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Local dynamics and bifurcation for a two-dimensional cubic Lotka-Volterra system (I)

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Abstract. A two-dimensional cubic Lotka-Volterra system depending on two parameters is considered. Local dynamics in a neighbourhood of the origin of the phase plane, when the parameters lay in a sufficiently small neighbourhood of the origin, is investigated. The study is performed when some additional hypotheses on the coefficients are satisfied. From one up to four different equilibria and several types of codimension one local bifurcations are found. For each of the identified cases, bifurcation diagrams are given.

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1. Introduction

Lotka-Volterra systems are widely used to model predator-pray type of interactions in biology and ecology [5], [8], as well as many other type of dynamical behavior [10], [2], [20],[16], [17]. They are a special case of Kolmogorov systems serving to model various phenomena in population modelling, biology, ecology, environment, engineering, economics or mechanics [4], [7], [11], [18], [19].

In this paper we study local bifurcation and dynamics for a two-dimensional cubic Lotka-Volterra system

$$\left\{\begin{array}{lcl} \frac{dx}{dt} & = & x\left[\mu_1+p_{11}x+p_{12}y+p_{13}x^2+p_{14}xy+p_{15}y^2\right] \\ \frac{dy}{dt} & = & y\left[\mu_2+p_{21}x+p_{22}y+p_{23}x^2+p_{24}xy+p_{25}y^2\right] \end{array}\right. \tag{1.1}$$

where $\mu=(\mu_1,\mu_2)\in\mathbb{R}^2$, and the coefficients $p_{ij}=p_{ij}(\mu)$, $i=1,2$, $j=\overline{1,5}$, are smooth functions of the parameter μ, while x,y are the state variables.

System (1.1) has an equilibrium point at the origin, which for the parameters $\mu_1=0,\mu_2=0$ is a nongeneric double zero singularity.

The local dynamics around the origin of system (1.1) was analyzed, when all the third order terms are present in [1], or only when some of these terms are present in [3], in relation to the double Hopf bifurcation. In [1], [3], the system was restricted to the invariant region $D=\left\{(x,y)\in\mathbb{R}^2, x\geq 0, y\geq 0\right\}$, and studied in the hypotheses (HH.1) $p_{11}(0)\neq 0$, (HH.2) $p_{12}(0)\neq 0$, (HH.3) $p_{21}(0)\neq 0$, (HH.4) $p_{22}(0)\neq 0$, (HH.5) $(p_{11}p_{22}-p_{12}p_{21})(0)\neq 0$.

Tigan et al., considering only some of the third order terms, analyzed the local dynamics of system (1.1), in the assumption $p_{12}(0)p_{22}(0)<0$, in two different hypotheses (i) $p_{11}(0)p_{21}(0)\neq 0$ [14] and (ii) either $p_{11}(0)=0$ or $p_{21}(0)=0$ [15]. In

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[22] the case $p_{12}(0)\,p_{22}(0) > 0$ was treated when one of the hypotheses (HH.1) or (HH.3) is not satisfied.

In a previous work [21], it was analyzed the dynamics of system (1.1) when $p_{12}(0)p_{21}(0) > 0$, and one of the hypotheses (HH.1) or (HH.4) is not fulfilled.

The present work is concerned with the study of the local dynamics of the system (1.1) when the hypotheses (HH.2) or (HH.3) are satisfied, and $p_{12}(0)p_{21}(0) < 0$.

Since Lotka-Volterra systems as (1.1) serve to model real-life phenomenas, from application point of view only the case of non-negative variables is of interest. Thus, we restrict the study to the first quadrant of the phase plane, respectively to the set

$$D = \left\{(x, y) \in \mathbb{R}^2, x \geq 0, y \geq 0\right\}.$$

Since the straight lines $x = 0$ and $y = 0$ are invariant curves for system (1.1), the set D is invariant with respect to the associated dynamical system.

The parameters $|\mu_1|$ and $|\mu_2|$ are considered infitesimally small, i.e. $|\mu| < \varepsilon$, for some $0 < \varepsilon \ll 1$ sufficiently small. The coefficients $p_{ij}(\mu)$ are assumed to be smooth functions on the open set $V_\varepsilon = \left\{(\mu_1, \mu_2) \in \mathbb{R}^2, |\mu| = \sqrt{\mu_1^2 + \mu_2^2} < \varepsilon\right\}$, such that $p_{12}(0)p_{21}(0) \neq 0$.

As $p_{12}(0)p_{21}(0) < 0$, assume $p_{12}(0) < 0$ and $p_{21}(0) > 0$. The change of variable

$$\xi_1 = -p_{12}(\mu)x, \xi_2 = p_{21}(\mu)y$$

is well defined and nonsingular for all $|\mu|$ small enough. In addition, the set D remains invariant.

Since $\frac{d\xi_1}{dt} = -p_{12}(\mu)\frac{dx}{dt}$ and $\frac{d\xi_2}{dt} = p_{21}(\mu)\frac{dy}{dt}$, system (1.1) is locally topologically equivalent near the origin to

$$\left\{\begin{array}{l} \frac{d\xi_1}{dt} = \xi_1\left(\mu_1 - \theta(\mu)\xi_1 + \gamma(\mu)\xi_2 + N(\mu)\xi_1^2 - M(\mu)\xi_1\xi_2 + L(\mu)\xi_2^2\right), \\ \frac{d\xi_2}{dt} = \xi_2\left(\mu_2 - \frac{1}{\gamma(\mu)}\xi_1 + \delta(\mu)\xi_2 + Q(\mu)\xi_1^2 - S(\mu)\xi_1\xi_2 + P(\mu)\xi_2^2\right), \end{array}\right. \tag{1.2}$$

where the coefficients are given by $\theta(\mu) = \frac{p_{11}(\mu)}{p_{12}(\mu)}$, $\gamma(\mu) = \frac{p_{12}(\mu)}{p_{21}(\mu)}$, $N(\mu) = \frac{p_{13}(\mu)}{p_{12}^2(\mu)}$, $M(\mu) = \frac{p_{14}}{p_{12}p_{21}}(\mu)$, $L(\mu) = \frac{p_{15}}{p_{21}^2}(\mu)$, $\delta(\mu) = \frac{p_{22}(\mu)}{p_{21}(\mu)}$, $Q(\mu) = \frac{p_{23}}{p_{12}^2}$, $S(\mu) = \frac{p_{24}}{p_{21}p_{12}}(\mu)$, and $P(\mu) = \frac{p_{25}}{p_{21}^2}(\mu)$, with $\theta = \theta(\mu)$, $\gamma = \gamma(\mu)$, $\delta = \delta(\mu)$, and similarly for the other coefficients. However, as some of these expressions are needed only at $\mu = 0$, we use the notations $\theta = \theta(0)$, $\gamma = \gamma(0)$, $\delta = \delta(0)$, $N = N(0)$ and so on.

Remark 1.1. 1) As $p_{12}(0)p_{21}(0) < 0$ it follows $\gamma(0) < 0$; we may consider ε such that $\gamma(\mu) < 0$ for $\mu \in V_\varepsilon$.

2) When $p_{12}(0) > 0$ and $p_{21}(0) < 0$, the change of variables $\xi_1 = p_{12}(\mu)x$, $\xi_2 = -p_{21}(\mu)y$, leads to system

$$\left\{\begin{array}{l} \frac{d\xi_1}{dt} = \xi_1\left(\mu_2 + \theta(\mu)\xi_1 + \gamma(\mu)\xi_2 + N(\mu)\xi_1^2 - M(\mu)\xi_1\xi_2 + L(\mu)\xi_2^2\right) \\ \frac{d\xi_2}{dt} = \xi_2\left(\mu_1 - \frac{1}{\gamma(\mu)}\xi_1 - \delta(\mu)\xi_2 + Q(\mu)\xi_1^2 - S(\mu)\xi_1\xi_2 + P(\mu)\xi_2^2\right) \end{array}\right.$$

and $(\theta, \delta) \longmapsto (-\theta, -\delta)$ lead to the same system (1.1).

The paper is organized as it follows. In Section 2 we determine the equilibrium points of the model near the origin and establish their topological type. In Section 3 we prove the exitence of codimension one local bifurcations. In addition to the transcritical bifurcation found in the case $\gamma > 0$ (studied in [21]), we found a richer dynamics, including limit cycles borned by supercritical or subcritical Hopf bifurcation. The parameter portraits and the generic local phase portraits are presented in section 4. Finally, some conclusions are formulated.

2. Local analysis of the system

We find that, in addition to the trivial equilibrium point in origin, the system (1.2) possesses at most three other equilibria, near to the origin. We establish the topological type of these equilibria and the parameter strata corresponding to nonhyperbolic points.

As $\theta(0)\delta(0) \neq 0$, we may assume $\theta(\mu)\delta(\mu) \neq 0$ for $\mu \in V_\varepsilon$.

System (1.2) has the trivial equilibrium $E_0 = (0,0)$, and two other equilibria $E_1 = \left(\hat{\xi}_1, 0\right)$ and $E_2 = \left(0, \hat{\xi}_2\right)$, close to E_0. The existence of these two equilibria is ensured by the Implicit Function Theorem (IFT) applied to equations $\mu_1 - \theta(\mu)\hat{\xi}_1 + L(\mu)\hat{\xi}_1^2 = 0$, and $\mu_2 + \delta(\mu)\hat{\xi}_2 + P(\mu)\hat{\xi}_2^2 = 0$, respectively.

As $\theta\delta \neq 0$, we find the solutions $\hat{\xi}_1 = \frac{1}{\theta}\mu_1 \left(1 + O\left(|\mu|\right)\right), \hat{\xi}_2 = -\frac{1}{\delta}\mu_2 \left(1 + O\left(|\mu|\right)\right),$ close to 0, with $|\mu|$ sufficiently small, respectively.

The existence of a third equilibrium $E_3 = (\xi_1^*, \xi_2^*)$ close to E_0 for $|\mu|$ small is also ensured by the IFT, applied to the system

$$\begin{cases} \mu_1 - \theta\left(\mu\right)\xi_1 + \gamma\left(\mu\right)\xi_2 + N(\mu)\xi_1^2 - M(\mu)\xi_1\xi_2 + L(\mu)\xi_2^2 = 0, \\ \mu_2 - \frac{1}{\gamma(\mu)}\xi_1 + \delta(\mu)\xi_2 + Q(\mu)\xi_1^2 - S(\mu)\xi_1\xi_2 + P(\mu)\xi_2^2 = 0, \end{cases}$$

in the hypothesis $\theta\delta - 1 \neq 0$. The coordinates of E_3 are

$$\xi_1^* = \frac{\delta\mu_1 - \gamma\mu_2}{\theta\delta - 1} + O\left(|\mu|^2\right), \quad \xi_2^* = \frac{\mu_1 - \theta\gamma\mu_2}{\gamma(\theta\delta - 1)} + O\left(|\mu|^2\right).$$

Remark that $E_1 \in D$ only if $\theta\mu_1 \geq 0$, $E_2 \in D$ if $\delta\mu_2 \leq 0$, while E_3 is inside D whenever the parameter (μ_1, μ_2) lies inside the region

$$R_1 = \left\{(\mu_1, \mu_2), \mu_1 - \theta\gamma\mu_2 < 0, \delta\mu_1 - \gamma\mu_2 > 0\right\} \tag{2.1}$$

if $\theta\delta > 1$, respectively, in

$$R_2 = \left\{(\mu_1, \mu_2), \mu_1 - \theta\gamma\mu_2 > 0, \delta\mu_1 - \gamma\mu_2 < 0\right\} \tag{2.2}$$

if $\theta\delta < 1$. This equilibrium exits D when (μ_1, μ_2) crosses the bifurcation curves

$$T_1 = \left\{(\mu_1, \mu_2), \mu_1 = \theta\gamma\mu_2 + O\left(\mu_2^2\right),\ \mu_2 < 0\right\} \tag{2.3}$$

or

$$T_2 = \left\{(\mu_1, \mu_2), \gamma\mu_2 = \delta\mu_1 + O\left(\mu_1^2\right),\ \mu_1 > 0\right\}. \tag{2.4}$$

For parameters in T_1 we have $\xi_2^* = 0$, thus E_3 collides with E_1, while for parameters in T_2 we have $\xi_1^* = 0$, thus E_3 collides with E_2. Note that only the lowest terms in (μ_1, μ_2) are used to describe regions R_1, R_2.

The following results concerning the topological type of equilibria E_0, E_1, E_2, E_3 can be easily obtained.

Lemma 2.1. *For $|\mu|$ sufficiently small the following hold, the trivial equilibrium point E_0 is:*
((i) a saddle as $\mu_1\mu_2 < 0$,
(ii) a repeller as $\mu_1 > 0, \mu_2 > 0$,

(iii) an attractor as $\mu_1 < 0, \mu_2 < 0$*, or*
(iv) nonhyperbolic of fold type as $\mu_1 = 0$ *or* $\mu_2 = 0$.

Proof. As the eigenvalues of the Jacobi matrix associated at E_0 are μ_1 and μ_2, the result is evident. □

Lemma 2.2. *For* $|\mu|$ *sufficiently small the following hold, whenever* E_1 *lies in* D*,* E_1 *is either:*
(i) a saddle as $\theta\mu_2 - \frac{1}{\gamma}\mu_1 > 0$,
(ii) a repeller as $\mu_2 - \frac{1}{\theta\gamma}\mu_1 > 0, \theta < 0$,
(iii) a stable node as $\mu_2 - \frac{1}{\theta\gamma}\mu_1 < 0, \theta > 0$*, or*
(iv) nonhyperbolic of fold type as $\mu_1 = 0$ *or* $\theta\gamma\mu_2 - \mu_1 = 0$.

Proof. The eigenvalues of the Jacobi matrix associated at E_1 are $-\mu_1\left(1 + O\left(|\mu|\right)\right)$ and $\left(\mu_2 - \frac{1}{\theta\gamma}\mu_1\right)\left(1 + O\left(|\mu|\right)\right)$, hence the results. □

Lemma 2.3. *For* $|\mu|$ *sufficiently small, whenever the equilibrium point* E_2 *lies in* D*,* E_2 *is either:*
(i) a saddle as $\delta\mu_1 - \gamma\mu_2 < 0$,
(ii) a stable node as $\mu_1 - \frac{\gamma}{\delta}\mu_2 < 0,\ \delta < 0$,
(iii) a repeller node as $\mu_1 - \frac{\gamma}{\delta}\mu_2 > 0,\ \delta > 0$*,or*
(iv) nonhyperbolic of fold type as $\mu_2 = 0$ *or* $\delta\mu_1 - \gamma\mu_2 = 0$.

Proof. The eigenvalues of the Jacobi matrix associated at E_1 are $\left(\mu_1 - \frac{\gamma}{\delta}\mu_2\right)\left(1 + O\left(|\mu|\right)\right)$, and $-\mu_2\left(1 + O\left(|\mu|\right)\right)$, hence the results. □

The topological type of the nontrivial equilibrium E_3 is established bellow.

Proposition 2.4. *Assume that the equilibrium point* E_3 *is inside* D*. The following assertions are true.*
1) If $\theta\delta - 1 > 0$*, then* E_3 *is a saddle.*
2) If $\theta\delta - 1 < 0$ *and* $\mu \in R_2$*, then* E_3 *is*
(i) an attractor, if either $\theta > 0, \delta < 0$ *or for* $\delta > 0, \theta > 0$*, and* $\mu_2 < \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)$*, or for* $\delta < 0, \theta < 0$ *and* $sign(\gamma - \delta)(\mu_2 - \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)) < 0$;
(ii) a repeller if either $\theta < 0, \delta > 0$ *or for* $\delta > 0, \theta > 0, \mu_2 > \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)$*, or for* $\delta < 0, \theta < 0$*,* $sign(\gamma - \delta)(\mu_2 - \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)) > 0$*;*
(iii) nohyperbolic of Hopf type if $0 < \theta\delta < 1$ *and* $\mu_2 = \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)$. .
3) If $\mu \in T_1 \cup T_2$*, then* E_3 *is a nonhyperbolic equilibrium of fold type.*

Proof. The equilibrium $E_3 = (\xi_1^*, \xi_2^*)$, which bifurcates from O along the curves T_1 and T_2.

Notice that E_3 collides with E_1 on T_1, respectively, with E_2 on T_2, for $|\mu|$ small, hence 3) is proved.

The eigenvalues $\lambda_{1,2}$ of E_3 satisfy the relations

$$\begin{aligned}
\lambda_1\lambda_2 &= \xi_1^*\xi_2^*\left(1 - \delta\theta + O(|\mu|)\right), \\
\lambda_1 + \lambda_2 &= -\frac{\delta(\theta\gamma - 1)}{\gamma(\delta\theta - 1)}\mu_1 + \frac{\theta(\gamma - \delta)}{\delta\theta - 1}\mu_2 + O(|\mu|^2).
\end{aligned}$$

Consequently, if $\delta\theta - 1 > 0$, then E_3 is a saddle.

If $\delta\theta - 1 < 0$, we have to evaluare the sign of the function $p(\mu) = -\frac{\delta(\theta\gamma-1)}{\gamma(\delta\theta-1)}\mu_1 + \frac{\theta(\gamma-\delta)}{\delta\theta-1}\mu_2 + O(|\mu|^2)$, for small $\|\mu\|$.

Using IFT applied to equation $p(\mu_1, \mu_2) = 0$, we obtain a curve

$$H = \left\{ (\mu_1, \mu_2), \mu_2 = \frac{\delta(\theta\gamma - 1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right) \right\} \tag{2.5}$$

provided that $\theta\gamma \neq 1$ and $\gamma \neq \delta$. This curve intersect region R_2, if $\delta\theta > 0$, while $R_2 \cap H = \varnothing$ if $\delta\theta < 0$. It is easy to prove that we have $p(\mu) < 0$ if $\theta > 0, \delta < 0$ and that $p(\mu) > 0$ if $\theta < 0, \delta > 0$.

If $\delta > 0, \theta > 0$ and $\mu \in R_2$, we obtain $p(\mu) < 0$ for $\mu_2 < \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)$, and $p(\mu) > 0$ for $\mu_2 > \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)$. If $\delta < 0, \theta < 0$ and $\mu \in R_2$, we obtain $p(\mu) > 0$ for $sign(\gamma-\delta)(\mu_2 - \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)) > 0$, and $p(\mu) < 0$ for $sign(\gamma-\delta)(\mu_2 - \frac{\delta(\theta\gamma-1)}{\theta\gamma(\gamma-\delta)}\mu_1 + O\left(\mu_1^2\right)) < 0$. □

3. Local bifurcations

Here we prove the existence of transcritical bifurcations and find necessary and sufficient conditions for the existence of Hopf bifurcation for the system (1.2).

Denote by

$$X_+ = \{(\mu_1, \mu_2), \mu_2 = 0,\ \mu_1 > 0\}, \quad X_- = \{(\mu_1, \mu_2), \mu_2 = 0,\ \mu_1 < 0\},$$
$$Y_+ = \{(\mu_1, \mu_2), \mu_1 = 0,\ \mu_2 > 0\}, \quad Y_- = \{(\mu_1, \mu_2), \mu_1 = 0,\ \mu_2 < 0\},$$

the four semiaxes of the (μ_1, μ_2) parameter plane.

Proposition 3.1. *The following transcritical bifurcations occur for system (1.2):*
(i) at the point E_0 as the parameter crosses the curves Y_+ or Y_- (when $E_0 = E_1$);
(ii) at the point E_0 as the parameter crosses the curves X_+ or X_- (when $E_0 = E_2$);
(iii) at the point E_1 as the parameter (μ_1, μ_2) crosses the curve T_1 (when $E_1 = E_3$);
(iv) at the point E_2 as the parameter (μ_1, μ_2) crosses the curve T_2 (when $E_2 = E_3$).

Proof. In order to prove these statements we apply a Sotomayor Theorem ([6], p. 338).

(i) The Jacobian matrix $Df(E_0, \mu_0)$ at $\mu_0 = (0, \mu_2)$, $\mu_2 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (1,0)^T$ and the left eigenvector $w = (1,0)^T$. It follows

$$w^T f_{\mu_1}(E_0, \mu_0) = 0, \quad w^T Df_{\mu_1}(E_0, \mu_0) = 1 \neq 0,$$
$$w^T[D^2 f(E_0, \mu_0)(v, v)] = -2\theta(1 + O(\mu_2)) \neq 0,$$

thus the transcritical bifurcation conditions are satisfied.

Here f is the vector field associated to system (1.2).

(ii) The Jacobian matrix $Df(E_0, \mu_0)$ at $\mu_0 = (\mu_1, 0)$, $\mu_1 \neq 0$, has a zero eigenvalue with the right eigenvector $v = (0,1)^T$ and the left eigenvector $w = (0,1)^T$. It follows

$$w^T f_{\mu_2}(E_0, \mu_0) = 0, \quad w^T Df_{\mu_2}(E_0, \mu_0) = 1 \neq 0,$$
$$w^T[D^2 f(E_0, \mu_0)(v, v)] = 2\delta(1 + O(\mu_1)) \neq 0,$$

ensuring the existence of a transcritical bifurcation.

(iii) Consider $\mu_0 = (\mu_1, \mu_2) \in T_1$, $\mu_1 \neq 0$, and μ_2 as a bifurcation parameter, thus $\mu_0 = (\mu_1, 1/\theta\gamma\mu_1)$. We find that $v = (\gamma, \theta)^T$, in its lowest terms, and $w = (0,1)^T$ are right and left eigenvectors of the Jacobian matrix $Df(E_1, \mu_0)$, respectively, corresponding to the zero eigenvalue, and

$$w^T f_{\mu_2}(E_1, \mu_0) = 0, \quad w^T Df_{\mu_2}(E_1, \mu_0) = \theta + O(\mu_1) \neq 0,$$
$$w^T[D^2 f(E_1, \mu_0)(v,v)] = 2\theta(\delta\theta - 1) + O(\mu_1) \neq 0,$$

consequently, for sufficiently small $|\mu|$, the conditions are satisfied.

(iv) Finally, consider $\mu_0 = (\mu_1, \mu_2) \in T_2$, $\mu_2 \neq 0$, and μ_1 as a bifurcation parameter, thus $\mu_0 = (\frac{\gamma}{\delta}\mu_2, \mu_2)$. We find the eigenvectors $v = (\delta\gamma, 1)^T$, in its lowest terms, and $w = (1,0)^T$, and

$$w^T f_{\mu_1}(E_2, \mu_0) = 0, \quad w^T Df_{\mu_1}(E_2, \mu_0) = \delta\gamma + O(\mu_2) \neq 0,$$
$$w^T[D^2 f(E_1, \mu_0)(v,v)] = -2\delta\gamma^2(\delta\theta - 1) + O(\mu_2) \neq 0,$$

for sufficiently small $|\mu|$. □

From Proposition 2.4 it follows that system (1.2) may exhibit a Hopf bifurcation only in the hypothesis $0 < \theta\delta < 1$.

Theorem 3.2. *For all $\gamma < 0$, and θ,δ, satisfying $0 < \theta\delta < 1$, a nondegenerated Hopf bifurcation takes place at E_3, when the parameters (μ_1, μ_2) transversally cross the curve H, for sufficiently small $|\mu|$, if the expression:*

$$V(\mu) = \theta\left(L\theta - 2M\delta\gamma\theta + 3N\delta\gamma^2 - 2S\delta\gamma\theta + 3P\theta + Q\delta\gamma^2\right)$$

is nonzero for $\mu \in H$. In addition,

1) if $V(\mu) < 0$ for $\mu \in H$, then the Hopf bifurcation is supercritical;

2) if $V(\mu) > 0$ for $\mu \in H$, then the Hopf bifurcation is subcritical.

Proof. To simplify the computation, we chose to cross the curve H in the direction of the $O\mu_2$ axis, thus $\mu_1 \neq 0$, is fixed, and μ_2 is the bifurcation parameter. Similar computations can be performed for other transversal directions.

The first condition for the Hopf bifurcation is satisfied, as

$$\frac{Re(\lambda_1)}{d\mu_2}_{|H} = \frac{\theta(\gamma - \delta)}{\theta\delta - 1}\left(1 + O(\mu_1)\right) \neq 0,$$

for sufficiently small $|\mu|$. For μ on H we obtain that the signum of first Lyapunov coefficient is the same with the signum of $V(\mu)$ and the results follows from the Andronov-Hopf Theorem. □

4. Bifurcation diagrams

For a fixed $\gamma < 0$, the curves $\theta\delta - 1 = 0$, $\theta = 0$, $\delta = 0$, determine six regions in the $(\theta, \delta)-$ plane, corresponding to the following cases:

A$_1$: $\theta\delta - 1 > 0, \theta > 0, \delta > 0$;
A$_2$: $\theta\delta - 1 > 0, \theta < 0, \delta < 0$;
A$_3$: $\theta > 0, \delta < 0$;
A$_4$: $\theta < 0, \delta > 0$;
A$_5$: $\theta\delta - 1 < 0, \theta > 0, \delta > 0$;
A$_6$: $\theta\delta - 1 < 0, \theta < 0, \delta < 0$.

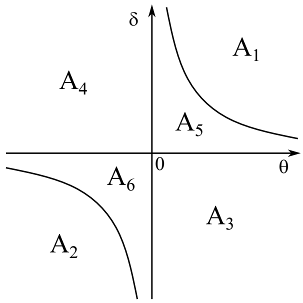

FIGURE 1. Six regions in the (θ, δ) plane, $\gamma < 0$, for system(1.2) (up-left).

For each region (see Fig. 1), in the parametric portraits in the (μ_1, μ_2)- plane, the parameter strata are determined by the origin and the codimension one bifurcation curves X_-, X_+, Y_-, Y_+, T_1, T_2, and H. As a consequence of Theorem 3.2, the curve H is present only in regions A_5 and A_6.

Gathering all of the above information, we can formulate the following result.

Theorem 4.1. *For all $\gamma < 0$, and θ, δ, in regions A_1, A_2, A_3, A_4 of the $(\theta, \delta)-$ plane, the parameter portraits consist of*

$$O \cup T_1 \cup T_2 \cup X_- \cup X_+ \cup Y_- \cup Y_+.$$

The parameter portraits and the corresponding generic phase portraits are shown in Fig. 2, 3, 4, 5.

In the above figures we used the following markers to emphasize the topological type of the equilibria: a black disc for an attractor, a black square for a repeller and a diamond for a saddle point.

Remark 4.1. As the curves $T_1, T_2, X_-, X_+, Y_-, Y_+$ correspond to transcritical bifurcations, the phase portrait for each of these strata is the one of the adjacent region where there exist fewer equilibria. For instance, in A_1, the regions adjacent to T_1 are 5 (with 4 equilibria) and 6 (with 3 equilibria). Thus the phase portrait corresponding to parameters in T_1 is the one in region 6.

Remark 4.2. For cases A_5 and A_6, a Hopf bifurcation occurs when parameters cross H, if the first Lyapunov coefficient is nonzero. As the parameters move away from H, the limit cycle born through this bifurcation may encounter a saddle equilibrium, transforming into a homoclinic loop, or it may exit the visible neighborhood of origin in D ("it blows up"), thus it disappears. In such cases there should exist o bifurcation curve L originating at $\mu = 0$, along which system (1.2) exhibits either a saddle homoclinic bifurcation or the limit cycle "blows up" [3]. Note that it is also possible that for parameters on the curve H the equilibrium E_3 could be a nonlinear center as the first Lyapunov coefficient vanishes.

Remark 4.3. As (θ, δ) belong to region A_5, A_6 for $\mu \in H$, sufficiently small, we have

$$sign(L_1(\mu)) = sign\left(\theta\left(L\theta - 2M\delta\gamma\theta + 3N\delta\gamma^2 - 2S\delta\gamma\theta + 3P\theta + Q\delta\gamma^2\right)\right).$$

As a consequence, we obtain the following result.

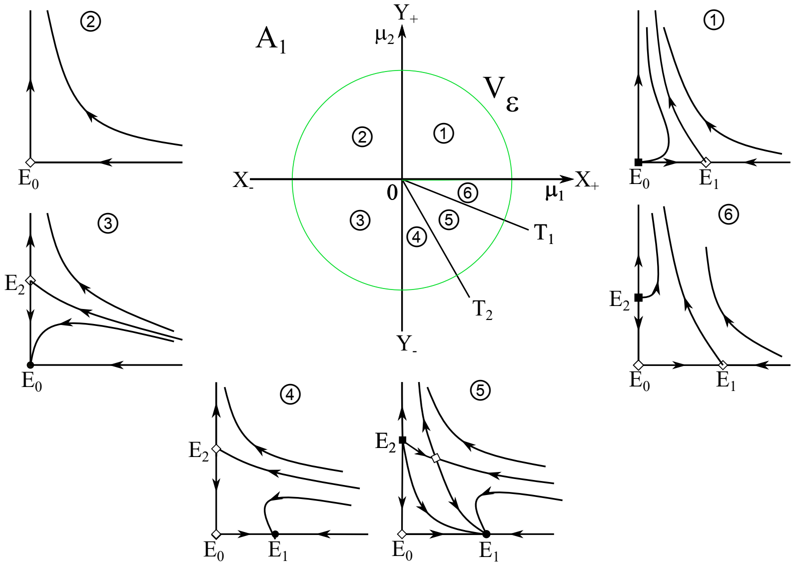

FIGURE 2. Parametric portrait and generic phase portraits in the case $\gamma < 0$, regions A_1.

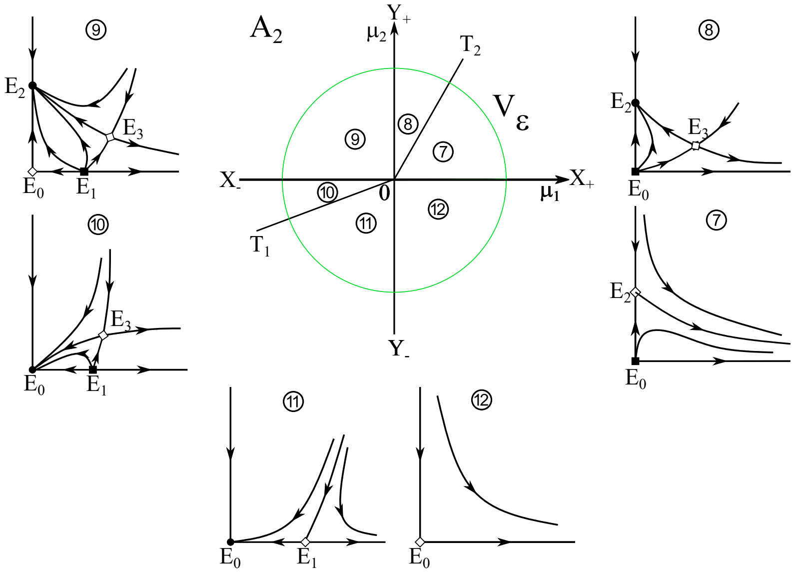

FIGURE 3. Parametric portrait and generic phase portraits in the case $\gamma < 0$, regions A_2.

Theorem 4.2. *For all $\gamma < 0$, and θ, δ, with $0 < \delta\theta < 1$ (in regions A_5 and A_6 of the $(\theta, \delta) -$ plane), the generic parameter portrait consists of*

$$O \cup T_1 \cup T_2 \cup X_- \cup X_+ \cup Y_- \cup Y_+ \cup H \cup L.$$

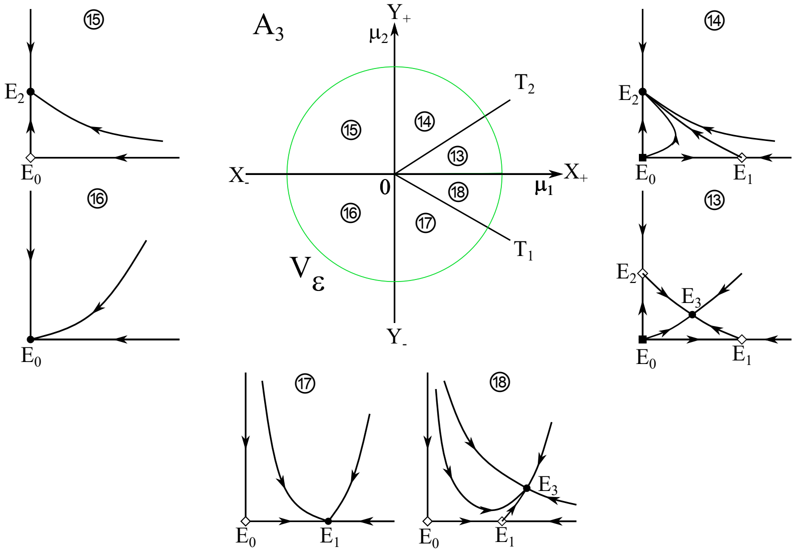

FIGURE 4. Parametric portrait and generic phase portraits in the case $\gamma < 0$, regions A_3.

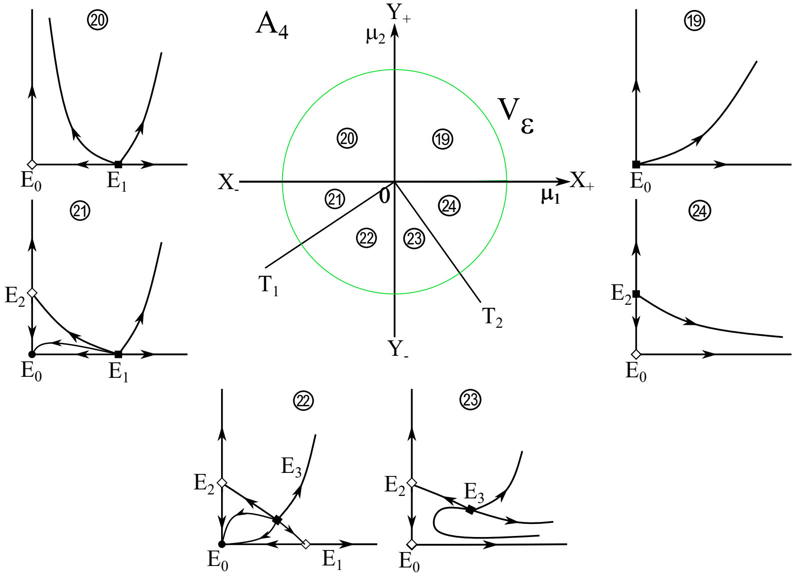

FIGURE 5. Parametric portrait and generic phase portraits in the case $\gamma < 0$, regions A_4.

The parameter portraits and the generic phase portraits are shown in Fig. 6, 7.

Remark that in Fig. 7, we considered only the cases when the Hopf bifurcation is supercritical. We found three different positions for the curve H, namely A_6(i) if $\gamma - \delta > 0$, A_6(ii) if $\gamma - \delta < 0$ and $\theta\gamma > 1$, A_6(iii) if $\gamma - \delta < 0$ and $\theta\gamma < 1$. In Fig. 6, we represented both cases when the Hopf bifurcation is supercritical A_5(i) or subcritical A_5(ii).

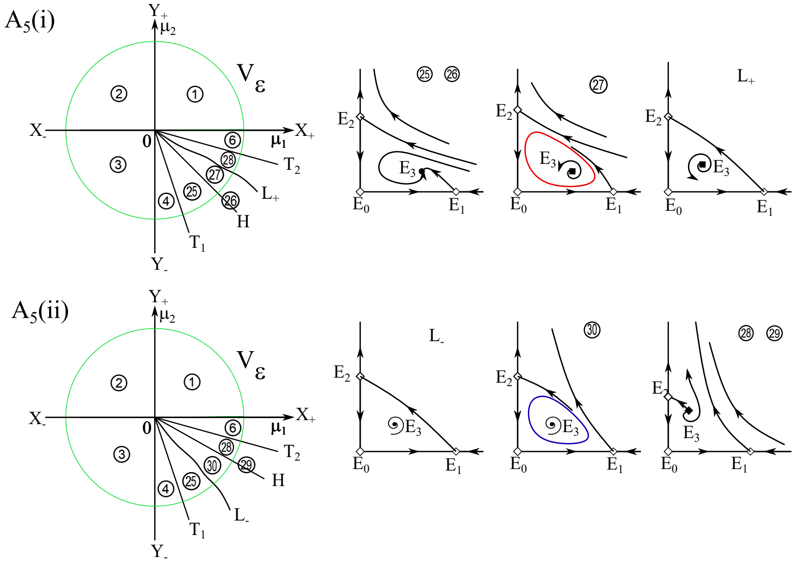

FIGURE 6. Parametric portrait and generic phase portraits in the case $\gamma < 0$, region A_5: (i) $L_1 < 0$, (ii) $L_1 > 0$.

5. Conclusions

In this paper we have studied local dynamics and bifurcation for the cubic Lotka-Volterra system (1.1), with coefficients depending on two parameters, in the hypothesis $p_{12}(0) \cdot p_{21}(0) < 0$. This study completes the one done in [21], where the case $p_{12}(0) \cdot p_{21}(0) > 0$ was investigated. Compared to the situation treated in [21] (called "the simple case"), we have obtained similar dynamics for certain parameter strata, but also bifurcations that are not present in the simple case. Such bifurcations arose mainly due to the presence of Hopf singularities.

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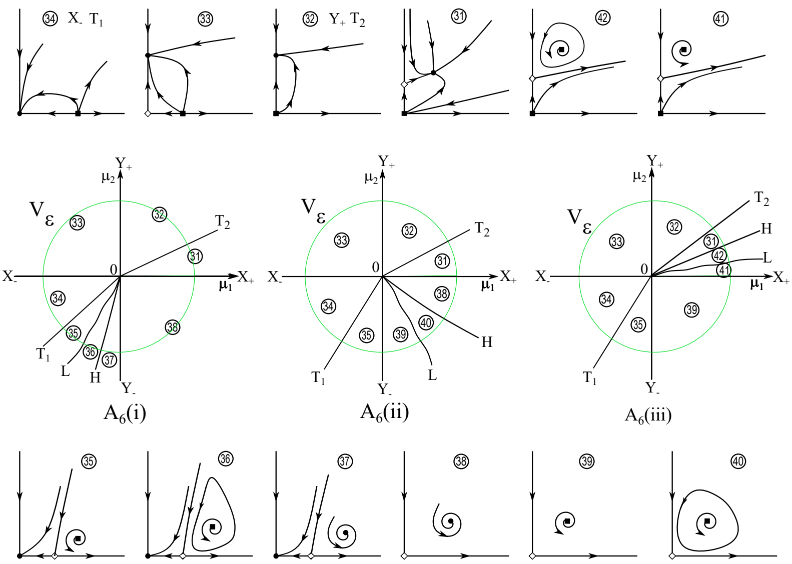

FIGURE 7. Parametric portrait and generic phase portraits in the case $\gamma < 0, \theta < 0, \delta < 0$, $0 < \theta\delta < 1$, region A_6, $L_1 < 0$: (i) $\gamma > \delta$; (ii) $\gamma < \delta$, $\theta\gamma > 1$; (iii) $\gamma < \delta$, $\theta\gamma < 1$.

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