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A sandwich theorem for functions defined on unbounded finite-simplicial sets, some inequalities and the moment problem

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Abstract. We recall the following sandwich-type problem: let X be a convex subset of a real vector space $E, f, g : X \to \mathbf{R}$ two maps, g convex, f concave, $g \leq f$. The problem is: in what conditions upon X, for any pair (g, f) as above, there exists $h : X \to \mathbf{R}$ affine, such that $g \leq h \leq f$.

If X is a compact convex subset of a locally convex space E and if the inequalities are strict (the functions g, f, h being supposed to be continuous), the problem is solved in [4] (X must be a simplex). Some other related results (for the case when X is a compact topological space and S is a convex cone of lower bounded, lower semicontinuous real functions on X for which axiom S_1 ([5], p. 493) is fulfilled), are proved in [5].

Going back to the case $X \subset E$, (E being a vector space), in Section 1 of the present work, we point out a sufficient condition on X for the existence of a solution h for any pair (g, f) as above, introducing the notion of a finite simplicial set. The novelty here is that such sets X are not supposed to be bounded in any locally convex topology on E. Our proof uses Theorem 4 [25], or Theorem 2.1 [26]. As applications, we point out two constants related to some interesting inequalities. In Section 2 we state two applications of Theorem 4 [25] to the classical moment problem in spaces of differentiable functions on $[0, b]$ and we also state a solution of a moment problem in the space of all real continuous functions on the unit sphere of $\mathbf{R}^n$. This last result can be proved without using Theorem 4 [25] or any Hahn-Banach type result (see [3]). In Section 3 we sketch the idea of the proof of Theorem 1.3 from the present work. All results of this work are proved in papers [3] and [28].

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1. A sandwich theorem for maps defined on finite-simplicial sets

It is well-known that if $X \subset E$ is a finite dimensional simplex, the sandwich problem stated in the abstract has a solution for any pair (g, f) as above. On the other hand, it is known that for an arbitrary set X such sandwich-type problems are solved "working" with inequalities involving finite sums (see [3] Theorem 2.3 or [15]). These facts lead naturally to the notion of finite-simplicial set.

Definition 1.1. A convex subset X of a real vector space E is said to be finite-simplicial if for any finite-dimensional compact convex subset $K \subset X$, there exists a finite dimensional simplex T such that

$$K \subset T \subset X.$$

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Examples. 1°. Any convex cone $X \subset \mathbf{R}^2$ is an unbounded finite-simplicial set.

2°. Let $E := \mathbf{R}^n$, $n \geq 3$, and let X be a closed convex cone in E, which has a compact base B which is a simplex. Then X is a finite-simplicial set.

3°. Let E be an arbitrary real vector space, $f \in E^*$ a linear functional, $f \neq 0$. Then the sets $X_1 := \{x \in E; f(x) > \alpha\}$, $\bar{X}_1 = \{x \in E; f(x) \geq \alpha\}$, $\alpha \in \mathbf{R}$, are finite-simplicial subsets. From here we deduce easily that the intersection of two finite-simplicial sets is not necessarily finite-simplicial.

4°. In $E = \mathbf{R}^n$, $n \geq 3$, the cone

$$X := \{(x_1, \ldots, x_n); x_n \geq \alpha(x_1^p + \ldots + x_{n-1}^p)^{1/p}\} \ (\alpha \in]0, \infty[,\ p \in]1, \infty[)$$

has a compact base, but is not finite-simplicial ($\Rightarrow$ the base is not a simplex).

To state the main result of this section, we have to mention some notations and a definition.

Let X be an arbitrary set and Y a vector space. Put

$$\mathcal{F} := \{f : X \to Y\}.$$

Let $x \in X$. Denote $\varepsilon_x(f) := f(x) \in Y, \quad f \in \mathcal{F}$.

Let

$$\varepsilon(X) := \{\varepsilon_x; x \in X\} \subset L(\mathcal{F}, Y),$$
$$V := Sp(\varepsilon(X)) \subset L(\mathcal{F}, Y).$$

Definition 1.2. Let $S \subset \mathcal{F}$ be a convex cone. A function $h \in \mathcal{F}$ is said to be S-affine if and only if for any $T \in V$ with

$$T(s) = 0 \quad \forall s \in S \cap (-S),$$

we have

$$T(h) = 0.$$

The following theorem is the first main result of this section.

Theorem 1.3. *Let E be a vector space, $X \subset E$ a finite-simplicial subset, Y an order-complete vector lattice. Let $f, g : X \to Y$, f concave, g convex, such that*

$$g(x) \leq f(x) \quad \forall x \in X.$$

Let S be the convex cone of all concave maps $f : X \to Y$.

Then there exists a S-affine map $h : X \to Y$ such that

$$g(x) \leq h(x) \leq f(x) \quad \forall x \in X.$$

Corollary 1.4. *Let E, X be as in Theorem 1.3. Assume that $Y = \mathbf{R}$, (and $f, g : X \to \mathbf{R}$, g convex, f concave,*

$$g \leq f \quad on \quad X).$$

Then there exists an affine map: $h : X \to \mathbf{R}$ such that

$$g \leq h \leq f \quad \text{on} \quad X.$$

Corollary 1.5. *Let $a \in \mathbf{R}$ and $g, f : [a, \infty[\to \mathbf{R}$, g convex, f concave,*

$$g \leq f \quad on \quad [a, \infty[\ .$$

Then there exists $\alpha, \beta \in \mathbf{R}$ such that

$$g(x) \leq \alpha x + \beta \leq f(x) \quad \forall x \in [a, \infty[\ .$$

Note that Corollary 1.5. can be obtained immediately from Corollary 1.4. (where $E = \mathbf{R}$, $X = [a, \infty[$). But one can give a direct very elementary proof of Corollary 1.5. which does not use separation or other Hahn-Banach-type arguments (see [28]).

In the following we state some applications of Corollary 1.5. to some scalar inequalities, which lead to some operatorial inequalities (via the spectral measure attached to a self-adjoint operator A acting on a Hilbert space H). In particular, one obtains matricial inequalities, which yield new scalar inequalitites (via Sylvester's criterion). Thus one applies the sketch

Corollary 1.5 $\Rightarrow$ scalar ineq. $\Rightarrow$ operator. ineq. $\Rightarrow$ new scalar ineq.

Here are some examples.

Theorem 1.6. *For any $a > 0$ and any $1 < p \leq 2$, there exists a constant $\rho = \rho(a,p)$ such that*

$$(x^p + a)^{1/p} \leq x + \rho \leq (x^p + x)^{1/p} \quad \forall x \in [a, \infty[.$$

The contant $\rho(a,p)$ is given by

$$\rho(a,p) = (a^p + a)^{1/p} - a.$$

Corollary 1.7. *Let $a > 0$, $n, k \in \mathbf{N}$, $1 \leq k \leq n$, $\rho := \left(a^{\frac{n+k}{n}} + a\right)^{\frac{n}{n+k}} - a$. Let A be a self-adjoint operator acting on a Hilbert space H, such that $\sigma(A) \subset [a, \infty[$ (where $\sigma(A)$ is the spectrum of A). Then we have*

$$\left(A^{\frac{n+k}{n}} + aI\right)^{\frac{n}{n+k}} \leq A + \rho I \leq \left(A^{\frac{n+k}{n}} + A\right)^{\frac{n}{n+k}}.$$

Corollary 1.8. *For any $n, k \in \mathbf{N}$,$1 \leq k \leq n$, the following inequalities hold*

$$\left(\frac{3^{n+k}+1}{2}\right)^n \leq 2^k \left(\frac{3^n - 1}{2} + 2^{-\frac{k}{n+k}}\right)^{n+k} \leq \left(\frac{3^{n+k}+3^n}{2}\right)^n.$$

For the proof of the latest four results see [28].

Theorem 1.9. *Let $n \in \mathbf{N}$, $n \geq 2$ and $a \geq 0$. Then there exists a unique constant*

$$\beta(a,n) \in [0,1]$$

such that for any self-adjoint operator A acting on the Hilbert space H, with $\sigma(A) \subset [a, \infty[$, we have

$$na^{n-1}I \leq (A + \beta(a,n)I)^n - A^n \leq nA^{n-1}.$$

The number $\beta(a,n)$ is given by

$$\beta(a,n) = a^{(n-1)/n}(n+a)^{1/n} - a.$$

In particular, we have

$$\lim_{a \to \infty} \beta(a,n) = 1.$$

Corollary 1.10. *For any $\varphi \in [0, \pi/4]$ and any $a \in [0, \cos\varphi - \sin\varphi]$, one has*

$$\begin{aligned}
&3\beta(a,3) + 3\cos\varphi\beta^2(a,3) + \beta^3(a,3) - 3a^2 \geq 0,\\
&3\beta(a,3) + 3\cos\varphi\beta^2(a,3) + \beta^3(a,3) - 3a^2 \geq\\
&\geq 3\sin(2\varphi)\beta(a,3) + 3\sin\varphi\beta^2(a,3);\\
&3[1 - \beta(a,3)] - 3\cos\varphi\beta^2(a,3) - \beta^3(a,3) \geq 0,\\
&\{3[1 - \beta(a,3)] - 3\cos\varphi\beta^2(a,3) - \beta^3(a,3)\}^2 \geq\\
&\geq \{3\sin(2\varphi)[1 - \beta(a,3)] - 3\sin\varphi\beta^2(a,3)\}^2.
\end{aligned}$$

Corollary 1.11. *For any self-adjoint operator A acting on H, with $\sigma(A) \subset [0,1]$, the following inequalities hold*

$$[I + \beta(A,n)]^n \leq (n+1)I, \quad n \geq 2,$$

where $\beta(A,n) := A^{(n-1)/n}(A + nI)^{1/n} - A$.

Corollary 1.12. *For any $n \in \mathbf{N}$, $n \geq 1$, we have*

$$(2^{1/n} - 1)n^{1/n} \leq (n+1)^{1/n} - 1.$$

Corollary 1.13. *For any self-adjoint operator A with the spectrum*

$$\sigma(A) \subset \{0\} \cup \left\{ \frac{1}{n}; n \in \mathbf{N},\ n \geq 1 \right\},$$

we have

$$(I + A)^A - 2^A \leq A^A - I.$$

For the proof of the latest five results see [3].

Open problems 1.14. (a) For $\dim_{\mathbf{R}} E \geq 2$, find examples of finite-simplicial subsets $X \subset E$ and of interesting pairs (g, f), $g, f : X \to Y$, g convex, f concave, $g \leq f$, for which Theorem 1.3. leads to substantial applications.

(b) Find further applications of the inequalities stated in the corollaries mentioned above.

2. New results concerning the classical moment problem

The following two results can be obtained as applications of Theorem 3.2. stated in Section 3.

Theorem 2.1. *Let $V := C^\infty([0, b])$ endowed with the order relation defined by the cone*

$$V_+ := \{v \in V; v^{(k)}(t) \geq 0, \quad \forall t \in [0, b], \quad \forall k \in \mathbf{N}\}.$$

Let $\{y_j\}_{j \in \mathbf{N}} \subset \mathbf{R}$, $a \in]0, b]$. The following assertions are equivalent:

(a) there exists a linear form $H \in V^*$ such that

$$H(v_j) = y_j, \quad j \in \mathbf{N} \tag{a_1}$$

and

$$0 \leq H(\varphi) \leq \varphi(a) \quad \forall \varphi \in V_+ , \tag{a_2}$$

where $v_j(t) := t^j$, $j \in \mathbf{N}$, $t \in [0, b]$;

(b) $$0 \leq y_j \leq v_j(a), \quad j \in \mathbf{N}.$$

In this type-results (see also the next statements), the main implication is (b) $\Rightarrow$ (a).

The y_j, $j \in \mathbf{N}$, are called the moments and equalities $H(v_j) = y_j$, $j \in \mathbf{N}$, are called the moment conditions. The linear functional H is called the solution of the moment problem defined by (a_1) and (a_2).

The problems stated in Theorems 2.1, 2.2, 2.3, 3.2, are similar to some Markov moment problems (see [17]).

Theorem 2.2. *Let $n \in \mathbf{N}$, $n \geq 2$, $V := C^n([0, b])$, $a \in [0, b]$, $m \in \{0, 1, \ldots, n-1\}$, $V_+ := \{v \in V; v^{(k)}(0) \geq 0, \forall k \in \{0, 1, \ldots, m\}\}$, $v_j(t) = t^j$, $j \in \mathbf{N}$. On V consider the norm*

$$||v||_m := \sup\{|v^{(k)}(t)|; t \in [0, b],\ k \in \{0, 1, \ldots, m\}\}.$$

Let $\{y_j : j \in \mathbf{N}\}$ be a sequence of real numbers. With these notations, one proves that $V = V_+ - V_+$, and the following assertions are equivalent:

(a) there exists a linear continuous functional H on V such that

$$H(v_j) = y_j, \quad j \in \{0, 1, \ldots, m\}$$

and

$$0 \le H(\varphi) \le \varphi(a) - \int_0^a \frac{\varphi^{(m+1)}(t)}{m!}(a-t)^m dt, \quad \varphi \in V_+;$$

(b) $0 \le y_j \le a^j$, $j \in \{0, 1, \dots, m\}$.

The following result is independent with respect to Theorem 3.2. It can be proved without using any Hahn-Banach type argument (see [3]).

Theorem 2.3. *Let $S \subset \mathbf{R}^n$ be the unit sphere, endowed with the $n-1$ surface measure $d\sigma$. Let $\{y_\alpha\}_\alpha \subset \mathbf{R}$ $(\alpha \in (2\mathbf{Z}_+)^n)$ be a sequence whose multiindices α have even components α_k. The following statements are equivalent:*

(a) there exists a nonnegative measurable function $h \le 1$ on S such that

$$y_\alpha = \int_S x_1^{\alpha_1} \dots x_n^{\alpha_n} h(x) d\sigma(x)$$

for all the multiindices $\alpha = (\alpha_1, \dots, \alpha_n) \in (2\mathbf{Z}_+)^n$;

(b) for any finitely supported sequence $\{t_\alpha\}_\alpha \subset \mathbf{R}$, we have

$$0 \le \sum_{\alpha,\beta} y_{\alpha+\beta} t_\alpha t_\beta \le \sum_{\alpha,\beta} c_{\alpha+\beta} t_\alpha t_\beta,$$

where $c_\gamma := \displaystyle\int_S x^\gamma d\sigma(x)$, $\gamma \in (2\mathbf{Z}_+)^n$.

In the proof of this theorem one uses a deep result of B. Reznick (1995) (see [32] and [3]). In the proof of Theorem 2.3 the extension-type results are avoided, being replaced by Stone-Weierstrass Theorem and other fundamental arguments.

3. The sketch of the proof of Theorem 1.3.

We recall the following known results.

Lemma 3.1 *Let E be an arbitrary real vector space and $T := co\{v_1, \dots, v_m\}$ a finite dimensional simplex ($v_1, \dots, v_m$ are affine independent). Let Y be an ordered vector space and $\tilde f, \tilde g : T \to Y$, $\tilde f$ concave, $\tilde g$ convex, with $\tilde g \le \tilde f$.*

Then there exists $\tilde h : T \to Y$ affine such that

$$\tilde g \le \tilde h \le \tilde f.$$

Moreover, if $\tilde g < \tilde f$, then $\tilde h$ may be chosen such that

$$\tilde g < \tilde h < \tilde f.$$

If we choose $y_k \in [\tilde g(v_k), \tilde f(v_k)]$, then the function

$$\tilde h\left(\sum_{k=1}^m \lambda_k v_k\right) := \sum_{k=1}^m \lambda_k y_k,\ \lambda_k \ge 0,\ \sum_{k=1}^m \lambda_k = 1$$

is well defined, affine and $\tilde g \le \tilde h \le \tilde f$ on T).

The following known result gives a solution of an abstract moment problem, but may be used to prove some of the results mentioned in Section 1.

Theorem 3.2. *Let V be a preordered vector space, Y an order-complete vector lattice, $\{v_j; j \in J\} \subset V$, $\{y_j; j \in J\} \subset Y$, $F, G \in L(V, Y)$ two linear operators.*

Consider the following assertions:

(a) there exists $H \in L(V, Y)$ such that

$$H(v_j) = y_j,\ j \in J,$$
$$G(\varphi) \le H(\varphi) \le F(\varphi),\ \varphi \in V_+;$$

(b) for any finite subset $J_1 \subset J$ and any $\{\beta_j; j \in J_1\} \subset \mathbf{R}$, the following implication holds:

$$\sum_{j \in J_1} \beta_j v_j = \varphi_2 - \varphi_1 \text{ with } \varphi_1, \varphi_2 \in V_+ \Rightarrow \sum_{j \in J_1} \beta_j y_j \leq F(\varphi_2) - G(\varphi_1).$$

If V is a vector lattice, we also consider the statement (b'):

(b') $G(\varphi) \leq F(\varphi)$, $\forall \varphi \in V_+$ and for any finite subset $J_1 \subset J$ and any $\{\beta_j; j \in J_1\} \subset \mathbf{R}$, we have

$$\sum_{j \in J_1} \beta_j y_j \leq F\left(\left(\sum_{j \in J_1} \beta_j v_j\right)^+\right) - G\left(\left(\sum_{j \in J_1} \beta_j v_j\right)^-\right).$$

Then (a) $\Leftrightarrow$ (b) holds, and, if V is a vector lattice, we have $(b') \Leftrightarrow (b) \Leftrightarrow (a)$. The basic implications are $(b) \Rightarrow (a)$ and $(b') \Rightarrow (a)$.

This theorem was published in 1991 (see [25]), without proof. Its proof was published in [26]. For some of its applications, see [3], [19], [26], [27], [29]. In [3] we applied this theorem to obtain Theorem 2.3 [3], which is a sandwich-type result for maps defined on an arbitrary set X. The idea is to imbed X in an ordered vector space V and then to apply Theorem 3.2. stated above. Here is the statement of this variant of a sandwich-type principle (which is well known cf. [15] and may be proved using some other Hahn-Banach type results instead of Theorem 3.2. mentioned above).

Theorem 3.3. *Let X be a nonempty arbitrary set, Y an order-complete vector lattice, $S \subset \mathcal{F}$ a convex cone (for notations and definitions see Section 1). Let $f, g \in \mathcal{F}$ be such that*

$$g(x) \leq f(x), \quad x \in X.$$

The following assertions are equivalent

(a) there exists a S-affine map $h \in \mathcal{F}$ such that

$$g(x) \leq h(x) \leq f(x) \quad \forall x \in X; \tag{1}$$

(b) for any $n \in \mathbf{N}$, $n \geq 1$, any $\{x_1, \ldots, x_n, x_1', \ldots, x_n'\} \subset X$ and any $\{\lambda_1, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_n\} \subset \mathbf{R}_+$ such that

$$\left(\sum_{j=1}^n \lambda_j \varepsilon_{x_j} - \sum_{j=1}^n \alpha_j \varepsilon_{x_j'}\right)(s) = 0 \quad \forall s \in S \cap (-S), \tag{2}$$

we must have

$$\sum_{j=1}^n \alpha_j g(x_j') \leq \sum_{j=1}^n \lambda_j f(x_j). \tag{3}$$

Now we are able to sketch the proof of Theorem 1.3. of the present work (for details see the proof of Theorem 3.4. [3]).

Sketch of the proof of Theorem 1.3.

One applies Theorem 3.3 stated above, (b) $\Rightarrow$ (a). We have to prove the implication (2) $\Rightarrow$ (3).

Let $\{x_1, \ldots, x_n, x_1', \ldots, x_n'\} \subset X$, $\{\lambda_1, \ldots, \lambda_n, \alpha_1, \ldots, \alpha_n\} \subset \mathbf{R}_+$ be as above, such that (2) holds. Let

$$K := co\{x_1, \ldots, x_n, x_1', \ldots, x_n'\}.$$

Obviously, K is a finite-dimensional compact convex subset of X (in any Hausdorff linear topology on E). Since X was supposed to be finite-simplicial, there exists a finite-dimensional simplex

$$T = co\{v_1, \dots, v_m\} \quad (v_1, \dots, v_m \text{ affine independent})$$

such that

$$K \subset T \subset X.$$

Let $\tilde{g} := g|_T$, $\tilde{f} := f|_T$. By Lemma 3.1., there exists $\tilde{h} : T \to Y$ affine, such that

$$\tilde{g} \leq \tilde{h} \leq \tilde{f} \quad \text{on} \quad T.$$

One can show, via a computation, that the existence of such an affine map $\tilde{h}$ implies

$$(1') \qquad \sum_{j=1}^{n} \alpha_j \tilde{g}(x'_j) \leq \sum_{j=1}^{n} \lambda_j \tilde{f}(x_j)$$

(see the proof of Theorem 3.4. [3]).

From this we deduce:

$$(1'') \qquad \sum_{j=1}^{n} \alpha_j g(x'_j) = \sum_{j=1}^{n} \alpha_j \tilde{g}(x'_j) \overset{(1')}{\leq} \sum_{j=1}^{n} \lambda_j \tilde{f}(x_j) = \sum_{j=1}^{n} \lambda_j f(x_j).$$

Thus (2) $\Rightarrow$ (3) is proved and by Theorem 3.3 (b) $\Rightarrow$ (a), the existence of an S-affine map $h \in \mathcal{F}$ such that (1) holds follows.

□

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