_2, we have

$$\begin{aligned}J_2&=\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \mathfrak{s}^{2\varsigma}(1-\mathfrak{s})^{\varsigma}\Upsilon^{(2\varsigma)}(\mathfrak{s}\ell_1+(1-\mathfrak{s})\mathfrak{m}\ell_2)(d\mathfrak{s})^{\varsigma}\\&=\frac{\Upsilon(\ell_1)}{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]\\&\quad+\frac{\Gamma(1+2\varsigma)}{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\times\frac{1}{\Gamma(1+\varsigma)}\int_0^1\Upsilon(\mathfrak{s}\ell_1+(1-\mathfrak{s})\mathfrak{m}\ell_2)(d\mathfrak{s})^{\varsigma}\\&\quad-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\frac{1}{\Gamma(1+\varsigma)}\int_0^1\mathfrak{s}^{\varsigma}\Upsilon(\mathfrak{s}\ell_1+(1-\mathfrak{s})\mathfrak{m}\ell_2)(d\mathfrak{s})^{\varsigma}.\end{aligned}$$

In this way you have

$$\begin{aligned}J_1+J_2&=\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]\\&\quad+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{(\mathfrak{m}\ell_2-\ell_1)^{3\varsigma}}\left[2^{\varsigma}\Gamma(1+2\varsigma)-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right].\end{aligned}\tag{45}$$

Multiplying this inequality, member by member, by the term $\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^\varsigma}$, leads us to (44). Thus the proof is completed. □

Remark 5.1. Putting $\varsigma = 1$ from (44) we have:

$$\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2} - \frac{1}{(\mathfrak{m}\ell_2-\ell_1)}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz = \frac{(\mathfrak{m}\ell_2-\ell_1)^2}{2}$$
$$\times\left(\int_0^1 \hbar(1-\hbar)^2\Upsilon''(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)d\hbar + \int_0^1 \hbar(1-\hbar)^2\Upsilon''((1-\hbar)\ell_1+\hbar\mathfrak{m}\ell_2)d\hbar\right).$$

Remark 5.2. The inequality [24, Lemma 6.1] is easily obtained considering $\mathfrak{m} = 1$ in (44).

Remark 5.3. The equality presented below is obtained from (44), considering $\varsigma = 1$ and $\mathfrak{m} = 1$:

$$\frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2} - \frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz = \frac{(\ell_2-\ell_1)^2}{2}\left(\int_0^1 \hbar(1-\hbar)^2\Upsilon''(\ell_1\hbar\right.$$
$$\left. + (1-\hbar)\ell_2)d\hbar + \int_0^1 \hbar(1-\hbar)^2\Upsilon''((1-\hbar)\ell_1+\hbar\ell_2)d\hbar\right).$$

Theorem 5.2. *Under the assumptions $(A_1),(A_2),(A_3)$ and $\Upsilon : I^o \subset \mathbb{R} \to \mathbb{R}^\varsigma$, is a funtion twice locally fractional differentiable function on I°. If $|\Upsilon^{(2\varsigma)}|$ is a generalized $(\mathfrak{m}-\mathrm{F})$-convex on $[\ell_1, \mathfrak{m}\ell_2]$, then the following inequality holds:*

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^\varsigma}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)} - \frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right] + \frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^\varsigma(\mathfrak{m}\ell_2-\ell_1)^\varsigma}U\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^\varsigma}\left[\frac{B_\varsigma(3,3)}{\Gamma(1+\varsigma)} + \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right]$$
$$\times\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right| + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right] - (\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B(3,4)}{\Gamma(1+\varsigma)}, \tag{46}$$

where

$$U = \left[2^\varsigma\Gamma(1+2\varsigma) - \frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right]$$

and

$$B_\varsigma = \int_o^1 \hbar^{(z-1)\varsigma}(1-\hbar)^{(y-1)\varsigma}(d\hbar)^\varsigma$$

is local beta function.

Proof. From Lemma 5.1 we get the following result, after using elemental properties:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^\varsigma}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)} - \frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right] + \frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^\varsigma(\mathfrak{m}\ell_2-\ell_1)^\varsigma}U\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^\varsigma}\left(|J_1|+|J_2|\right). \tag{47}$$

Taking into account that $\Upsilon^{(2\varsigma)}$ is a generalized $(\mathfrak{m}-\mathrm{F})$-convex function we obtain, from the first integral, the following

$$\begin{aligned}
|J_1| \leq & \frac{1}{\Gamma(1+\varsigma)} \int_0^1 \hbar^{\varsigma}(1-\hbar)^{2\varsigma} \left|\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)\right| (d\hbar)^{\varsigma} \\
\leq & \frac{\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|}{\Gamma(1+\varsigma)} \int_0^1 \hbar^{2\varsigma}(1-\hbar)^{2\varsigma}(d\hbar)^{\varsigma} + \frac{\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|}{\Gamma(1+\varsigma)} \int_0^1 \hbar^{\varsigma}(1-\hbar)^{2\varsigma+\varsigma}(d\hbar)^{\varsigma} \\
& - \frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)}{\Gamma(1+\varsigma)} \int_0^1 \hbar^{2\varsigma}(1-\hbar)^{3\varsigma}(d\hbar)^{\varsigma} \\
= & \frac{B_{\varsigma}(3,3)\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|}{\Gamma(1+\varsigma)} + \left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right) \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right| \\
& - \frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B(3,4)}{\Gamma(1+\varsigma)}. \qquad (48)
\end{aligned}$$

Analogously, we have from the second integral, this result

$$\begin{aligned}
|J_2| \leq & \left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right) \left|\Upsilon^{(2\varsigma)}(\ell_1)\right| \\
& - \frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B(3,4)}{\Gamma(1+\varsigma)} + \frac{B_{\varsigma}(3,3)\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|}{\Gamma(1+\varsigma)}. \qquad (49)
\end{aligned}$$

Using these last two inequalities (48) and (49), we have

$$\begin{aligned}
|J_1|+|J_2| \leq & \left[\frac{B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)} + \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right] \\
& \times \left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right| + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right] - \frac{2^{\varsigma}\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B(3,4)}{\Gamma(1+\varsigma)}. \qquad (50)
\end{aligned}$$

The required inequality (46) is obtained after multiplying this last inequality by the term $\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}$. □

Remark 5.4. (i) In the class of $(\mathfrak{m}-\mathrm{F})$-convex function, the following result, a new inequality, is obtained from the above result, putting $\varsigma = 1$ in (46):

$$\begin{aligned}
& \left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2} - \frac{1}{\mathfrak{m}\ell_2-\ell_1}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz\right| \\
& \leq \frac{(\mathfrak{m}\ell_2-\ell_1)^2}{24}[|\Upsilon''(\ell_1)|+\mathfrak{m}|\Upsilon''(\ell_2)|] - (\mathfrak{m}\ell_2-\ell_1)^2\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)\left(\frac{1}{60}\right).
\end{aligned}$$

(ii) A new inequality, for functions of the class generalized strong $\mathfrak{m}$-convex, is obtained considering $\mathrm{F}(\ell) = c^{\varsigma}|\ell|^{2\varsigma}$ in (46):

$$\begin{aligned}
& \left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)} - \frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right] + \frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}U\right| \\
& \leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left[\frac{B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)} + \frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)} - \frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right] \\
& \quad \times \left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right| + \mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right] - (\mathfrak{m}\ell_2-\ell_1)^{\varsigma}\mathfrak{m}c^{\varsigma}(\ell_1-\ell_2)^{2\varsigma}\frac{B(3,4)}{\Gamma(1+\varsigma)}. \qquad (51)
\end{aligned}$$

(iii) Taking $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=0$ in (46), we have the inequality of [24, Remak (6.1)].
(iv) In the class of generalized $\mathfrak{m}$-convex, taking $\mathrm{F}(\ell)=0$ in (46), we have the following result:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}U\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left[\frac{B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)}+\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right]$$
$$\times\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|+\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|\right], \tag{52}$$

a new inequality not reported in the literature.
(v) The inequality of [27, Proposition 2], is easily obtained from (46), considering $\varsigma=1$, $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=0$:

$$\left|\frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2}-\frac{1}{(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz\right|\leq\frac{(\ell_2-\ell_1)^2}{24}[|\Upsilon''(\ell_1)|+|\Upsilon''(\ell_2)|].$$

Theorem 5.3. *Considering the assumptions $(A_1),(A_2),(A_3)$. If the function $\Upsilon: I^0\subset\mathbb{R}\to\mathbb{R}^{\varsigma}$ is twice locally fractional differentiable on I° and assuming that $|\Upsilon^{(2\varsigma)}|^{\mathrm{q}}$ is a generalized $(\mathfrak{m}-\mathrm{F})$-convex function on $[\ell_1,\mathfrak{m}\ell_2]$, then we have the following result, with $\mathrm{p},\mathrm{q}\geq 1$, $\frac{1}{\mathrm{p}}+\frac{1}{\mathrm{q}}=1$:*

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}U\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{\frac{1}{\mathrm{p}}}H, \tag{53}$$

where

$$U=\left[2^{\varsigma}\Gamma(1+2\varsigma)-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right],$$

and

$$H=\left[\frac{B_{\varsigma}(3,2\mathrm{q}+1)\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}+\xi\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B_{\varsigma}(3,2\mathrm{q}+2)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathrm{q}}}$$
$$+\left[\xi\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}+\frac{B_{\varsigma}(3,2\mathrm{q}+1)\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B_{\varsigma}(3,2\mathrm{q}+2)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathrm{q}}},$$

whith

$$\xi=\left[\frac{\Gamma(1+(2\mathrm{q}+1)\varsigma)}{\Gamma(1+(2\mathrm{q}+2)\varsigma)}-\frac{\Gamma(1+(2\mathrm{q}+2)\varsigma)}{\Gamma(1+(2\mathrm{q}+3)\varsigma)}\right].$$

Proof. Using the triangular inequality, we have from Lemma 5.1:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}U\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(|J_1|+|J_2|\right). \tag{54}$$

If in this last inequality, we use the $(\mathfrak{m}-\mathrm{F})$-generalized convexity of $\left|\Upsilon^{(2\varsigma)}\right|^{\mathrm{q}}$, and the well-known Hölder's Inequality, then we obtain the following result:

$$\begin{aligned}|J_1| &= \left|\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{\varsigma}(1-\hbar)^{2\varsigma}\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}\right| \\ &\le \frac{1}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{\frac{\varsigma}{\mathrm{p}}}\hbar^{\frac{\varsigma}{\mathrm{q}}}(1-\hbar)^{2\varsigma}|\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)|(d\hbar)^{\varsigma} \\ &\le \left(\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{\frac{\varsigma}{\mathrm{p}}\mathrm{p}}(d\hbar)^{\varsigma}\right)^{\frac{1}{\mathrm{p}}}\left[\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{2\varsigma}(1-\hbar)^{2\varsigma\mathrm{q}}(d\hbar)^{\varsigma}\right] \\ &\quad\times\left[\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}\frac{1}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{\varsigma}(1-\hbar)^{\varsigma}(1-\hbar)^{2\varsigma\mathrm{q}}(d\hbar)^{\varsigma}\right. \\ &\quad\left.-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)}{\Gamma(1+\varsigma)}\int_0^1 \hbar^{2\varsigma}(1-\hbar)^{2\varsigma\mathrm{q}+\varsigma}(d\hbar)^{\varsigma}\right]^{\frac{1}{\mathrm{q}}},\end{aligned}$$

or

$$\begin{aligned}|J_1| \le \left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{\frac{1}{\mathrm{p}}}&\left[\frac{B_{\varsigma}(3,2\mathrm{q}+1)\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}+\xi\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}\right. \\ &\left.-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B_{\varsigma}(3,2\mathrm{q}+2)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathrm{q}}}.\end{aligned} \tag{55}$$

Analogously for the second integral, $|J_2|$, we have:

$$\begin{aligned}|J_2| \le \left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{\frac{1}{\mathrm{p}}}&\left[\xi\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}+\frac{B_{\varsigma}(3,2\mathrm{q}+1)\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}\right. \\ &\left.-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B_{\varsigma}(3,2\mathrm{q}+2)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathrm{q}}}.\end{aligned} \tag{56}$$

If we add the inequalities (55) and (56), we have

$$|J_1|+|J_2| \le \left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{\frac{1}{\mathrm{p}}}H. \tag{57}$$

The sought inequality (53) is easily obtained, if we put (57) in (54). □

Remark 5.5. (i) In the case of functions of the class $(\mathfrak{m}-\mathrm{F})$-convex, we have the new result, putting $\varsigma=1$ in (53):

$$\begin{aligned}&\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2}-\frac{1}{(\mathfrak{m}\ell_2-\ell_1)}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz\right| \\ &\le\left[B(3,2\mathrm{q}+1)\left|\Upsilon^2(\ell_1)\right|^{\mathrm{q}}+\xi\mathfrak{m}\left|\Upsilon^2(\ell_2)\right|^{\mathrm{q}}-\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B(3,2\mathrm{q}+2)\right]^{\frac{1}{\mathrm{q}}} \\ &\quad+\left[\xi\left|\Upsilon^2(\ell_1)\right|^{\mathrm{q}}+B(3,2\mathrm{q}+1)\mathfrak{m}\left|\Upsilon^2(\ell_2)\right|^{\mathrm{q}}-\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B(3,2\mathrm{q}+2)\right]^{\frac{1}{\mathrm{q}}},\end{aligned}$$

being

$$\xi=\left[\frac{\Gamma(1+(2\mathrm{q}+1))}{\Gamma(1+(2\mathrm{q}+2))}-\frac{\Gamma(1+(2\mathrm{q}+2))}{\Gamma(1+(2\mathrm{q}+3))}\right].$$

(ii) Taking $\mathrm{F}(\ell)=c^{\varsigma}|\ell|^{2\varsigma}$ in (53), we have the following result:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}U\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{\frac{1}{\mathrm{p}}}\left[\frac{B_{\varsigma}(3,2\mathrm{q}+1)\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}+\xi\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}\right.$$
$$\left.-\frac{\mathfrak{m}c^{\varsigma}(\ell_1-\ell_2)^{2\varsigma}B_{\varsigma}(3,2\mathrm{q}+2)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathrm{q}}}$$
$$+\left[\xi\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}+\frac{B_{\varsigma}(3,2\mathrm{q}+1)\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}-\frac{\mathfrak{m}c^{\varsigma}(\ell_1-\ell_2)^{2\varsigma}B_{\varsigma}(3,2\mathrm{q}+2)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathrm{q}}},$$

a new inequality in literature for generalized strongly $\mathfrak{m}$-convex function.

(iii) Considering $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=c^{\varsigma}|\ell|^{2\varsigma}$ in (53), we have the inequality of [24, Remark (6.3)].

(iv) The following result, is derived from (53), putting $\mathrm{F}(\ell)=0$:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}U\right|$$
$$\leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+\varsigma)}{\Gamma(1+2\varsigma)}\right)^{\frac{1}{\mathrm{p}}}V, \tag{58}$$

where

$$V=\left[\frac{B_{\varsigma}(3,2\mathrm{q}+1)\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}+\xi\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}\right]^{\frac{1}{\mathrm{q}}}$$
$$+\left[\xi\left|\Upsilon^{(2\varsigma)}(\ell_1)\right|^{\mathrm{q}}+\frac{B_{\varsigma}(3,2\mathrm{q}+1)\mathfrak{m}\left|\Upsilon^{(2\varsigma)}(\ell_2)\right|^{\mathrm{q}}}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathrm{q}}}. \tag{59}$$

(v) Considering $\varsigma=1$, $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=0$ in (53), the following result is easily derived:

$$\left|\frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2}-\frac{1}{\ell_2-\ell_1}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz\right|\leq\frac{(\ell_2-\ell_1)^2}{24}[|\Upsilon''(\ell_1)|^{\mathrm{q}}+|\Upsilon''(\ell_2)|^{\mathrm{q}}]^{\frac{1}{\mathrm{q}}}.$$

Theorem 5.4. *Under the assumptions* $(A_1),(A_2),(A_3)$ *and let* $|\Upsilon^{(2\varsigma)}|^{\mathrm{q}}$ *a generalized* $(\mathfrak{m}-\mathrm{F})$*-convexity on* $[\ell_1,\mathfrak{m}\ell_2]$*, then the following result is fulfilled with* $\mathrm{q}\geq 1$*:*

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]\right.$$
$$\left.+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[2^{\varsigma}\Gamma(1+2\varsigma)-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right]\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\right)^{1-\frac{1}{\mathrm{q}}}B, \tag{60}$$

where

$$B=\left[\frac{B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)}+\left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right)\right.$$
$$\left.-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B_{\varsigma}(3,4)}{\Gamma(1+\varsigma)}(|\Upsilon^{(2\varsigma)}(\ell_1)|^{\mathfrak{q}}+\mathfrak{m}|\Upsilon^{(2\varsigma)}(\ell_2)|^{\mathfrak{q}})\right]^{\frac{1}{\mathfrak{q}}}.$$

Proof. Taking into consideration the Lemma 5.1, the following result is obtained:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[2^{\varsigma}\Gamma(1+2\varsigma)-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right]\right|$$
$$\leq\frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(J_1+J_2\right). \tag{61}$$

If we now use the generalized power mean integral inequality of $|\Upsilon^{(2\varsigma)}|^{\mathfrak{q}}$ and known properties of the module we will have the following:

$$\begin{aligned}|J_1|&\leq\left|\frac{1}{\Gamma(1+\varsigma)}\int_0^1\hbar^{\varsigma}(1-\hbar)^{2\varsigma}\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)(d\hbar)^{\varsigma}\right|\\&\leq\left(\frac{1}{\Gamma(1+\varsigma)}\int_0^1\hbar^{\varsigma}(1-\hbar)^{2\varsigma}(d\hbar)^{\varsigma}\right)^{1-\frac{1}{\mathfrak{q}}}\\&\quad\times\left(\frac{1}{\Gamma(1+\varsigma)}\int_0^1\hbar^{\varsigma}(1-\hbar)^{2\varsigma}|\Upsilon^{(2\varsigma)}(\ell_1\hbar+(1-\hbar)\mathfrak{m}\ell_2)|(d\hbar)^{\varsigma}\right)^{\frac{1}{\mathfrak{q}}}.\end{aligned}$$

From the F-convexity and integrating, we have the following result:

$$\begin{aligned}|J_1|&\leq\left(\frac{1}{\Gamma(1+\varsigma)}\int_0^1\hbar^{\varsigma}(1-\hbar)^{2\varsigma}(d\hbar)^{\varsigma}\right)^{1-\frac{1}{\mathfrak{q}}}\\&\quad\times\left(\frac{|\Upsilon^{(2\varsigma)}(\ell_1)|^{\mathfrak{q}}}{\Gamma(1+\varsigma)}\int_0^1\hbar^{2\varsigma}(1-\hbar)^{2\varsigma}+\frac{\mathfrak{m}|\Upsilon^{(2\varsigma)}(\ell_2)|}{\Gamma(1+\varsigma)}\int_0^1\hbar^{\varsigma}(1-\hbar)^{3\varsigma}\right.\\&\quad\left.-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)}{\Gamma(1+\varsigma)}\int_0^1\hbar^{2\varsigma}(1-\hbar)^{3\varsigma}(d\hbar)^{\varsigma}\right)^{\frac{1}{\mathfrak{q}}},\end{aligned}$$

or

$$\begin{aligned}|J_1|&\leq\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\right)^{1-\frac{1}{\mathfrak{q}}}\\&\quad\times\left[\frac{|\Upsilon^{(2\varsigma)}(\ell_1)|^{\mathfrak{q}}B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)}+\mathfrak{m}|\Upsilon^{(2\varsigma)}(\ell_2)|^{\mathfrak{q}}\left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right)\right.\\&\quad\left.-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B_{\varsigma}(3,4)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathfrak{q}}}.\end{aligned} \tag{62}$$

Similarly,

$$\begin{aligned}|J_2|&\leq\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\right)^{1-\frac{1}{\mathfrak{q}}}\left[|\Upsilon^{(2\varsigma)}(\ell_1)|^{\mathfrak{q}}\left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right)\right.\\&\quad\left.+\frac{\mathfrak{m}|\Upsilon^{(2\varsigma)}(\ell_2)|^{\mathfrak{q}}B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)}-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)B_{\varsigma}(3,4)}{\Gamma(1+\varsigma)}\right]^{\frac{1}{\mathfrak{q}}}.\end{aligned} \tag{63}$$

The following result, is obtained after adding inequalities (62) and (63):

$$|J_1|+|J_2| \leq \left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\right)^{1-\frac{1}{\mathfrak{q}}} B. \tag{64}$$

The inequality (60), is reached after put (64) in (61). This completes the proof. □

Remark 5.6. (i) In the class of $(\mathfrak{m}-\mathrm{F})$-convex functions, putting $\varsigma=1$ in (60), we have the following result:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2}-\frac{1}{\mathfrak{m}\ell_2-\ell_1}\int_{\ell_1}^{\mathfrak{m}\ell_2}\Upsilon(z)dz\right| \leq \frac{(\mathfrak{m}\ell_2-\ell_1)^2}{2}\left(\frac{1}{12}\right)^{1-\frac{1}{\mathfrak{q}}}\left[\left(\frac{5}{60}\right)-\frac{\mathfrak{m}\mathrm{F}(\ell_1-\ell_2)}{60}(|\Upsilon^2(\ell_1)|^{\mathfrak{q}}+\mathfrak{m}|\Upsilon^2(\ell_2)|^{\mathfrak{q}})\right]^{\frac{1}{\mathfrak{q}}},$$

a new inequality not known in the literature.

(ii) If we take $\mathrm{F}(\ell)=c^{\varsigma}|\ell|^{2\varsigma}$ in (60), the following result is derived, for generalized strongly $\mathfrak{m}$-convex functions:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[2^{\varsigma}\Gamma(1+2\varsigma)-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right]\right| \leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\right)^{1-\frac{1}{\mathfrak{q}}}\left[\frac{B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)}+\left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right)-\frac{\mathfrak{m}c^{\varsigma}(\ell_1-\ell_2)^{2\varsigma}B_{\varsigma}(3,4)}{\Gamma(1+\varsigma)}(|\Upsilon^{(2\varsigma)}(\ell_1)|^{\mathfrak{q}}+\mathfrak{m}|\Upsilon^{(2\varsigma)}(\ell_2)|^{\mathfrak{q}})\right]^{\frac{1}{\mathfrak{q}}},$$

a new inequality in literature for this class of functions.

(iii) If we take $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=c^{\varsigma}|\ell|^{2\varsigma}$ in (60), then we obtain the inequality, which appeared in [24, Remark (6.4)].

(iv) Putting $\mathrm{F}(\ell)=0$ in (60), the following result is easily obtained:

$$\left|\frac{\Upsilon(\ell_1)+\mathfrak{m}\Upsilon(\ell_2)}{2^{\varsigma}}\left[\frac{\Gamma(1+3\varsigma)}{\Gamma(1+2\varsigma)}-\frac{\Gamma(1+2\varsigma)}{\Gamma(1+\varsigma)}\right]+\frac{{}_{\ell_1}I^{(\varsigma)}_{\mathfrak{m}\ell_2}\Upsilon(z)}{2^{\varsigma}(\mathfrak{m}\ell_2-\ell_1)^{\varsigma}}\left[2^{\varsigma}\Gamma(1+2\varsigma)-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+\varsigma)}\right]\right| \leq \frac{(\mathfrak{m}\ell_2-\ell_1)^{2\varsigma}}{2^{\varsigma}}\left(\frac{\Gamma(1+2\varsigma)}{\Gamma(1+3\varsigma)}-\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}\right)^{1-\frac{1}{\mathfrak{q}}} B,$$

where

$$B=\left[\frac{B_{\varsigma}(3,3)}{\Gamma(1+\varsigma)}+\left(\frac{\Gamma(1+3\varsigma)}{\Gamma(1+4\varsigma)}-\frac{\Gamma(1+4\varsigma)}{\Gamma(1+5\varsigma)}\right)(|\Upsilon^{(2\varsigma)}(\ell_1)|^{\mathfrak{q}}+\mathfrak{m}|\Upsilon^{(2\varsigma)}(\ell_2)|^{\mathfrak{q}})\right]^{\frac{1}{\mathfrak{q}}},$$

(v) Putting $\varsigma=1$, $\mathfrak{m}=1$ and $\mathrm{F}(\ell)=0$ in (60), we have the following result:

$$\left|\frac{\Upsilon(\ell_1)+\Upsilon(\ell_2)}{2}-\frac{1}{(\ell_2-\ell_1)}\int_{\ell_1}^{\ell_2}\Upsilon(z)dz\right| \leq \frac{(\ell_2-\ell_1)^2}{24}[|\Upsilon''(\ell_1)|^{\mathfrak{q}}+|\Upsilon''(\ell_2)|^{\mathfrak{q}}]^{\frac{1}{\mathfrak{q}}}.$$

Conclusions

In this paper, we have obtained various results referring to inequalities of the Hermite-Hadamard type for the class of $(\mathfrak{m}-\mathrm{F})$-convex functions using local fractional calculus.

Throughout the work we have shown that many results known from the literature are particular cases of ours. We also derive new, unpublished inequalities, as a consequence of various considerations in our results. All this shows the scope and generality of them.

Of course our methods illustrate the possibility of obtaining new generalizations for other classes of convex functions.

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