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Relative Uniform Convergence of Quantum Difference Sequence of Functions Related to the ℓ_p-Space

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Abstract. The notion of relative uniform convergence of a sequence of functions was introduced and investigated by Moore, followed by E. W. Chittenden in the twentieth century. The concept attracted the researcher at the beginning, of the twenty-first century. Jackson F. H. initiated the concept of the notion of the quantum difference operator. We introduced the quantum difference sequence space $m(\phi, ru, \nabla_p), p \geq 0$ of W.L.C. Sargent type. We examine its various properties such as solidity, convergence-free, completeness, etc. Also, some induction results have been established.

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1. Introduction

Essentially, limits in standard calculus are used to derive the derivatives of real functions. Today, the term "calculus without boundaries" is used to refer to quantum calculus, also referred to as q-calculus. The fundamental q-calculus formulas were discovered by Euler in the seventeenth century. However, Jackson [1] may have been the first to introduce the notion of the definite q-difference operator which is defined by,

$$D_q\zeta(x) = \frac{\zeta(qx) - \zeta(x)}{x(q-1)}, x \neq 0.$$

and $D_q\zeta(0) = \zeta^{'}(0)$, where q is a fixed number, $q \in (0,1)$. The function ζ is defined on a q-geometric set $A \subseteq \mathbb{R}$(or $\mathbb{C}$) such that whenever x and $qx \in A$. Due to its use in a variety of mathematical fields, including orthogonal polynomials, fundamental hypergeometric functions, combinatorics, the calculus of variations, and the theory of relativity, quantum difference operators play an intriguing role in mathematics.
Throughout the paper $\omega_f, \omega(ru), \ell_\infty(\nabla_q, ru), c(\nabla_q, ru), c_0(\nabla_q, ru), \ell_p(\nabla_q, ru)$ denotes the spaces of all sequence of functions, relative uniform, bounded quantum difference, convergent quantum difference and null quantum difference sequence of function.

Moore[2] first introduced the notion of relative uniform convergence of the sequence of functions relative to a scale function. Thereafter Chittenden[3] formulated a definition of the notion of relative uniform convergence of a sequence of functions. Throughout this article we consider D to be a compact domain on which the functions are defined.

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The definition of the relative uniform convergence which formulated by Chittenden [3] as follows:

Definition 1.1. A sequence of functions (ζ_n), defined on a compact domain D converges relatively uniformly to a limit function ζ such that for every small positive number ε, there is an integer n_ε such that for every $n \geq n_\varepsilon$ the inequality

$$|\zeta(x) - \zeta_n(x)| \leq \varepsilon|\mu(x)|,$$

holds uniformly in x on the compact domain D.

The function $\mu(x)$ of the definition above is called the scale function.

The notion was further studied by many other researchers like Demirci et.al. [4], Demirci and Orhan [5], Sahin and Dirik [6], Devi and Tripathy([7],[8],[9]) and others. For the details of basics on the sequence spaces and summability theory, one may refer to the monograph by Kamthan and Gupta [10].

2. Definitions and examples

We obtain certain fundamental concepts in this section, which will be applied to determining this article's findings.

Definition 2.1. A quantum difference sequence of function $(\nabla_q \zeta_n)$ of real single-valued function defined over a compact domain D of real numbers convergence relative uniformly on D if there exists a function $\eta(x)$ such that for every small positive number τ, there is an integer $n_0 = n_0(\tau)$ such that for every $n \geq n_0$ the inequality

$$|\eta(x) - \nabla_q \zeta_n(x)| < \tau|\mu(x)|.$$

Where, $\nabla_q \zeta_n(x) = \zeta_n(x) - q\zeta_{n-1}(x)$ hold for every $x \in D$.

Example 2.1. Consider the sequence of function (ζ_n) define by

$$\zeta_n(x) = \begin{cases} \frac{x}{n^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

$$\nabla_q \zeta_n(x) = \begin{cases} x\left[\frac{1-q}{(n-1)^2} + \frac{(1-2n)}{n^2(n-1)^2}\right], & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

which is not a uniformly convergent sequence of function, but converges relatively uniformly w.r.t the scale function $\mu(x)$ defined by,

$$\mu(x) = \begin{cases} x, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Remark 2.1. If $q = 1$, then the above definition is converted into the difference sequence of functions space.

Definition 2.2. A quantum difference sequence of the function $(\nabla_q \zeta_n)$ of single-valued functions ranging over a compact domain D of a real number is said to be relative uniformly Cauchy if there exists a function $\mu(x)$ defined on D and for every $\tau > 0$ there exists an integer $n_0 = n_0(\tau)$ such that

$$|\nabla_q \zeta_n(x) - \nabla_q \zeta_m(x)| < \tau|\mu(x)|,$$

for all $n, m \geq n_0$, holds for every element $x \in D$.

Definition 2.3. The classes of relative uniform quantum difference sequences of functions $\ell_\infty(\nabla_q, ru), c(\nabla_q, ru), c_0(\nabla_q, ru), \ell_p(\nabla_q, ru)$ on a compact domain D are defined as follows,
(1) $\ell_\infty(\nabla_q, ru) = \{(\zeta_n) \in \omega(ru) : (\nabla_q \zeta) \in \ell_\infty\}$.
(2) $c(\nabla_q, ru) = \{(\zeta_n) \in \omega(ru) : (\nabla_q \zeta) \in c\}$.
(3) $c_0(\nabla_q, ru) = \{(\zeta_n) \in \omega(ru) : (\nabla_q \zeta) \in c_0\}$.
(4) $\ell_p(\nabla_q, ru) = \{(\zeta_n) \in \omega(ru) : (\nabla_q \zeta) \in \ell_p\}$.
The above spaces are normed by,

$$||\zeta||_{(\nabla_q, ru)} = |\zeta_1(x)| + \sup_{x \in D, n \in \mathbb{N}} |\nabla_q \zeta_n(x) \mu(x)|.$$

Definition 2.4. A subset $Z_f \subset \omega_f$ is said to be solid or normal, if $(\zeta_n) \in Z_f$ implies $(\eta_n) \in Z_f$ for all sequence (η_n) such that, $|\eta_n(x)| \leq |\zeta_n(x)|$, for all $n \in \mathbb{N}$, and $x \in D$.

Definition 2.5. A subset $Z_f \subset \omega_f$ is said to be convergence free, if $(\zeta_n) \in Z_f$ then, $\zeta_n(x) = 0 \implies \eta_n(x) = 0$ on $x \in D$ and if $\zeta_n(x) \neq 0 \implies \eta_n(x)$ can be any thing, then $(\eta_n) \in Z_f$.

Definition 2.6. A subset $Z_f \subset \omega_f$ is said to be symmetric if $(\zeta_n) \in Z_f \implies (\zeta_{\pi(n)}) \in Z_f$, where π is a permutation of $\mathbb{N}$.

3. Main results

In this section, we establish the results of this article.

The following results can be established using standard techniques. Hence we state the theorem without proof.

Theorem 3.1. *The class of sequence of functions $\ell_\infty(\nabla_q, ru), c(\nabla_q, ru), c_0(\nabla_q, ru)$ are normed linear space and also Banach space normed by*

$$||\zeta||_{(\nabla_q, ru)} = |\zeta_1(x)| + \sup_{x \in D, n \in N} |\nabla_q \zeta_n(x) \mu(x)|$$

Following Sargent [11], we introduce the following definition in this article.

Definition 3.1. The space of relative uniform convergence of $q-$difference sequence of function related to ℓ_p space is denoted by

$$m(\phi, ru, \nabla_q, p) = \left\{ (\zeta_n) \in \omega(ru) : \sup_{s \geq 1, \sigma \in \xi_s, x \in D} \frac{1}{\phi_s} \sum_{n \in \sigma} |\nabla_q \zeta_n(x)|^p < \tau |\mu(x)| \right\}.$$

Theorem 3.2. *The space $m(\phi, ru, \nabla_q, p)$ is a Banach space with the norm $||.||_{m(\phi, ru, \nabla_q, p)}$ defined by,*

$$||\zeta||_{m(\phi, ru, \nabla_q, p)} = |\zeta_1(x)| + \sup_{x \in D, s \geq 1, \sigma \in \xi_s} \left\{ \frac{1}{\phi_s} \sum_{n \in \sigma} |\nabla_q \zeta_n(x) \mu(x)|^p \right\}^{\frac{1}{p}}, 1 \leq p < \infty. \quad (1)$$

and a complete p-normed space by $p-$ norm

$$||\zeta||_{m(\phi, ru, \nabla_q, p)} = |\zeta_1(x)|^p + \sup_{x \in D, s \geq 1, \sigma \in \xi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} |\nabla_q \zeta_n(x) \mu(x)|^p, 0 < p < 1. \quad (2)$$

Proof. Let (ζ_n) and $(\eta_n) \in m(\phi, ru, \nabla_q, p)$ and δ_1, δ_2 be two scalars.
Then we have,

$$\sup_{x\in D, s\geq 1, \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\nabla_q \zeta_n(x)\mu_1(x)|^p \right\}^{\frac{1}{p}} < \tau_1, 1 \leq p < \infty,$$

and

$$\sup_{x\in D, s\geq 1, \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\nabla_q \eta_n(x)\mu_2(x)|^p \right\}^{\frac{1}{p}} < \tau_2, 1 \leq p < \infty.$$

We have,

$$\sup_{x\in D, s\geq 1, \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\nabla_q (\delta_1\zeta_n(x) + \delta_2\eta_n(x))|^p \right\}^{\frac{1}{p}} < \max(\tau_1, \tau_2)\max(|\mu_1(x)|, |\mu_2(x)|),$$

$1 \leq p < \infty$. Hence, $(\delta_1\zeta_n + \delta_2\eta_n) \in m(\phi, ru, \nabla_q, p)$.
Therefore, $m(\phi, ru, \nabla_q, p)$ is a linear space.
It is clear that $m(\phi, ru, \nabla_q, p)$ is a normed space by the norm defined by norm defined by (1) for $1 \leq p < \infty$ and a $p-$normed space by (2) for $0 < p < 1$.

Now, we need to show that $m(\phi, ru, \nabla_q, p)$ is complete.
Let (ζ^i) be a Cauchy sequence in $m(\phi, ru, \nabla_q, p)$.
Where, $(\zeta^i) = (\zeta^i_n) = ((\zeta^i_1), (\zeta^i_2), ...) \in m(\phi, ru, \nabla_q, p)$ for each $i \in \mathbb{N}$. Then for a given $\tau > 0$ there exist $n_0 \in \mathbb{N}$ such that

$$||\zeta^i - \zeta^j||_{\nabla_q} = \sup_{x\in D} |\zeta^i_1(x) - \zeta^j_1(x)| + \sup_{x\in D, s\geq 1, \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\nabla_q (\zeta^i_n(x) - \zeta^j_n(x))|\mu(x)|^p \right\}^{\frac{1}{p}}$$
$$< \tau, \text{ for all } i, j > n_0.$$

$\implies$ $(\nabla_q \zeta^i_n)_{i=1}^\infty$ is a Cauchy sequences in D for each $n \in \mathbb{N}$.
$\implies$ $(\nabla_q \zeta^i_n)$ is convergent in D w.r.t. the scale function $\mu(x)$, for each $n \in \mathbb{N}$.

Let, $(\zeta^i_1)_{i=1}^\infty$ converges to ζ_1 and $(\nabla_q \zeta^i_n)_{i=1}^\infty$ converges to ζ_n for all $n \in \mathbb{N}$. Taking limit as $j \to \infty$ in (1), we get

$$\sup_{x\in D} |\zeta^i_1(x) - \zeta_1(x)| + \sup_{x\in D, s\geq 1, \sigma\in\xi_s} \left\{ \frac{1}{\phi_s} \sum_{n\in\sigma} |\nabla_q (\zeta^i_n(x) - \zeta_n(x))|\mu(x)|^p \right\}^{\frac{1}{p}} < \tau, \text{for all } i \geq n_0.$$

Hence, $(\zeta^i_n - \zeta_n) \in m(\phi, ru, \nabla_q, p)$.
We know that $m(\phi, ru, \nabla_q, p)$ is a linear space and (ζ^i_n), $(\zeta^i_n - \zeta_n) \in m(\phi, ru, \nabla_q, p)$, it follows that,

$$\zeta_n(x) = \zeta^i_n(x) - (\zeta^i_n(x) - \zeta_n(x)).$$

Therefore, $(\zeta_n) \in m(\phi, ru, \nabla_q, p)$ Hence, $m(\phi, ru, \nabla_q, p)$ is complete. Similarly we can prove that $m(\phi, ru, \nabla_q, p)$ is complete space $p-$normed by (2) for $0 < p < 1$. □

Theorem 3.3. *The space $m(\phi, ru, \nabla_q, p)$ is a $K-$space.*

Proof. Let $\nabla_q(\zeta_n - \zeta) \to 0$, as $n \to \infty$. Then for a given $\tau > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\nabla_q(\zeta_n - \zeta) < \tau, \text{ for all } n \leq n_0.$$

$$\sup_{x\in D}|\zeta_1^i(x)-\zeta_1(x)|^p+\sup_{x\in D,s\geq 1,\sigma\in\xi_s}\frac{1}{\phi_s}\sum_{n\in\sigma}\left|\nabla_q(\zeta_n^i(x)-\zeta_n(x))\mu(x)\right|^p<\tau,\ \text{for all } i\leq n_0. \tag{3}$$

From (3) it follows that

$$|\zeta_1^i(x)-\zeta_1(x)|^p<\tau, \text{for all } i\geq n_0.$$

$$\implies \zeta_1^i\to\zeta_1, \text{as } n\to\infty. \tag{4}$$

From (3) consider $s=1$ and $n=1$, we have

$$|(\zeta_1^i(x)-\zeta_1(x))-q(\zeta_0^i(x)-\zeta_0(x))|<\tau\phi_1, \text{for all } i\geq n_0. \tag{5}$$

From (4) and (5) it follows that

$$|\zeta_0^i(x)-\zeta_0(x)|<\frac{\tau(1+\phi_1)}{q}, \text{for all } i\geq n_0.$$

$$\implies \lim_{i\to\infty}\zeta_0^i=\zeta_0.$$

Proceeding in this way inductively, we have

$$\lim_{i\to\infty}\zeta_n^i=\zeta_n, \text{for all } n\in\mathbb{N}.$$

Hence, $m(\phi,ru,\nabla_q,p)$ is a $K-$space. □

Theorem 3.4. *For any two sequence (ϕ_s) and (ψ_s) of real numbers, we have*

$$m(\phi,ru,\nabla_q,p)\subseteq m(\psi,ru,\nabla_q,p)$$

if and only if

$$\sup_{s\geq 1}\left\{\frac{\phi_s}{\psi_s}\right\}<\infty.$$

Proof. Let $(\zeta_n)\in m(\phi,ru,\nabla_q,p)$. Then there exists a $\tau>0$ such that,

$$|\zeta_1(x)|^p+\sup_{x\in D,s\geq 1,\sigma\in\xi_s}\frac{1}{\phi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p<\tau.$$

Suppose,

$$\sup_{s\geq 1}\left\{\frac{\phi_s}{\psi_s}\right\}<\infty.$$

Then $\phi_s\leq K\psi_s$ and so that $\frac{1}{\psi_s}\leq\frac{K}{\phi_s}$, for some positive number K and for all s. Therefore we have,

$$\frac{1}{\psi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p\leq\frac{K}{\phi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p$$

$$\sup_{x\in D,s\geq 1,\sigma\in\xi_s}\frac{1}{\psi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p<\sup_{x\in D,s\geq 1,\sigma\in\xi_s}\frac{K}{\phi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p<K\tau.$$

Hence,

$$\sup_{x\in D,s\geq 1,\sigma\in\xi_s}\frac{1}{\psi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p<\tau'.$$

Therefore, $(\zeta_n)\in m(\psi,ru,\nabla_q,p)$.
Hence, $m(\phi,ru,\nabla_q,p)\subset m(\psi,ru,\nabla_q,p)$.

Conversely, let $m(\phi, ru, \nabla_q, p) \subset m(\psi, ru, \nabla_q, p)$. Suppose that, $\sup_{s\geq 1}\left\{\frac{\phi_s}{\psi_s}\right\} \to \infty$. Then there exists a sequence of natural numbers s_i such that,

$$\lim_{i\to\infty}\frac{\phi_{s_i}}{\psi_{s_i}} = \infty.$$

Let $(\zeta_n) \in m(\phi, ru, \nabla_q, p)$. Then there exists a $\tau > 0$ such that,

$$|\zeta_1(x)|^p + \sup_{x\in D, s\geq 1, \sigma\in\xi_s}\frac{1}{\phi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p < \tau.$$

Now we have,

$$\begin{aligned}&\sup_{x\in D, s\geq 1, \sigma\in\xi_s}\frac{1}{\psi_s}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p\\ &\leq \left\{\sup_{i\geq 1}\frac{\phi_{s_i}}{\psi_{s_i}}\right\}\sup_{x\in D, s\geq 1, \sigma\in\xi_s}\frac{K}{\phi_{s_i}}\sum_{n\in\sigma}|\nabla_q\zeta_n(x)\mu(x)|^p \to \infty.\end{aligned}$$

Therefore, $(\zeta_n) \notin m(\psi, ru, \nabla_q, p)$.

As such we arrive at our contradiction.

Hence,

$$\sup_{s\geq 1}\left\{\frac{\phi_s}{\psi_s}\right\} < \infty.$$

□

In view of Theorem 3.4, we formulate the following result without proof.

Theorem 3.5. *Let $0 < p < 1$, then for (ϕ_n) and (ψ_n) two sequences of real numbers,*

$$m(\phi, ru, \nabla_q, p) = m(\psi, ru, \nabla_q, p),$$

if and only if

$$\sup_{s\geq 1}\eta_s < \infty \text{ and } \sup_{s\geq 1}\eta_s^{-1} < \infty, \text{ where } \eta_s = \frac{\phi_s}{\psi_s}.$$

Lemma 3.6. (1) $\ell_p(\nabla_q, ru) \subseteq m(\phi, ru, \nabla_q, p) \subseteq \ell_\infty(\nabla_q, ru)$.

(2) $m(\phi, ru, \nabla_q, p) = \ell_p(\nabla_q, ru)$ *if and only if* $\lim_{s\to\infty}\phi_s < \infty$.

(3) $m(\phi, ru, \nabla_q, p) = \ell_\infty(\nabla_q, ru)$ *if and only if* $\lim_{s\to\infty}\frac{\phi_s}{s} > 0$.

The results follow from lemma of Sargent [11] by taking

$$\eta_n(x) = \nabla_q\zeta_n(x)\mu(x), x \in D.$$

Result 3.1. $m(\phi, ru, \nabla_q, p)$ is not solid.

The proposed result follows from the following example.

Example 3.1. Let us consider a sequence of functions $(\zeta_n(x))$ defined by

$$\zeta_n(x) = \begin{cases}\frac{x}{n^2}, & \text{for } x \neq 0;\\ 0, & \text{for } x = 0.\end{cases}$$

Then,

$$\nabla_q(\zeta_n(x)) = \begin{cases} \frac{x\{n^2(1-q)+1-2n\}}{n^2(n-1)^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Let, $\phi_n = 1$ for all $n \in \mathbb{N}$.
Then, $\nabla_q(\zeta_n)$ is convergent uniformly with respect to the scale function $\mu(x)$ defined by,

$$\mu(x) = \begin{cases} x, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Hence, $(\nabla_q(\zeta_n)) \in m(\phi, ru, \nabla_q, p)$. Let, (λ_n) be defined as, $\lambda_n(x) = (-1)^n$ for all $n \in \mathbb{N}$. Also, $|\lambda_n(x)| \leq 1$, for all $n \in \mathbb{N}$.

Then, $(\nabla_q(\lambda_n \zeta_n)) \notin m(\phi, ru, \nabla_q, p)$. Hence, $m(\phi, ru, \nabla_q, p)$ is not solid.

Result 3.2. The space $m(\phi, ru, \nabla_q, p)$ is not symmetric.

The proposed result follows from the following example.

Example 3.2. Let, $\phi_n = n$, for all $n \in \mathbb{N}$. Consider the sequence of functions space (ζ_n) defined by

$$\zeta_n(x) = nx, \text{ for all } n \in \mathbb{N}, x \in [0,1].$$

and

$$\nabla_q(\zeta_n(x)) = (n(1-q)+q)x \text{ for all } n \in \mathbb{N} \text{ and } x \in [0,1].$$

Then, $(\zeta_n) \in m(\phi, ru, \nabla_q, p)$ with respect to the scale function $\mu(x) = x$, for each $x \in [0,1]$.
Now consider the rearrangement (η_n) of (ζ_n) defined as follows, $(\eta_n) = (\zeta_2, \zeta_7, \zeta_{19}, \zeta_1, \zeta_{70}, \ldots\ldots)$.
Then, $(\eta_n) \notin m(\phi, ru, \nabla_q, p)$.
Hence, $m(\phi, ru, \nabla_q, p)$ is not symmetric.

Result 3.3. The class of sequence of functions $m(\phi, ru, \nabla_q, p)$ is not sequence algebra and also not convergence free.

This remark follows from the following examples.

Example 3.3. Let, $\zeta_n(x) = (-1)^n x$ and $\eta_n(x) = (-1)^{n+1} 2x$, where $x \in [1,2]$ and $n \in \mathbb{N}$. Then,

$$\nabla_q \zeta_n(x) = (-1)^n x(1+q) \text{ and } \nabla_q \eta_n(x) = (-1)^{n+1} 2x(1+q).$$

Taking $\phi_n = n$, for all $n \in \mathbb{N}$ then, $(\nabla_q \zeta_n), (\nabla_q \eta_n) \in m(\phi, ru, \nabla_q, p)$ with respect to the scale function $\mu(x) = x$, where $x \in [1,2]$.
But $(\nabla_q \zeta_n * \nabla_q \eta_n) \notin m(\phi, ru, \nabla_q, p)$ with respect to the same scale function.
Hence, $m(\phi, ru, \nabla_q, p)$ is not sequence algebra.

Example 3.4. Let us take a sequence of functions defined by,

$$\zeta_n(x) = \begin{cases} \frac{1}{n^2 x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then,

$$\nabla_q \zeta_n(x) = \begin{cases} \frac{1}{x}\left[\frac{1-q}{(n-1)^2} - \frac{2}{n(n-1)^2} + \frac{1}{n^2(n-1)^2}\right], & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then it is clear that $(\nabla_q \zeta_n)$ does not converge uniformly but converges relatively uniformly to zero function with respect to a scale function $\mu(x)$ defined by,

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Consider the sequence (ϕ_n) defined by $\phi_n = 1$, for all $n \in \mathbb{N}$ and $p = 1$.
Then,

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D} \sum_{n \in \sigma} |\nabla_q \zeta_n(x)| < \tau |\mu(x)|$$

Hence, $(\nabla_q \zeta_n) \in m(\phi, ru, \nabla_q, p)$.
Now we consider another sequence of function defined by,

$$\eta_n(x) = \begin{cases} nx, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then,

$$\nabla_q \eta_n(x) = \begin{cases} x[n(1-q)+q], & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

But in that case

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D} \sum_{n \in \sigma} \frac{|\nabla_q \eta_n(x)|}{|\mu(x)|} \to \infty, \text{ as } n \to \infty.$$

Therefore, $(\eta_n) \notin m(\phi, ru, \nabla_q, p)$.
Hence, $m(\phi, ru, \nabla_q, p)$ is not convergence free.

4. Relevance of the work done

In this article the quantum difference operator on the class of sequences of $m(\phi)$ type has been introduced and its properties have been investigated. The work done is related to the field of sequence spaces and Summability theory. It aims to introduce new classes of sequences and investigate their different properties. It deals with the problems related to function spaces and topological spaces too.

5. Applications

The results obtained in this article can be applied to introducing different classes of sequences of functions and investigating their different algebraic and topological properties. The techniques can be applied for further investigation of other classes of sequences of functions.

6. Further results

Sequence spaces of functions defined by Orlicz functions can be introduced and their properties will be investigated. The results can be generalized in the setting of n-normed spaces. Also, different types of difference operators can be introduced as defined by Orlicz functions.

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