Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 51(1), 2024, Pages 223–233, DOI: 10.52846/ami.v51i1.1791
ISSN: 1223-6934

A Sequence of Weak Solutions for a Nonlinear Equation Involving Hardy Potential

Saeid Shokooh

Abstract. In this work, we find an interval for a parameter λ for which a functional $J - \lambda I$ possesses a sequence of critical points. It should be noted that members of this sequence will be weak solutions to an elliptic problem with Hardy potential.

2010 Mathematics Subject Classification. Primary 35A15, 35J65; Secondary 35H30, 35J75.
Key words and phrases. variational method, Hardy potential, boundary value problem, weak solutions.

1. Introduction

Considering the applications of $p-$Laplacian and biharmonic equations in physics, Medicine and engineering, many authors have studied such equations under suitable conditions in past years. Among them, we cite the readers to Afrouzi et al [1], Bonanno et al [4], Cammaroto [10], Li [11], Shokooh [14], Xu et al [15] and the references therein.

Recently, in [10], Cammaroto proved the existence results for following boundary value problem:

$$\begin{cases} \Delta_p^2 u - \Delta_p u + V(x)|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 1)$, $p > \max\{1, \frac{N}{2}\}$, $\lambda \in \mathbb{R}$, $\mu > 0$, f, g are Carathéodory functions with suitable behaviors, $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ and $\Delta_p u := \nabla(|\nabla u|^{p-2}\nabla u)$ are p-biharmonic and p-Laplacian operators, respectively, and $V \in C(\overline{\Omega})$ such that $\inf\{V(x) : x \in \overline{\Omega}\} > 0$.

Also, in [15], some existence results are obtained for the following fourth order equation involving Hardy potential:

$$\begin{cases} M\left(\int_\Omega |\Delta u|^p dx\right) \Delta_p^2 u - \frac{d^*}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $1 < p < \frac{N}{2}$, λ, μ, d^* are constants, $f, g \in C(\overline{\Omega} \times \mathbb{R})$.

Afrouzi with his coauthor [1] have investigated the following $p(x)-$biharmonic problem

$$\begin{cases} \Delta_{p(x)}^2 u + V(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u), & x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{cases}$$

Received July 14, 2023. Accepted January 13, 2024.

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, $p^- := \inf_{x\in\overline{\Omega}} p(x) > \frac{N}{2}$, $\lambda > 0$, $\mu \geq 0$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $V \in L^\infty(\Omega)$ satisfying $\inf_{\overline{\Omega}} V > 0$.

Inspired by the above papers, we are interested in studying the following p-biharmonic elliptic equation with Hardy potential:

$$\begin{cases} \Delta_p^2 u - \Delta_p u + a(x)|u|^{p-2}u = \frac{d}{|x|^{2p}}|u|^{p-2}u + \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = 0, \qquad x \in \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $1 < p < \frac{N}{2}$, λ is a positive parameter, d is a constant, $f \in C(\overline{\Omega} \times \mathbb{R})$ with suitable conditions and $a \in C(\overline{\Omega})$ satisfying $\inf_{\overline{\Omega}} a > 0$.

The equation (1) is singular and has two operators, $\Delta_p^2 u$ and $\Delta_p u$, as distinctive features. The singularity arises due to the presence of the Hardy potential term $\frac{1}{|x|^{2p}}$, which can make the analysis of solutions more challenging. Singular equations like (1) are encountered in many areas of science and engineering, and their study is important for the development of new technologies and innovations, see [9].

Also, the control parameter λ in the equation (1) allows for the study of the behavior of solutions under different values of λ. This can lead to interesting insights into the properties of the system, and can provide a more accurate and realistic representation of the problem. Control parameters are commonly used in mathematical models to capture the effect of external factors that influence the behavior of the system, and their presence can help to better understand the system's behavior.

We organize the structure of this article as follows. In the second section, the required concepts of the variational structure and also the main tool (Lemma 2.1) are stated. In Section 3, we give the main results and their proof along with an example.

2. Variational framework

In this section, we introduce the function space in which the problem (1) will be studied. Also, for the convenience of the reader, we state the main tool to proof the results.

From now on, let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be an open bounded set, $1 < p < \frac{N}{2}$, and $a \in C(\overline{\Omega})$ with $\inf\{a(x) : x \in \overline{\Omega}\} > 0$. The function space $E = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ with the following standard norm

$$\|u\| = \left(\int_\Omega |\Delta u|^p \, dx\right)^{\frac{1}{p}} \tag{2}$$

is a reflexive Banach space. According to the assumptions on the function a, the norm (2) is equivalent to the following norm

$$\|u\|_a = \left(\int_\Omega (|\Delta u|^p + |\nabla u|^p + a(x)|u|^p) \, dx\right)^{\frac{1}{p}} \tag{3}$$

for all $u \in E$.

To prove the main results, we need a sequence of functions as test functions. For optimal selection, we introduce a class of these functions as follows: let $\{\vartheta_n\}$ is a positive real sequence and $\{\eta_n\}, \{\theta_n\}$ are two real sequences such that $0 < \eta_n < \theta_n$ for all $n \in \mathbb{N}$. Put

$$\mathcal{T}(\{\eta_n\}, \{\theta_n\}, \{\vartheta_n\}) = \{\{\phi_n\} \subset W^{2,p}(]\eta_n, \theta_n[);\ \forall n \in \mathbb{N}\}$$

where ϕ_n satisfies in following conditions

(t_1) $0 \leq \phi_n(x) \leq \vartheta_n$ for a.e. $x \in]\eta_n, \theta_n[$;

(t_2) $\lim_{x\to\eta_n^+} \phi_n(x) = \vartheta_n$, $\lim_{x\to\theta_n^-} \phi_n(x) = 0$;

(t_3) $\lim_{x\to\eta_n^+} \phi'_n(x) = \lim_{x\to\theta_n^-} \phi'_n(x) = 0$;

(t_4) there exist $z_1, z_2 > 0$, such that

$$|\phi'_n(x)| \leq \frac{z_1\vartheta_n}{\theta_n - \eta_n}, \qquad |\phi''_n(x)| \leq \frac{z_2\vartheta_n}{(\theta_n - \eta_n)^2}. \tag{4}$$

For example, the sequences

$$\phi^1_n(x) = 4\vartheta_n(4x^3 - 9x^2 + 6x - 1)$$

and

$$\phi^2_n(x) = \frac{\vartheta_n}{2}\cos(2\pi x - \pi + 1),$$

where $x \in \left]\frac{1}{2}, 1\right[$, are in the space $\mathcal{T}\left(\{\frac{1}{2}, 1, \{\vartheta_n\}\right)$. Additionally,

$$|{\phi^1_n}'(x)| \leq 3\vartheta_n, \quad |{\phi^1_n}''(x)| \leq 24\vartheta_n, \quad |{\phi^2_n}'(x)| \leq \pi\vartheta_n$$

and $|{\phi^2_n}''(x)| \leq 2\pi^2\vartheta_n$ for all $x \in \left]\frac{1}{2}, 1\right[$. So, in view of (4),

$$z_1(\{\phi^1_n\}) = \frac{3}{2}, \;\; z_2(\{\phi^1_n\}) = 6, \;\; z_1(\{\phi^2_n\}) = \frac{\pi}{2}, \;\; z_2(\{\phi^2_n\}) = \frac{\pi^2}{2}.$$

Here, we state two inequalities (5) and (6) that will be used to prove theorems in the next section. If $q \in [1, p^* := \frac{pN}{N-2p})$, by Sobolev embedding

$$\exists c_q > 0;\ \|u\|_{L^q(\Omega)} \leq c_q\|u\| \leq c_q\|u\|_a \tag{5}$$

for all $u \in E$. Hence, the embedding $E \hookrightarrow L^q(\Omega)$ is compact, see [15].

Hardy inequality in the space E is as follows:

$$\int_\Omega \frac{|u(x)|^p}{|x|^{2p}}dx \leq \frac{1}{H}\int_\Omega |\Delta u(x)|^p dx \leq \frac{1}{H}\int_\Omega (|\Delta u|^p + |\nabla u|^p + a(x)|u|^p)\,dx, \tag{6}$$

where $H = \left(\frac{(p-1)N(N-2p))}{p^2}\right)^p$, see [12].

A function $u \in E$ is a weak solution of problem (1) if

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v dx + \int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx + \int_\Omega |u|^{p-2}uv dx$$
$$- \int_\Omega \frac{|u|^{p-2}}{|x|^{2p}}uv dx - \lambda\int_\Omega f(x, u(x))v(x)\,dx = 0$$

for all $v \in E$.

Ricceri in [13] proved a critical point theorem that it is our main tool for proving the results. Later, another version of this theorem was stated in [5] which we recall here for the convenience of the reader.

Lemma 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\left(\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \textit{and} \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(Φ_1) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(Φ_2) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either the functional I_λ possesses a global minimum, or there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(Φ_3) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either there is a global minimum of Φ which is a local minimum of I_λ, or there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

Lemma 2.1 and its different versions are useful tools to obtain exitance results for various equations. Whenever researchers want to find a sequence $\{u_n\}$ of weak solutions of an equation in a reflexive real Banach space X such that

- $\{u_n\}$ is unbounded in X, or
- $u_n \to 0$ strongly in X,

they can benefit from the above lemma. In the last two decades, various problems have been investigated with this method and the existence of infinitely many weak solutions for them has been proved (see, for example, [1, 2, 3, 6, 7, 8, 15] and references therein).

3. Main results

This section is devoted to our main results.

Theorem 3.1. *Suppose that $a \in C(\overline{\Omega})$ satisfy $\inf_\Omega a > 0$, $1 < p < \frac{N}{2}$, $0 < d < H$ (where H comes from (6)) and let $f \in C(\overline{\Omega} \times \mathbb{R})$ such that*

(A_1) *$F(x,t)$ is non-negative for all $(x,t) \in \Omega \times [0, +\infty[$;*

(A_2) *there exist $x_0 \in \Omega$ and $\rho > 0$, $p_0 \geq p$ such that $B(x_0, \rho) \subseteq \Omega$ and*

$$\alpha := \liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq t} \int_\Omega F(x,\xi)\,dx}{t^p} < +\infty,$$

$$\beta := \limsup_{t\to+\infty} \frac{\displaystyle\int_{B(x_0,\rho)} F(x,t)\,dx}{t^{p_0}} > 0.$$

Then, following facts hold:

(i$_1$) *if* $p < p_0$*, for all* $\lambda \in \left]0, \frac{H-d}{pHc_q^p\alpha}\right[$*, the problem* (1) *has an unbounded sequence of nonzero weak solutions;*

(i$_2$) *if* $p = p_0$*, for all* $\lambda \in \left]\frac{H-d}{pHc_q^pR\beta}, \frac{H-d}{pHc_q^p\alpha}\right[$ *where* $R = \frac{1}{c_q^p\omega c_{a,\varrho,\rho}}$*,* $\varrho > 1$ *and* $\alpha < R\beta$*, the problem* (1) *has an unbounded sequence of nonzero weak solutions.*

Proof. We want to apply part (Φ_2) of Lemma 2.1 to prove (i$_1$) with $X = E$ furnished with the norm introduced in (3). For fix $\lambda \in \left]0, \frac{H-d}{pHc_q^p\alpha}\right[$ define the functionals $J, I : E \to \mathbb{R}$, for all $u \in E$, by

$$J(u) = \frac{1}{p}\|u\|_a^p - \frac{d}{p}\int_\Omega \frac{|u|^p}{|x|^{2p}}dx, \qquad I(u) = \int_\Omega F(x,u(x))dx,$$

where $F(x,t) = \int_0^t f(x,\xi)d\xi$ for every $(x,t) \in \Omega \times \mathbb{R}$.

The functionals J and I are Gâteaux differentiable and whose derivative are

$$\begin{aligned} J'(u)(v) &= \int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta v dx + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx \\ &\quad + \int_\Omega |u|^{p-2}u\,v dx - \int_\Omega \frac{|u|^{p-2}}{|x|^{2p}}u\,v dx \\ I'(u)(v) &= \int_\Omega f(x,u(x))v(x)\,dx \end{aligned}$$

for any $u, v \in E$.

Obviously, the functional J is sequentially weakly lower semi-continuous and strongly continuous. In view of (6) one has $J(u) \geq \frac{1}{p}\left(1 - \frac{d}{H}\right)\|u\|_a^p$, it follows J is coercive. Also, since I' is compact, one has it is sequentially weakly upper semi-continuous. Taking (5) and (6) into account, we observe that

$$\begin{aligned} \{u \in E \,:\, J(u) < r\} &\subseteq \left\{u \in E \,:\, \left(1 - \tfrac{d}{H}\right)\|u\|_a^p < pr\right\} \\ &\subseteq \left\{u \in E \,:\, \|u\|_{L^q(\Omega)} < c_q\left(\tfrac{pHr}{H-d}\right)^{\frac{1}{p}}\right\}. \end{aligned} \tag{7}$$

By $J(0) = I(0) = 0$ and (7), for $r > 0$ one has

$$\begin{aligned} \varphi(r) &= \inf_{u\in J^{-1}(-\infty,r)} \frac{\left(\sup_{v\in J^{-1}(-\infty,r)} I(v)\right) - I(u)}{r - J(u)} \\ &\leq \frac{\sup_{v\in J^{-1}(-\infty,r)} I(v)}{r} \\ &\leq \frac{\sup_{\|\xi\|_{L^q(\Omega)}\leq l}\int_\Omega F(x,\xi)\,dx}{r}, \end{aligned} \tag{8}$$

that $l = c_q \left(\frac{pHr}{H-d}\right)^{\frac{1}{p}}$. Assume that $\{\zeta_n\} \subset]0,+\infty[$ be a sequence and $\zeta_n \to +\infty$. Set $r_n = \frac{H-d}{pHc_q^p}\zeta_n^p$ for all $n \in \mathbb{N}$. From (A$_1$), (A$_2$) and (8), we have

$$\begin{aligned}
\gamma &= \liminf_{r\to+\infty} \varphi(r) \le \liminf_{n\to+\infty} \varphi(r_n) \\
&\le \frac{pHc_q^p}{H-d} \liminf_{n\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le\zeta_n} \int_\Omega F(x,\xi)\,dx}{\zeta_n^p} \\
&\le \frac{pHc_q^p\alpha}{H-d} < +\infty,
\end{aligned}$$

and consequently $\lambda < \frac{1}{\gamma}$. Now, we prove $J - \lambda I$ for $\lambda \in]0, \frac{H-d}{pHc_q^p\alpha}[$ is unbounded from below. By (A$_2$), fixed $0 < \overline{\beta} < \beta$ we get $\tau_n \in]0,+\infty[$ with $\tau_n \ge n$ such that

$$\int_{B(x_0,\rho)} F(x,\tau_n)dx \ge \overline{\beta}\tau_n^{p_0}$$

for all $n \in \mathbb{N}$. By choosing $\varrho > 1$ such that $B(x_0,\rho\varrho) \subseteq \Omega$ and a sequence $\{\phi_n\} \in \mathcal{T}(\rho,\rho\varrho,\{\alpha_n\})$, we define

$$w_n(x) := \begin{cases} 0 & \text{if} \quad x \in \overline{\Omega} \setminus B(x_0,\rho\varrho), \\ \tau_n & \text{if} \quad x \in B(x_0,\rho), \\ \phi_n(|x-x_0|) & \text{if} \quad x \in B(x_0,\rho\varrho) \setminus B(x_0,\rho), \end{cases} \tag{9}$$

for each $n \in \mathbb{N}$. With simple computations, for $n \in \mathbb{N}$ and for $1 \le i \le N$, one has

$$\frac{\partial w_n(x)}{\partial x_i} = \begin{cases} 0 & \text{if} \quad x \in \overline{\Omega} \setminus B(x_0,\rho\varrho), \\ 0 & \text{if} \quad x \in B(x_0,\rho), \\ \phi_n'(|x-x_0|)\frac{x_i - x_i^0}{|x-x_0|} & \text{if} \quad x \in B(x_0,\rho\varrho) \setminus B(x_0,\rho) \end{cases}$$

and

$$\frac{\partial^2 w_n(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x_0,\rho\varrho), \\ 0 & \text{if } x \in B(x_0,\rho), \\ T(x) & \text{if } x \in B(x_0,\rho\varrho) \setminus B(x_0,\rho), \end{cases}$$

where $T(x) = \phi_n''(|x-x_0|)\frac{(x_i-x_i^0)^2}{|x-x_0|^2} + \phi_n'(|x-x_0|)\frac{|x-x_0|^2-(x_i-x_i^0)^2}{|x-x_0|^3}$.

So, by (4), one has

$$|\nabla w_n(x)| \le |\phi_n'(|x-x_0|)| \le z_1\frac{\tau_n}{\rho\varrho-\rho}$$

and

$$\begin{aligned}
|\Delta w_n(x)| &\le |\phi_n''(|x-x_0|)| + |\phi_n'(|x-x_0|)|\frac{n-1}{|x-x_0|} \\
&\le \frac{z_2\tau_n}{\rho^2(\varrho-1)^2} + \frac{z_1(n-1)\tau_n}{\rho^2(\varrho-1)}.
\end{aligned}$$

These inequalities allow us to estimate the norm of the functions w_n as follows:

$$\begin{aligned}\|w_n\|_a^p &= \int_\Omega (|\Delta w_n|^p + |\nabla w_n|^p + a(x)|w_n|^p)dx \\ &= \int_{B(x_0,\rho\varrho)\setminus B(x_0,\rho)} |\Delta w_n(x)|^p dx + \int_{B(x_0,\rho\varrho)\setminus B(x_0,\rho)} |\nabla w_n(x)|^p dx \\ &\quad + \int_{B(x_0,\rho\varrho)} a(x)|w_n(x)|^p dx \\ &\leq \omega\tau_n^p \left[\frac{2^{p-1}(\varrho^n-1)}{\rho^{2p-n}(\varrho-1)^{2p}} z_2^p + \frac{(2^{p-1}(n-1)^p+\rho^p)(\varrho^n-1)}{\rho^{2p-n}(\varrho-1)^p} z_1^p\right] \\ &\quad + \omega\tau_n^p\rho^n\varrho^n \max_{x\in B(x_0,\rho\varrho)} a(x).\end{aligned}$$

If we put

$$C_{a,\rho,\varrho} = \frac{2^{p-1}(\varrho^n-1)}{\rho^{2p-n}(\varrho-1)^{2p}} z_2^p + \frac{(2^{p-1}(n-1)^p+\rho^p)(\varrho^n-1)}{\rho^{2p-n}(\varrho-1)^p} z_1^p + \rho^n\varrho^n \max_{x\in B(x_0,\rho\varrho)} a(x)$$

we have

$$\begin{aligned}J(w_n) - \lambda I(w_n) &= \frac{1}{p}\|w_n\|_a^p - \frac{d}{p}\int_\Omega \frac{|u|^p}{|x|^{2p}} dx - \lambda\int_\Omega F(x,w_n)dx \\ &\leq \frac{\omega C_{a,\rho,\varrho}}{p}\tau_n^p - \lambda\int_{B(x_0,\rho)} F(x,\tau_n)dx \\ &\leq \frac{\omega C_{a,\rho,\varrho}}{p}\tau_n^p - \lambda\overline{\beta}\tau_n^{p_0}.\end{aligned}$$

By the assumption $p < p_0$ in (i_1) and $\lim_{n\to\infty}\tau_n = +\infty$, we get

$$\lim_{n\to+\infty}(J(w_n) - \lambda I(w_n)) = -\infty.$$

So, the functional $J(u) - \lambda I(u)$ has no global minimum. According part (Φ_2) of Lemma 2.1, the problem (1) has a sequence of weak solutions in E that it is not bounded. This concludes the proof of (r_1). Now, we check (r_2). Since $\frac{1}{\lambda} < \frac{pHc_q^pR\beta}{H-d}$ there exist a sequence $\{\tau_n\} \subset]0,+\infty[$ and $\underline{\beta} > 0$ such that $\tau_n \to +\infty$,

$$\int_{B(x_0,\rho)} F(x,\tau_n)dx \geq \underline{\beta}\tau_n^{p_0}$$

and

$$\frac{1}{\lambda} < \underline{\beta} < \frac{p}{\omega C_{a,\rho,\varrho}}\frac{\int_{B(x_0,\rho)} F(x,\tau_n)dx}{\tau_n^p}.$$

As (9), we define the sequence $\{w_n\}$, so we obtain

$$\begin{aligned}J(w_n) - \lambda I(w_n) &= \frac{1}{p}\|w_n\|_a^p - \frac{d}{p}\int_\Omega \frac{|u|^p}{|x|^{2p}} dx - \lambda\int_\Omega F(x,w_n)dx \\ &\leq \frac{\omega C_{a,\rho,\varrho}}{p}\tau_n^p - \lambda\int_{B(x_0,\rho)} F(x,\tau_n)dx \\ &\leq \frac{\omega C_{a,\rho,\varrho}}{p}(1-\lambda\underline{\beta})\tau_n^p.\end{aligned}$$

Hence,

$$\lim_{n\to+\infty}(J(w_n)-\lambda I(w_n))=-\infty.$$

consequently, the functional $J-\lambda I$ is unbounded from below, and it follows that $J-\lambda I$ has no global minimum. So, by part (Φ_2) of Lemma 2.1, there exists an unbounded sequence $\{u_n\}$ in E such that $J'(u_n)-\lambda I'(u_n)=0$ for all $n\in\mathbb{N}$, and the proof of the theorem is achieved. $\square$

In Theorem 3.1, taking $\lambda=1$, we have following result.

Theorem 3.2. *Let* (A_1) *in the Theorem 3.1 and following assumptions hold:*

$$\alpha<\frac{H-d}{pHc_q^p},\qquad \beta>\frac{H-d}{pHc_q^pR}.$$

Then, the problem

$$\begin{cases}\Delta_p^2u-\Delta_pu+a(x)|u|^{p-2}u=\frac{d}{|x|^{2p}}|u|^{p-2}u+f(x,u), & x\in\Omega,\\ u=\Delta u=0, \qquad x\in\partial\Omega,\end{cases}$$

has an unbounded sequence of weak solutions in E.

Now, we present another consequence of Theorem 3.1.

Corollary 3.3. *Assume that* $f_1\in C(\mathbb{R})$ *be non-negative. Set*

$$F_1(\xi):=\int_0^\xi f_1(t)\,dt$$

for every $\xi\in\mathbb{R}$ *and let*

(A3) $\liminf\limits_{t\to+\infty}\frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t}F_1(\xi)}{t^p}<+\infty$;

(A4) $\limsup\limits_{t\to+\infty}\frac{F_1(t)}{t^p}=+\infty$.

Suppose further that for all $1\le i\le n$, $\alpha_i\in L^1(\Omega)$ *with* $\min_{x\in\Omega}\alpha_i(x)\ge 0$ *and* $\alpha_1\ne 0$.

Additionally, let for $2\le i\le n$, $f_i\in C(\mathbb{R})$, $f_i\ge 0$,

$$\max\left\{\sup_{\xi\in\mathbb{R}}F_i(\xi):2\le i\le n\right\}\le 0$$

and

$$\min\left\{\liminf_{t\to+\infty}\frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t}F_i(\xi)}{t^p}:2\le i\le n\right\}>-\infty,$$

where $F_i(\xi):=\int_0^\xi f_i(t)\,dt$ *for all* $\xi\in\mathbb{R}$. *Then, for each*

$$\lambda\in\left]0,\frac{H-d}{pHc_q^p\liminf_{t\to+\infty}\frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t}F_1(\xi)}{t^p}\int_\Omega\alpha_1(x)\,dx}\right[,$$

the problem

$$\begin{cases}\Delta_p^2u-\Delta_pu+a(x)|u|^{p-2}u=\frac{d}{|x|^{2p}}|u|^{p-2}u+\lambda\sum_{i=1}^n\alpha_i(x)f_i(u), & x\in\Omega,\\ u=\Delta u=0, \qquad x\in\partial\Omega,\end{cases}$$

has infinitely many weak solutions in the space E.

Proof. Put $f(x,t) = \sum_{i=1}^{n} \alpha_i(x) f_i(t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$. Take (A4) and the condition

$$\min\left\{\liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t} F_i(\xi)}{t^p} : 2 \le i \le n\right\} > -\infty,$$

into account, we conclude

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t} \int_\Omega F(x,\xi)\,dx}{t^p}\\
&= \limsup_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t} \sum_{i=1}^{n}\left(F_i(\xi)\int_\Omega \alpha_i(x)\,dx\right)}{t^p} = +\infty.
\end{aligned}$$

In addition, In view of (A3) and the assumption

$$\max\left\{\sup_{\xi\in\mathbb{R}} F_i(\xi) : 2 \le i \le n\right\} \le 0,$$

one has

$$\begin{aligned}
&\liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t} \int_\Omega F(x,\xi)\,dx}{t^p}\\
&\le \left(\int_\Omega \alpha_1(x)\,dx\right) \liminf_{t\to+\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t} F_1(\xi)}{t^p} < +\infty.
\end{aligned}$$

In light of the Theorem 3.1, the conclusion is achieved. □

Theorem 3.4. *Let $b \in C(\overline{\Omega})$ satisfy $\inf_\Omega b > 0$, $1 < p < \frac{N}{2}$, $0 < d < H$ (where H comes from (6)) and let $g \in C(\overline{\Omega} \times \mathbb{R})$ such that*

(A_5) *$G(x,t) := \int_0^t g(x,\xi)d\xi \ge 0$ for every $(x,t) \in \Omega \times [0,+\infty[$;*

(A_6) *there exist $x_0 \in \Omega$ and $\rho' > 0$, $p_0 \ge p$ such that $B(x_0,\rho') \subseteq \Omega$ and*

$$\alpha' := \liminf_{t\to0^+} \frac{\sup_{\|\xi\|_{L^q(\Omega)}\le t} \int_\Omega G(x,\xi)\,dx}{t^p} < +\infty,$$

$$\beta' := \limsup_{t\to0^+} \frac{\int_{B(x_0,\rho')} G(x,t)\,dx}{t^{p_0}} > 0.$$

Then, following facts hold:

(i_3) *if $p < p_0$, for all $\lambda \in \left]0, \frac{H-d}{pHc_q^p\alpha'}\right[$, the problem (1) possesses a sequence of weak solutions that strongly converges to 0 in E;*

(i_4) *if $p = p_0$, for all $\lambda \in \left]\frac{H-d}{pHc_q^pR\beta'}, \frac{H-d}{pHc_q^p\alpha}\right[$ where $R_0 = \frac{1}{c_q^p\omega c_{a,\varrho,\rho}}$, $\varrho' > 1$ and $\alpha' < R_0\beta'$, the problem (1) possesses a sequence of weak solutions that strongly converges to 0 in E.*

Proof. Using part (Φ_3) of Lemma 2.1 and by similar argument mentioned in the proof of Theorem 3.1, the result is achieved. □

Remark 3.1. In Theorem 3.4, if $\alpha' = 0$ and $\beta' = +\infty$, we can deduce for every $\lambda > 0$, problem (1) possesses a sequence of weak solutions that strongly converges to 0 in E.

We end this article with an example which shows an evidence of our abstract results.

Example 3.1. Suppose $\Omega = \{x \in \mathbb{R}^6; |x|_{\mathbb{R}^6} < 3\}$. Consider the problem

$$\begin{cases} \Delta^2_{\frac{5}{2}} u - \Delta_{\frac{5}{2}} u + u\sqrt{|u|} = \frac{1}{|x|^{2p}}|u|^{p-2}u + \lambda f(x,u), & x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega, \end{cases} \tag{10}$$

where

$$f(x,t) = \begin{cases} f^*(x)t^6\Big(7 - 7\cos(\ln(|t|)) + \sin(\ln(|t|))\Big) & \text{if} \quad (x,t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0 & \text{if} \quad (x,t) \in \Omega \times \{0\}, \end{cases}$$

where $f^* \in C(\Omega)$ and $f^*(x) \geq 0$ for all $x \in \Omega$. By a direct calculation, one has

$$F(x,t) = \begin{cases} f^*(x)t^7\Big(1 - \cos(\ln(|t|))\Big) & \text{if} \quad (x,t) \in \Omega \times (\mathbb{R} - \{0\}), \\ 0 & \text{if} \quad (x,t) \in \Omega \times \{0\}. \end{cases}$$

Hence,

$$\liminf_{t \to +\infty} \frac{\sup_{\|\xi\|_{L^q(\Omega)} \leq t} \displaystyle\int_\Omega F(x,\xi)\,dx}{t^{\frac{5}{2}}} = 0,$$

and

$$\limsup_{t \to +\infty} \frac{\displaystyle\int_\Omega F(x,t)\,dx}{t^{\frac{5}{2}}} = +\infty.$$

Hence, using Theorem 3.1, for every $\lambda > 0$ the problem (10) admits infinitely many weak solutions in E.

References

[1] G.A. Afrouzi, S. Shokooh, Existence of infinitely many solutions for quasilinear problems with a $p(x)$-biharmonic operator, *Electron. J. Differ. Equ.* **317** (2015), 1–14.

[2] G. Bonanno, G. D'Agui, A Neumann boundary value problem for the Sturm-Liouville equation, *Appl. Math. Comput.* **208** (2009), 318–327.

[3] G. Bonanno, B.Di Bella, Infinitely many solutions for a fourth-order elastic beam equation, *Nodea Nonlinear Differential Equations Appl.* **18** (2011), 357–368.

[4] G. Bonanno, A. Chinnì, D. O'Regan, Existence of two non-zero weak solutions for a nonlinear Navier boundary value problem involving the p-biharmonic, *Acta Appl. Math.* **166** (2020), 1–10.

[5] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.* **2009** (2009), 1–20.

[6] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a Dirichlet problem involving the p-Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **140** (2010), 737–752.

[7] G. Bonanno, G. Molica Bisci, D. O'Regan, Infinitely many weak solutions for a class of quasilinear elliptic systems, *Math. Comput. Modelling* **52** (2010), 152–160.

[8] G. Bonanno, G. Molica Bisci, V. Rădulescu, Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlics-Sobolev spaces, *C. R. Acad. Sci. Paris., Ser. I* **349** (2011), 263–268.

[9] A. Callegari, A. Nachman, A nonlinear singular boundary value problem in the theory of pseudoplastic fluids, *SIAM J. Appl. Math.* **38** (1980), 275–281.
[10] F. Cammaroto, Infinitely many solutions for a nonlinear Navier problem involving the p-biharmonic operator, *Cubo* **24** (2022), 501–519.
[11] L. Li, Two weak solutions for some singular fourth order elliptic problems, *Electron. J. Qual. Theory Differ. Equ.* **2016** (2016), 1–9.
[12] E. Mitidieri, A simple approach to Hardy inequalities, *Mathematical Notes* **67** (2000), 479–486.
[13] B. Ricceri, A general variational principle and some of its applications, *J. Comput. Appl. Math.* **113** (2000), 401–410.
[14] S. Shokooh, Existence and multiplicity results for elliptic equations involving the $p-$Laplacian-like, *Ann. Univ. Craiova Math. Comput. Sci. Ser.* **44** (2017), 249–278.
[15] M. Xu, Ch. Bai, Existence of infinitely many solutions for perturbed Kirchhoff type elliptic problems with Hardy potential, *Electron. J. Differ. Equ.* **2015** (2015), 1–9.

(Saeid Shokooh) Department of Mathematics, Faculty of Basic Sciences, Gonbad Kavous University, Gonbad Kavous, Iran
E-mail address: shokooh@gonbad.ac.ir