Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 51(2), 2024, Pages 303–314, DOI: 10.52846/ami.v51i2.1798
ISSN: 1223-6934

A Study of Triple Sequences Statistical Convergence in Neutrosophic Normed Spaces

Hari Shankar, Ayaz Ahmad, and Ayhan Esi

Abstract. Neutrosophic logic, probability, and sets are all included in this discipline. The generalization of conventional sets, fuzzy sets, intuitionistic fuzzy sets, and other related ideas is the neutrosophic set theory. It is a mathematical concept that deals with situations involving inconsistent, ambiguous, and imprecise data. To fully understand sequence spaces, statistical convergence is a crucial concept. In this particular scientific work, we introduce the notion of statistical convergence for triple sequences in the neutrosophic normed space. In this neutrosophic normed space, we also explore the statistical properties of completeness and triple Cauchy sequences.

2020 *Mathematics Subject Classification.*
Key words and phrases. Neutrosophic normed spaces, Statistical triple convergence, t-norm, t-conorm.

1. Introduction

To address ambiguity and imprecision in practical issues, L. Zadeh [22] introduced the flexible mathematical framework of fuzzy set theory in 1965. As we know, fuzzy set theory is an extension of classical set theory. Significant studies have been done on fuzzy theory over the past 50 years. Many authors made a substantial contribution to the discovery of the idea in addition to developing the theoretical frameworks that are still widely used today, such as fuzzy physics [12], fuzzy programming [9], and similarity relations[23]. In a relatively short period, authors accomplished this feat, which was astonishing. Since the beginning, fuzzy set theory has gone through lots of changes. In their research, Esi and Hazarika [4] investigated the lacunary summable sequence of fuzzy numbers. In 1986, Atanassov [2] developed the notion of intuitionistic fuzzy sets as an extension of fuzzy sets. These sets cope with each element's degree of membership as well as its degree of non-membership.

Park's [18] study of the theory of intuitionistic fuzzy metric spaces utilizing continuous t-norm and continuous t-conorm operations expanded the amount of knowledge in this field. The neutrosophic set (NS), an innovative utilization of the classical set notion built on fuzzy sets, was introduced by Smarandache [21]. Since its initial introduction in 1998, neutrosophy has received a lot of written attention.
Furthermore, Kaleva and Seikkala [10] established the idea of fuzzy metric spaces (FMS), in which the separation between two points is specified as a non-negative fuzzy integer. The Baire Category Theorem for FMS was then convincingly established in [6] after extensive investigation of numerous fundamental characteristics of FMS. The

Received July 30, 2023. Accepted January 13, 2024.

findings of the research are that FMS has a wide range of useful applications in the applied sciences, including fixed point theory, image and signal processing, medical imaging, decision-making, and more. FMS is a useful tool for handling complicated issues and boosting analytical skills in various fields because of its adaptability and versatility. In their paper, Bera and Mahapatra [3] presented the idea of neutrosophic soft normed linear spaces (NSNLS). In their research, they described and studied a lot of NSNLS-related concepts, such as the neutrosophic norm, Cauchy sequences in NSNLS, the convexity of NSNLS, and metrics in NSNLS. In their latest study published in 2020, Kirisci and Simsek [11] put up and explored the notion of statistical convergence in neutrosophic normed spaces. Their research yielded significant findings in this area. Further statistical convergence of double sequences in neutrosophic normed spaces is given by C. Granados and A. Dhital [8]. Triple sequences in neutrosophic normed spaces have been included in our study's advanced concept of statistical convergence. We not only establish the characteristics and qualities of this expanded idea, but we also offer a thorough organizational framework for the study. Some authors [17, 15] recently examined the notions of different kinds of sequence spaces in intuitionistic and neutrosophic fuzzy normed spaces and established some key results that are useful in further study of the current findings in this work.

The following outlines the structure of the article: We begin with an overview of well-known concepts and terminologies that are essential for the development of this study. In Section 2, these concepts will serve as the foundation for our further analysis. The notion of statistical convergence and the completeness of triple sequences in neutrosophic normed spaces will be defined and thoroughly examined, aiming to gain a deeper understanding of their behavior in the context of neutrosophic normed spaces (NNS), as covered in Section 3. Finally, in Section 4, we present our conclusions.

2. Definitions and Preliminaries

The idea of statistical convergence was first put up by Fast and Steinhaus [5] on their behalf, and it has subsequently been thoroughly explored by several researchers. Assuming that set $\mathbf{S}$ is a subset of $\mathbb{N}$, the following definition applies to the asymptotic density of the set, represented by :

$$d(\mathbf{S}) = \lim_k \frac{1}{k}|\{r \leq k : r \in \mathbf{S}\}|,$$

where the number of elements in the given set is indicated by the vertical bars. The sequence $y = (y_r)$ is said to be statistically convergent to the number $\mathcal{L}_1$ if the set $A = \{r \leq k : |y_r - \mathcal{L}_1| > \epsilon_1\}$ has asymptotic density 0 for each $\epsilon_1 > 0$. In light of this notion, Mursaleen and Edely [14] developed the theory of statistical convergence of double sequences. Moricz [13] on his own define Tauberian theorems for cesaro summable double sequence. Sahiner et al.[20] studied the statistical convergence for triple sequence. Let $\mathbf{R} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ be a three-dimensional set of positive integers. Let us consider the set $\mathbf{R}(m, n, r) = \{(i, j, k) \in \mathbf{R} : i \leq m, j \leq n, k \leq r\}$.

The triple natural density of a given set is defined by Sahiner et al.[20] as,

$$\delta_3(\mathbf{R}) = \lim_{m,n,r\to\infty} \frac{|\mathbf{R}(m,n,r)|}{mnr} \text{ (here limit taken in the Pringsheim's sense)},$$

where the number of elements in the given set is indicated by the vertical bars. Also, the statistical convergence of the triple sequence is as follows:

A real triple sequence $y = (y_{nkl})$ is said to be statistically convergent to the number $\mathcal{L}_1$ if for each $\epsilon_1 > 0$

$$\delta_3(\{(n,k,l) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : |y_{nkl} - \mathcal{L}_1| \geq \epsilon_1\}) = 0.$$

In this case, the notation $S_3 - \lim y_{nkl} = \mathcal{L}_1$ is used to denote the statistical convergence of the triple sequence $y = (y_{nkl})$ to the number $\mathcal{L}_1$.

On a different note, the concept of t-norm was initially introduced by Menger[12]. Menger suggested an alternate strategy that involved using probability distributions rather than numerical distances to determine the distance between two items in space. This concept utilizes t-norms (triangular norms) to generalize the probability distribution while considering the conditions of a metric space and the triangle inequality. t-norms and t-conorms play a crucial role in fuzzy operations, such as intersections and unions. Additionally, t-conorms, which are the dual operations of t-norms, are also significant in this context. Both t-norms(triangular norms) and t-conorms (triangular conorms) contribute significantly to fuzzy operations and find practical applications in various fields.

Definition 2.1. ([17]) Define a binary operation as, $* : [0,1]^2 \to [0,1]$ if meets the requirements listed below, then it is considered to be a continuous t-norm,

(i) $x_1 * 1 = x_1, \forall\, x_1 \in [0,1]$,
(ii) $x_1 * x_2 \leq x_3 * x_4$ whenever $x_1 \leq x_3$ and $x_2 \leq x_4$ for each $x_1, x_2, x_3, x_4 \in [0,1]$,
(iii) $*$ is associative and commutative,
(iv) $*$ is continuous.

Definition 2.2. ([17]) Define a binary operation as, $\boxdot : [0,1]^2 \to [0,1]$ if meets the requirements listed below, then it is considered to be a continuous t-conorm,

(i) $x_1 \boxdot 0 = x_1, \forall\, x_1 \in [0,1]$,
(ii) $x_1 \boxdot x_2 \leq x_3 \boxdot x_4$ whenever $x_1 \leq x_3$ and $x_2 \leq x_4$ for each $x_1, x_2, x_3, x_4 \in [0,1]$,
(iii) $\boxdot$ is associative and commutative,
(iv) $\boxdot$ is continuous.

Remark 2.3. ([11]) If $0 < \nu_j < 1$, $j = 1$ to 7, $*$ and $\boxdot$ are continuous- t-norm continuous- t-conorm respectively. Then,

(i) If we take $0 \leq \nu_1, \nu_2 < 1$ for $\nu_1 > \nu_2$ then there exists $0 \leq \nu_3, \nu_4 < 1$ such that $\nu_1 * \nu_3 \geq \nu_2$, $\nu_1 \geq \nu_4 \boxdot \nu_2$.
(ii) If we take $0 \leq \nu_5 < 1$ then there exists $0 \leq \nu_6, \nu_7 < 1$ such that $\nu_6 * \nu_6 \geq \nu_5$, $\nu_7 \boxdot \nu_7 \leq \nu_5$.

Definition 2.4. ([11]) Let $\mathbb{E}$ be any arbitrary set, and let

$$\mathcal{M} = \{< x_1, \Omega(x_1), \Phi(x_1), \Psi(x_1) >: x_1 \in \mathbb{E}\},$$

be a neutrosophic set (NS) such that $\mathcal{M} : \mathbb{E} \times \mathbb{E} \times \mathbb{R}^+ \to [0,1]$. The continuous t-norm and continuous t-conorm can be illustrated using $*$ and $\boxdot$, respectively. The four-tuple $(\mathbb{E}, \mathcal{M}, *, \boxdot)$ is referred to as neutrosophic metric space (NMS) if the following conditions are satisfied:

(1) $0 \leq \Omega(x_1, x_2, \kappa_1) \leq 1$, $0 \leq \Phi(x_1, x_2, \kappa_1) \leq 1$, $0 \leq \Psi(x_1, x_2, \kappa_1) \leq 1\ \forall\, \kappa_1 \in \mathbb{R}^+$,
(2) $\Omega(x_1, x_2, \kappa_1) + \Phi(x_1, x_2, \kappa_1) + \Psi(x_1, x_2, \kappa_1) \leq 3$ for $\kappa_1 \in \mathbb{R}^+$,
(3) $\Omega(x_1, x_2, \kappa_1) = 1$, for $\kappa_1 > 0$,
(4) $\Omega(x_1, x_2, \kappa_1) = \Omega(x_2, x_1, \kappa_1)$, for $\kappa_1 > 0$ iff $x_1 = x_2$,
(5) $\Omega(x_1, x_2, \kappa_1) * \Omega(x_2, x_3, \kappa_2) \leq \Omega(x_1, x_3, \kappa_1 + \kappa_2)$, $\forall\, \kappa_1, \kappa_2 > 0$,

(6) $\Omega(x_1, x_2, .) : [0, \infty) \to [0, 1]$ is continuous,
(7) $\lim_{\kappa_1 \to \infty} \Omega(x_1, x_2, \kappa_1) = 1, \forall\ \kappa_1 > 0$,
(8) $\Phi(x_1, x_2, \kappa_1) = 0$, for $\kappa_1 > 0$, iff $x_1 = x_2$,
(9) $\Phi(x_1, x_2, \kappa_1) = \Phi(x_2, x_1, \kappa_1)$, for $\kappa_1 > 0$,
(10) $\Phi(x_1, x_2, \kappa_1) \boxdot \Phi(x_2, x_3, \kappa_2) \geq \Phi(x_1, x_3, \kappa_1 + \kappa_2), \forall\ \kappa_1, \kappa_2 > 0$,
(11) $\Phi(x_1, x_2, .) : [0, \infty) \to [0, 1]$ is continuous,
(12) $\lim_{\kappa_1 \to \infty} \Phi(x_1, x_2, \kappa_1) = 0, \forall\ \kappa_1 > 0$,
(13) $\Psi(x_1, x_2, \kappa_1) = 0$, for $\kappa_1 > 0$ iff $x_1 = x_2$,
(14) $\Psi(x_1, x_2, \kappa_1) = \Psi(x_2, x_1, \kappa_1)$ for $\kappa_1 > 0$,
(15) $\Psi(x_1, x_2, \kappa_1) \boxdot \Psi(x_2, x_3, \kappa_2) \geq \Psi(x_1, x_3, \kappa_1 + \kappa_2), \forall\ \kappa_1, \kappa_2 > 0$,
(16) $\Psi(x_1, x_2, .) : [0, \infty) \to [0, 1]$ is continuous,
(17) $\lim_{\kappa_1 \to \infty} \Psi(x_1, x_2, \kappa_1) = 0, \forall\ \kappa_1 > 0$,
(18) if $\kappa_1 \leq 0$, then $\Omega(x_1, x_2, \kappa_1) = 0$, $\Phi(x_1, x_2, \kappa_1) = 1$, $\Psi(x_1, x_2, \kappa_1) = 1$.

$\forall\ x_1, x_2, x_3 \in \mathbb{E}$. Then, $\mathcal{M} = (\Omega, \Phi, \Psi)$ is called Neutrosophic metric (NM) on $\mathbb{E}$.

Both the concept of Neutrosophic normed space (NNS) and statistical convergence in NNS was defined by [11].

Definition 2.5. ([11]) Let $\mathbb{E}$ be the vector space, $\mathcal{M} = \{< x_1, \Lambda(x_1), \Gamma(x_1), \Pi(x_1) >: x_1 \in \mathbb{E}\}$ be an normed space such that $\mathbb{M} : \mathbb{E} \times \mathbb{R}^+ \to [0, 1]$. Let the continuous t-norm and continuous t-conorm be represented by $*$ and $\boxdot$, respectively. The four-tuple $V = (\mathbb{E}, \mathcal{M}, *, \boxdot)$ is referred to as NNS for every $x_1, x_2 \in \mathbb{E}, \kappa_1, \kappa_2 > 0$ and for each $\tau \neq 0$ if the following conditions are satisfied:
(1) $0 \leq \Lambda(x_1, \kappa_1) \leq 1$, $0 \leq \Gamma(x_1, \kappa_1) \leq 1$, $0 \leq \Pi(x, \kappa_1) \leq 1\ \forall\ \kappa_1 \in \mathbb{R}^+$,
(2) $\Lambda(x_1, \kappa_1) + \Gamma(x_1, \kappa_1) + \Pi(x_1, \kappa_1) \leq 3$ for $\kappa_1 \in \mathbb{R}^+$,
(3) $\Lambda(x_1, \kappa_1) = 1$, for $\kappa_1 > 0$ iff $x_1 = 0$,
(4) $\Lambda(\tau x_1, \kappa_1) = \Lambda(x_1, \frac{\kappa_1}{|\tau|})$,
(5) $\Lambda(x_1, \kappa_1) * \Lambda(x_2, \kappa_2) \leq \Lambda(x_1 + x_2, \kappa_1 + \kappa_2)$,
(6) $\Lambda(x_1, .)$ is a continuous non-decreasing function,
(7) $\lim_{\kappa_1 \to \infty} \Lambda(x_1, \kappa_1) = 1$,
(8) $\Gamma(x_1, \kappa_1) = 0$, or $\kappa_1 > 0$ iff $x_1 = 0$,
(9) $\Gamma(\tau x_1, \kappa_1) = \Gamma(x_1, \frac{\kappa_1}{|\tau|})$,
(10) $\Gamma(x_1, \kappa_1) \boxdot \Gamma(x_2, \kappa_2) \geq \Gamma(x_1 + x_2, \kappa_1 + \kappa_2)$,
(11) $\Gamma(x_1, .)$ is a continuous, non-increasing function,
(12) $\lim_{\kappa_1 \to \infty} \Gamma(x_1, \kappa_1) = 0$,
(13) $\Pi(x_1, \kappa_1) = 0$, for $\kappa_1 > 0$ iff $x_1 = 0$,
(14) $\Pi(\tau x_1, \kappa_1) = \Pi(x_1, \frac{\kappa_1}{|\tau|})$,
(15) $\Pi(x_1, \kappa_1) \boxdot \Pi(x_2, \kappa_2) \geq \Pi(x_1 + x_2, \kappa_1 + \kappa_2)$,
(16) $\Pi(x_1, .)$ is a continuous, non-increasing function,
(17) $\lim_{\kappa_1 \to \infty} \Pi(x_1, \kappa_1) = 0$,
(18) If $\kappa_1 \leq 0$ then $\Lambda(x_1, \kappa_1) = 0$, $\Gamma(x_1, \kappa_1) = 1$, $\Pi(x_1, \kappa_1) = 1$.

Then the neutrosophic norm (NN) is defined as $\mathcal{M} = (\Lambda, \Gamma, \Pi)$.

Example 2.1. ([11]) Let us Suppose $(\mathbb{E}, ||.||)$ is an NS. Assign the operation $*$ and $\boxdot$ as t-norm $x_1 * x_2 = x_1 x_2$, t-conorm $x_1 \boxdot x_2 = x_1 + x_2 - x_1 x_2$ for $\kappa_1 > ||x_1||$.

$$\Lambda(x_1, \kappa_1) = \frac{\kappa_1}{\kappa_1 + ||x_1||},\ \Gamma(x_1, \kappa_1) = \frac{||x_1||}{\kappa_1 + ||x_1||},\ \Pi(x_1, \kappa_1) = \frac{||x_1||}{\kappa_1},$$

$\forall\, x_1, x_2 \in \mathbb{E}$ and $\kappa_1 > 0$. If we consider $\kappa_1 \leq ||x_1||$, then $\Lambda(x_1, \kappa_1) = 0$, $\Gamma(x_1, \kappa_1) = 1$ and $\Pi(x_1, \kappa_1) = 1$. Then, $(\mathbb{E}, \mathcal{M}, *, \boxdot)$ is NNS such that $\mathcal{M} : \mathbb{E} \times \mathbb{R}^+ \to [0, 1]$.

Definition 2.6. ([11]) Let V be a NNS and (y_n) is a sequence in V such that $0 < \epsilon_1 < 1$ and $\kappa_1 > 0$. Then, (y_n) converges to $\mathcal{L}_1$ iff there exists $n_1 \in \mathbb{N}$ such that $\Lambda(y_n - \mathcal{L}_1, \kappa_1) > 1 - \epsilon_1$, $\Gamma(y_n - \mathcal{L}_1, \kappa_1) < \epsilon_1$ and $\Pi(y_n - \mathcal{L}_1, \kappa_1) < \epsilon_1$. That is $\lim_{n\to\infty} \Lambda(y_n - \mathcal{L}_1, \kappa_1) = 1$, $\lim_{n\to\infty} \Gamma(y_n - \mathcal{L}_1, \kappa_1) = 0$, and $\lim_{n\to\infty} \Pi(y_n - \mathcal{L}_1, \kappa_1) = 0$ as $\kappa_1 > 0$. In this case, the sequence (y_n) is said to be convergent sequence in V. The convergence in NNS is denoted by $\mathcal{M} - \lim y_n = \mathcal{L}_1$.

Definition 2.7. ([11]) Let V be a NNS and (y_n) represent a sequence in V such that $0 < \epsilon_1 < 1$ and $\kappa_1 > 0$. Then, (y_n) is Cauchy in a NNS V if there exists $n_1 \in \mathbb{N}$ such that $\Lambda(y_n - y_m, \kappa_1) > 1 - \epsilon_1$, $\Gamma(y_n - y_m, \kappa_1) < \epsilon_1$ and $\Pi(y_n - y_m, \kappa_1) < \epsilon_1$ for $n, m \geq n_1$.

Definition 2.8. ([11]) Let V be NNS. For $\kappa_1 > 0$, $x_1 \in F$ and $0 < \epsilon_1 < 1$,

$$\begin{aligned}\mathcal{O}(x_1, \epsilon_1, \kappa_1) = &\{x_2 \in \mathbb{E} : \Lambda(x_1 - x_2, \kappa_1) > 1 - \epsilon_1, \\ &\Gamma(x_1 - x_2, \kappa_1) < \epsilon_1, \Pi(x_1 - x_2, \kappa_1) < \epsilon_1\}\end{aligned}$$

is known as an open ball (OB) with centre x_1 and radius ϵ_1.

Definition 2.9. ([11]) In NNS V, the set $A \subset \mathbb{E}$ is known as neutrosophic-bounded (NB) if there exists $\kappa_1 > 0$, and $\epsilon_1 \in (0, 1)$ such that $\Lambda(x_1, \kappa_1) > 1 - \epsilon_1$, $\Gamma(x_1, \kappa_1) < \epsilon_1$ and $\Pi(x_1, \kappa_1) < \epsilon_1$ $\forall\, x_1 \in A$.

3. Main Results

In this section, we define and study the notion of statistical triple convergence and completeness in neutrosophic normed space.

Definition 3.1. Let V be a NNS and (y_{ijk}) be a triple sequence in V such that $0 < \epsilon_1 < 1$ and $\kappa_1 > 0$. Then, (y_{ijk}) converges to $\mathcal{L}_1$ iff there exists $n_1 \in \mathbb{N}$ such that $\Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) > 1 - \epsilon_1$, $\Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) < \epsilon_1$ and $\Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) < \epsilon_1$. That is $\lim_{n\to\infty} \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) = 1$, $\lim_{n\to\infty} \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) = 0$ and $\lim_{n\to\infty} \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) = 0$ as $\kappa_1 > 0$. In this case, the triple sequence (y_{ijk}) is regarded as a convergent sequence in the NNS space V. $\mathcal{M}_3 - \lim y_{ijk} = \mathcal{L}_1$ indicates the triple convergence in NNS.

Theorem 3.2. *Let V be a NNS and (y_{ijk}) be the triple sequence in V. Consequently, the following statements are true:*

(i) *If (y_{ijk})) in V is convergent then the limit point is unique.*

(ii) *In V, if $\lim_{i,j,k\to\infty} y_{ijk} = \mathcal{L}_1$ and $\lim_{i,j,k\to\infty} z_{ijk} = \mathcal{L}_2$, then $\lim_{i,j,k\to\infty}(y_{ijk} + z_{ijk}) = \mathcal{L}_1 + \mathcal{L}_2$.*

(iii) *If $\lim_{i,j,k\to\infty} y_{ijk} = \mathcal{L}_1$ and $a \neq 0$, then $\lim_{i,j,k\to\infty} a y_{ijk} = a\mathcal{L}_1$.*

Proof. Since the proof of this theorem is straightforward, we have chosen to omit it. □

Definition 3.3. Let V be a NNS and (y_{ijk}) represents a triple sequence in V such that $0 < \epsilon_1 < 1$ and $\kappa_1 > 0$. Then, the triple sequence (y_{ijk}) is Cauchy in a NNS V

if there is a $n_1 \in \mathbb{N}$ such that $\Lambda(y_{ijk} - z_{pqr}, \kappa_1) > 1 - \epsilon_1$, $\Lambda(y_{ijk} - z_{pqr}, \kappa_1) < \epsilon_1$ and $\Pi(y_{ijk} - z_{pqr}, \kappa_1) < \epsilon_1$ for $i, j, k, p, q, r \geq n_1$. A NNS V is said to be complete iff every triple Cauchy sequence (y_{ijk}) converges to $\mathcal{L}_1$ in NNS V.

Example 3.1. Let Λ, Γ and Π be the values from Example 2.1; in this case, V is a NNS. Further,

$$\lim_{i,j,k,p,q,r\to\infty} \frac{\kappa_1}{\kappa_1 + ||y_{ijk} - y_{pqr}||} = 1, \lim_{i,j,k,p,q,r\to\infty} \frac{||y_{ijk} - y_{pqr}||}{\kappa_1 + ||y_{ijk} - y_{pqr}||} = 0 \textit{ and}$$

$$\lim_{i,j,k,p,q,r\to\infty} \frac{||y_{ijk} - y_{pqr}||}{\kappa_1} = 0,$$

that is

$$\lim_{i,j,k,p,q,r\to\infty} \Lambda(y_{ijk} - y_{pqr}) = 1, \lim_{i,j,k,p,q,r\to\infty} \Gamma(y_{ijk} - y_{pqr}) = 0 \textit{ and}$$

$$\lim_{i,j,k,p,q,r\to\infty} \Pi(y_{ijk} - y_{pqr}) = 0.$$

Therefore, we can say that the triple sequence (y_{ijk}) is a triple Cauchy sequence in NNS V.

Remark 3.4. Every triple convergent sequence in NNS V is a triple Cauchy sequence. However, converse is not true.

Theorem 3.5. *Assume that V is a NNS and that (y_{ijk}) is a triple sequence in the NNS V. Then, the following statements are true:*

(i) *If we choose the continuous t-norm $x_1 * x_2 = min\{x_1, x_2\}$ and the continuous t-conorm $x_1 \boxdot x_2 = max\{x_1, x_2\}$ for $x_1, x_2 \in [0, 1]$, then every triple Cauchy sequence is bounded in NNS V.*

(ii) *Let (y_{ijk}) and (z_{pqr}) be triple Cauchy sequences, and (a_{ijk}) be scalars in NNS V. Then, the triple sequences $(y_{ijk} + z_{pqr})$ and $(a_{ijk}y_{ijk})$ are triple Cauchy in NNS V.*

(iii) *If every triple Cauchy sequence in NNS V has a triple convergent subsequence, then V is a complete NNS.*

Proof. As we can easily prove Λ, Γ, and Π, Cauchy triple sequence in V and completeness, it is followed by the definitions of NNS. $\square$

Definition 3.6. Let V be a NNS. A triple sequence (y_{ijk}) is said to be statistical convergence with respect to the neutrosophic norm (TSC-NN), if there exists $\mathcal{L}_1 \in \mathbb{E}$ such that the set,

$$\mathcal{R}_{\epsilon_3} = \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) \leq 1 - \epsilon_1 \text{ or } \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \epsilon_1,\ \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \epsilon_1\}$$

or, equivalently,

$$\mathcal{R}_{\epsilon_3} = \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) > 1 - \epsilon_1 \text{ and } \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) < \epsilon_1, \Pi(y_{ijk} - \mathcal{L}_1,\ \kappa_1) < \epsilon_1\},$$

has a triple natural density (TND) zero, for every $\epsilon_1 > 0$ and $\kappa_1 > 0$. That is $d(\mathcal{R}_{\epsilon_3}) = 0$ or equivalently,

$$\lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) \leq 1 - \epsilon_1 \text{ or } \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \epsilon_1, \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \epsilon_1\}| = 0.$$

Therefore, we can write $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$ or $y_{ijk} \to \mathcal{L}_1(S_{\mathcal{M}_3})$. The set of TSC-NN will be denoted by $S_{\mathcal{M}_3}$. If $\mathcal{L}_1 = 0$, then we can write $S^0_{\mathcal{M}_3}$.

Lemma 3.7. *Let V be a NNS. Then the following statements are equivalent, $\forall\ \epsilon_1 > 0$ and $\kappa_1 > 0$,*

(i) $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$,

(ii) $\lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) \leq 1 - \epsilon_1\}| = \lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \epsilon_1\}| = \lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \epsilon_1\}| = 0$,

(iii) $\lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) > 1 - \epsilon_1$ *and* $\Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) < \epsilon_1, \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) < \epsilon_1\}| = 1$,

(iv) $\lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) > 1 - \epsilon_1\}| = \lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) < \epsilon_1\}| = \lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) < \epsilon_1\}| = 0$,

(v) $S_{\mathcal{M}_3} - \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) = 1$ *and* $S_{\mathcal{M}_3} - \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) = 0$, $S_{\mathcal{M}_3} - \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) = 0$.

Theorem 3.8. *Let V be a NNS. If (y_{ijk}) is TSC-NN, then $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$ is unique.*

Proof. Let us suppose that $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$ and $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_2$ for $\mathcal{L}_1 \neq \mathcal{L}_2$. Then take $\epsilon_1 > 0$ and for a given $\xi > 0$, $(1 - \epsilon_1) * (1 - \epsilon_1) > 1 - \xi$ and $\epsilon_1 \boxdot \epsilon_1 < \xi$. For any $\kappa_1 > 0$. The following sets should be written as:

$$\mathcal{R}_{\Lambda_1}(\epsilon_1, \kappa_1) := \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) \leq 1 - \epsilon_1\},$$

$$\mathcal{R}_{\Lambda_2}(\epsilon_1, \kappa_1) := \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_2, \frac{\kappa_1}{2}) \leq 1 - \epsilon_1\},$$

$$\mathcal{R}_{\Gamma 1}(\epsilon_1, \kappa_1) := \{i \leq m, j \leq n, k \leq r : \Gamma(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) \geq \epsilon_1\},$$

$$\mathcal{R}_{\Gamma_2}(\epsilon_1, \kappa_1) := \{i \leq m, j \leq n, k \leq r : \Gamma(y_{ijk} - \mathcal{L}_2, \frac{\kappa_1}{2}) \geq \epsilon_1\},$$

$$\mathcal{R}_{\Pi_1}(\epsilon_1, \kappa_1) := \{i \leq m, j \leq n, k \leq r : \Pi(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) \geq \epsilon_1\},$$

$$\mathcal{R}_{\Pi_2}(\epsilon_1, \kappa_1) := \{i \leq m, j \leq n, k \leq r : \Pi(y_{ijk} - \mathcal{L}_2, \frac{\kappa_1}{2}) \geq \epsilon_1\}.$$

Since that $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$. Then by Lemma 3.7, for all $\kappa_1 > 0$,

$$d(\mathcal{R}_{\Lambda_1}(\epsilon_1, \kappa_1)) = d(\mathcal{R}_{\Gamma_1}(\epsilon_1, \kappa_1)) = d(\mathcal{R}_{\Pi_1}(\epsilon_1, \kappa_1)) = 0.$$

Moreover, since we have $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_2$. Then by the Lemma 3.7, for $\kappa_1 > 0$,

$$d(\mathcal{R}_{\Lambda_2}(\epsilon_1, \kappa_1)) = d(\mathcal{R}_{\Gamma_2}(\epsilon_1, \kappa_1)) = d(\mathcal{R}_{\Pi_2}(\epsilon_1, \kappa_1)) = 0.$$

Now, let

$$\mathcal{R}_{\mathcal{M}_3}(\xi,\kappa_1) := \{\mathcal{R}_{\Lambda_1}(\epsilon_1,\kappa_1) \cup \{\mathcal{R}_{\Lambda_2}(\epsilon_1,\kappa_1)\} \cap \{\mathcal{R}_{\Gamma_1}(\epsilon_1,\kappa_1) \cup \{\mathcal{R}_{\Gamma_2}(\epsilon_1,\kappa_1)\} \cap \{\mathcal{R}_{\Pi_1}(\epsilon_1,\kappa_1) \cup \{\mathcal{R}_{\Pi_2}(\epsilon_1,\kappa_1)\}.$$

Then, we can see that $d(\mathcal{R}_{\mathcal{M}_3}(\xi,\kappa_1)) = 0$ which implies $d(\mathbb{N}\times\mathbb{N}\times\mathbb{N} - \mathcal{R}_{\mathcal{M}_3}(\epsilon_1,\kappa_1)) = 1$. Then there are the following possibilities when we take $(i,j,k) \in (\mathbb{N}\times\mathbb{N}\times\mathbb{N} - \mathcal{R}_{\mathcal{M}_3}(\epsilon_1,\kappa_1))$:
(i) $(i,j,k) \in (\mathbb{N}\times\mathbb{N}\times\mathbb{N} - (\mathcal{R}_{\Gamma_1}(\epsilon_1,\kappa_1) \cup \mathcal{R}_{\Gamma_2}(\epsilon_1,\kappa_1))$,
(ii) $(i,j,k) \in (\mathbb{N}\times\mathbb{N}\times\mathbb{N} - (\mathcal{R}_{\Lambda_1}(\epsilon_1,\kappa_1) \cup \mathcal{R}_{\Lambda_2}(\epsilon_1,\kappa_1))$,
(iii) $(i,j,k) \in (\mathbb{N}\times\mathbb{N}\times\mathbb{N} - (\mathcal{R}_{\Pi_1}(\epsilon_1,\kappa_1) \cup \mathcal{R}_{\Pi_2}(\epsilon_1,\kappa_1))$,

First of all, consider (i). Then, we have

$$\Gamma(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) \geq \Gamma(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) * \Gamma(y_{ijk} - \mathcal{L}_2, \frac{\kappa_1}{2}) > (1-\epsilon_1) * (1-\epsilon_1).$$

Since we have $(1-\epsilon_1) * (1-\epsilon_1) > (1-\xi)$,

$$\Gamma(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) > (1-\xi). \tag{1}$$

So by (1), For all $\kappa_1 > 0$, we have that $\Gamma(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) = 1$, where $\xi > 0$ is arbitrary. So we obtain $\mathcal{L}_1 = \mathcal{L}_2$. For case (ii) if we select $(i,j,k) \in (\mathbb{N}\times\mathbb{N}\times\mathbb{N} - (\mathcal{R}_{\Lambda_1}(\epsilon_1,\kappa_1) \cup \mathcal{R}_{\Lambda_2}(\epsilon_1,\kappa_1))$, then we can write,

$$\Lambda(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) \leq \Lambda(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) \boxdot \Lambda(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1 \boxdot \epsilon_1.$$

Now using $\epsilon_1 \boxdot \epsilon_1 < \xi$, We observe that $\Lambda(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) < \xi$. $\forall\ \kappa_1 > 0$, we get $\Lambda(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) = 0$, where $\xi > 0$ is arbitrary. Therefore $\mathcal{L}_1 = \mathcal{L}_2$.

Lastly, in the same manner, for case (iii), if we choose $(i,j,k) \in (\mathbb{N}\times\mathbb{N}\times\mathbb{N} - (\mathcal{R}_{\Pi_1}(\epsilon_1,\kappa_1) \cup \mathcal{R}_{\Pi_2}(\epsilon_1,\kappa_1))$, then we can write,

$$\Pi(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) \leq \Pi(y_{ijk} - y_1, \frac{\kappa_1}{2}) \boxdot \Pi(y_{ijk} - y_2, \frac{\kappa_1}{2}) < \epsilon_1 \boxdot \epsilon_1.$$

Now using $\epsilon_1 \boxdot \epsilon_1 < \xi$, we can observe that $\Pi(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) < \xi$. $\forall\ \kappa_1 > 0$, we get $\Pi(\mathcal{L}_1 - \mathcal{L}_2,\kappa_1) = 0$, where $\xi > 0$ is arbitrary. Therefore $\mathcal{L}_1 = \mathcal{L}_2$.
Hence the proof is over. □

Theorem 3.9. *If $\mathcal{M}_3 - \lim y_{ijk} = \mathcal{L}_1$ for a NNS V, then $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$.*

Proof. Let $\mathcal{M}_3 - \lim y_{ijk} = \mathcal{L}_1$. Then , $\forall\ \epsilon_1 > 0$ and $\kappa_1 > 0$, there exists a number $n_1 \in \mathbb{N}$ such that $\Lambda(y_{ijk} - \mathcal{L}_1,\kappa_1) > 1 - \epsilon_1$ *and* $\Gamma(y_{ijk} - \mathcal{L}_1,\kappa_1) < \epsilon_1, \Pi(y_{ijk} - \mathcal{L}_1,\ \kappa_1) < \epsilon_1$, $\forall\ i,j,k \geq n_1$.
Hence the set,

$$\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1,\kappa_1) > 1 - \epsilon_1 \ \textit{and}\ \Gamma(y_{ijk} - \mathcal{L}_1,\kappa_1) < \epsilon_1, \Pi(y_{ijk} - y,\kappa_1) < \epsilon_1\},$$

has a finite number of terms. This means that every finite subset of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ has a triple natural density zero, indicating that,

$\lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1,\kappa_1) \leq 1 - \epsilon_1 \ \textit{or}\ \Gamma(y_{ijk} - \mathcal{L}_1,\kappa_1) \geq \epsilon_1,\ \Pi(y_{ijk} - \mathcal{L}_1,\kappa_1) \geq \epsilon_1\}| = 0.$

Hence the proof is over. □

Theorem 3.10. *Let V be a NNS. $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$ iff there exists an increasing index triple sequence $\mathfrak{P} = \{p_1, p_2, ...p_n, p_1, p_2...p_m, p_1, p_2..., p_r\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ while $d(\mathfrak{P}) = 1$, $\mathcal{M}_3 - \lim y_{nmr} = \mathcal{L}_1$.*

Proof. Let us suppose that $S_{\Gamma_{\mathcal{M}_3}} - \lim y_{ijk} = \mathcal{L}_1$. For any $\kappa_1 > 0$ and $\xi = 1, 2, 3, ...,$

$$\mathfrak{Q}_{\mathcal{M}_3}(\xi, \kappa_1) = \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) > 1 - \frac{1}{\xi} \text{ and } \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) < \frac{1}{\xi}, \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) < \frac{1}{\xi}\},$$

and

$$\mathfrak{R}_{\mathcal{M}_3}(\xi, \kappa_1) = \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) \leq 1 - \frac{1}{\xi} \text{ or } \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \frac{1}{\xi}, \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \frac{1}{\xi}\}.$$

Then, $d(\mathfrak{R}_{\mathcal{M}_3}(\xi, \kappa_1)) = 0$, since $S_{\mathcal{M}_3} - \lim y_{ijk} = \mathcal{L}_1$. Apart from, for $\kappa_1 > 0$ and $\xi = 1, 2, 3, ...$, $\mathfrak{Q}_{\mathcal{M}_3}(\xi + 1, \kappa_1) \subset \mathfrak{Q}_{\mathcal{M}_3}(\xi, \kappa_1)$

$$d(\mathfrak{Q}_{\mathcal{M}_3}(\xi, \kappa_1)) = 1. \tag{2}$$

We will now show that for $(i, j, k) \in \mathfrak{Q}_{\mathcal{M}_3}(\xi, \kappa_1)$, $\mathcal{M}_3 - \lim y_{nmr} = \mathcal{L}_1$. Let us take $\mathcal{M}_3 - \lim y_{nmr} \neq \mathcal{L}_1$, for some $(i, j, k) \in \mathfrak{Q}_{\mathcal{M}_3}(\xi, \kappa_1)$. Then, there is $\sigma > 0$ and a integer n_1 such that $\Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) \leq 1 - \sigma$ *or* $\Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \sigma, \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \sigma$, $\forall\ i, j, k \geq n_1$. Hence,

$$\lim_{n,m,r} \frac{1}{nmr} |\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) > 1 - \sigma \text{ and } \Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) < \sigma, \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) < \sigma\}| = 0.$$

Since $\sigma > \frac{1}{\xi}$, we have that $d(\mathfrak{Q}_{\mathcal{M}_3}(\xi, \kappa_1)) = 0$, we get a contradiction from (2). Therefore, $\mathcal{M}_3 - \lim y_{ijk} = \mathcal{L}_1$.

Let us now suppose that there is a subset $\mathfrak{P} = \{p_1, p_2, ...p_n, p_1, p_2...p_m, p_1, p_2..., p_r\} \subset \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that $d(\mathfrak{P}) = 1$ and $\mathcal{M}_3 - \lim y_{ijk} = \mathcal{L}_1$, this implies that there exist $n_1 \in \mathbb{N}$ such that $\Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) > 1 - \xi$ *and* $\Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) < \xi$, $\Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) < \xi$, for every $\xi > 0$ and $\kappa_1 > 0$. In this case,

$\mathfrak{R}_{\mathcal{M}_3}(\xi, \kappa_1) := \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - \mathcal{L}_1, \kappa_1) \leq 1 - \xi$ *or* $\Gamma(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \xi, \Pi(y_{ijk} - \mathcal{L}_1, \kappa_1) \geq \xi\} \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} - \{p_{n+1}, p_{n+2}, ...; p_{m+1}, p_{m+}, ...; p_{r+1}, p_{r+1}, p_{r+2}...\}$. Therefore, $\mathfrak{R}_{\mathcal{M}_3}(\xi, \kappa_1) \leq 1 - 1 = 0$. Hence $\mathcal{M}_3 - \lim y_{ijk} = \mathcal{L}_1$. □

4. Statistical completeness of triple sequence in NNS

Definition 4.1. If a triple sequence y_{ijk} is statistically Cauchy with respect to NN(TSCa-NN) in NNS V, if there exits $N = N(\epsilon_1)$, $M = M(\epsilon_1)$ and $R = R(\epsilon_1)$, for every $\epsilon_1 > 0$ and $\kappa_1 > 0$ such that

$$\mathcal{RC}_{\epsilon_1} := \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk} - y_{NMR}, \kappa_1) \leq 1 - \epsilon_1 \text{ or } \Gamma(y_{ijk} - y_{NMR}, \kappa_1) \geq \epsilon_1, \Pi(y_{ijk} - y_{NMR}, \kappa_1) \geq \epsilon_1\},$$

has triple natural density zero. That is $d(\mathcal{RC}_{\epsilon_1}) = 0$.

Theorem 4.2. *If triple sequence (y_{ijk}) is TSC-NN in NNS V. Then it becomes TSCa-NN.*

Proof. Let (y_{ijk}) be TSC-NN. We have that $(1-\epsilon_1)*(1-\epsilon_1) > 1-\xi$ and $\epsilon_1 \boxdot \epsilon_1 < \xi$, for a given $\epsilon_1 > 0$, take $\xi > 0$ then, we have

$$d(\mathfrak{A}(\epsilon_1,\xi)) = d(\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk}-\mathcal{L}_1, \frac{\kappa_1}{2}) \leq 1-\epsilon_1 \ or$$
$$\Gamma(y_{ijk}-\mathcal{L}_1, \frac{\kappa_1}{2}) \geq \epsilon_1 \ , \Pi(y_{ijk}-\mathcal{L}_1, \frac{\kappa_1}{2}) \geq \epsilon_1\}) = 0$$

or

$$d(\mathfrak{A}^{\mathfrak{c}}(\epsilon_1,\xi)) = d(\{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk}-\mathcal{L}_1, \frac{\kappa_1}{2}) > 1-\epsilon_1 \ and$$
$$\Gamma(y_{ijk}-\mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1, \ \Pi(y_{ijk}-\mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1\}) = 1,$$

for $\kappa_1 > 0$. Let us suppose $s,t,u \in \mathfrak{A}^{\mathfrak{c}}(\epsilon_1,\xi)$. Then,

$$\Lambda(y_{stu}-\mathcal{L}_1,\kappa_1) > 1-\epsilon_1 \ and \ \Gamma(y_{stu}-\mathcal{L}_1,\kappa_1) < \epsilon_1, \Pi(y_{stu}-\mathcal{L}_1,\kappa_1) < \epsilon_1.$$

Also, let

$$\mathfrak{B}(\epsilon_1,\xi) = \{i \leq m, j \leq n, k \leq r : \Lambda(y_{ijk}-y_{stu},\kappa_1) \leq 1-\xi \ or$$
$$\Gamma(y_{ijk}-y_{stu},\kappa_1) \geq \xi, \Pi(y_{ijk}-y_{stu},\kappa_1) \geq \xi\}.$$

Now we claim that $\mathfrak{B}(\epsilon_1,\xi) \subset \mathfrak{A}^{\mathfrak{c}}(\epsilon_1,\xi)$. Let $a,b,c \in \mathfrak{B}(\epsilon_1,\xi) - \mathfrak{A}(\epsilon_1,\xi)$. Then

$$\Lambda(y_{abc}-y_{stu},\kappa_1) \leq 1-\xi \ and \ \Lambda(y_{abc}-\mathcal{L}_1, \frac{\kappa_1}{2}) > 1-\xi,$$

in particular $\Lambda(y_{stu}-\mathcal{L}_1,\kappa_1) > 1-\epsilon_1$. Then

$$\begin{aligned} 1-\xi &\geq \Lambda(y_{abc}-y_{stu},\kappa_1) \\ &\geq \Lambda(y_{abc}-\mathcal{L}_1, \frac{\kappa_1}{2}) * \Lambda(y_{stu}-\mathcal{L}_1, \frac{\kappa_1}{2}) \\ &> (1-\epsilon_1)*(1-\epsilon_1) \\ &> 1-\xi, \end{aligned}$$

which is not possible. In addition,

$$\Gamma(y_{abc}-y_{stu},\kappa_1) \geq \xi \ and \ \Gamma(y_{abc}-\mathcal{L}_1,\kappa_1) < \xi,$$

in particular $\Gamma(y_{stu}-\mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1$. Then,

$$\xi \leq \Gamma(y_{abc}-y_{stu},\kappa_1) \leq \Gamma(y_{abc}-\mathcal{L}_1, \frac{\kappa_1}{2}) \boxdot \Gamma(y_{stu}-\mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1 \boxdot \epsilon_1 < \xi,$$

which is not possible. Similarly

$$\Pi(y_{abc}-y_{stu},\kappa_1) \geq \xi \ and \ \Pi(y_{abc}-\mathcal{L}_1,\kappa_1) < \xi,$$

in particular $\Pi(y_{stu}-\mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1$. Then,

$$\xi \leq \Pi(y_{abc}-y_{stu},\kappa_1) \leq \Pi(y_{abc}-\mathcal{L}_1, \frac{\kappa_1}{2}) \boxdot \Pi(y_{stu}-\mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1 \boxdot \epsilon_1 < \xi,$$

which is not possible. In this case $\mathfrak{B}(\epsilon_1,\xi) \subset \mathfrak{A}(\epsilon_1,\xi)$. Then by equation (3) $d(\mathfrak{A}(\epsilon_1,\xi)) = 0$, so the triple sequaence (y_{ijk}) is TSCa-NN. □

Definition 4.3. Let V be a NNS. Then, V is called triple statistically complete (TSC-NN) if every TSCa-NN is TSC-NN.

Theorem 4.4. *Every NNS V is (TSC-NN)-complete.*

Proof. Let us suppose that (y_{ijk}) be TSCa-NN but not TSC-NN. Now take $\xi > 0$. We have that $(1-\epsilon_1) * (1-\epsilon_1) > (1-\xi)$ and $\epsilon_1 \boxdot \epsilon_1 < \xi$, for a given $\epsilon_1 > 0$ and $\kappa_1 > 0$, since (y_{ijk}) is not TSC-NN,

$$\begin{aligned}\Lambda(y_{ijk} - y_{NMR}, \kappa_1) &\geq \Lambda(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) * \Lambda(y_{NMR} - \mathcal{L}_1, \frac{\kappa_1}{2}) \\ &> (1-\epsilon_1) * (1-\epsilon_1) > (1-\xi),\end{aligned}$$

$$\Gamma(y_{ijk} - y_{NMR}, \kappa_1) \leq \Gamma(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) \boxdot \Gamma(y_{NMR} - \mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1 \boxdot \epsilon_1 < \xi,$$

$$\Pi(y_{ijk} - y_{NMR}, \kappa_1) \leq \Pi(y_{ijk} - \mathcal{L}_1, \frac{\kappa_1}{2}) \boxdot \Pi(y_{NMR} - \mathcal{L}_1, \frac{\kappa_1}{2}) < \epsilon_1 \boxdot \epsilon_1 < \xi,$$

For

$$\mathfrak{T}(\epsilon_1, \kappa_1) = \{i \leq N, j \leq M, k \leq R : \Lambda_{y_{ijk} - y_{NMR}}(\epsilon_1) \leq 1 - \xi\},$$

$d(\mathfrak{T}^c(\epsilon_1, \kappa_1) = 0$ and hence $d(\mathfrak{T}(\epsilon_1, \kappa_1) = 1$, so we have a contradiction, since (y_{ijk}) is TSCa-NN. Therefore, (y_{ijk}) must be TSC-NN. As a result, we get, every NNS is (TSC-NN)-complete. □

Lemma 4.5. *Let V be a NNS. Then, for any triple sequence $(y_{ijk}) \in \mathbb{E}$,the following conditions are equivalent:*

(i) *(y_{ijk}) is TSC-NN.*
(ii) *(y_{ijk}) is TSCa-NN.*
(iii) *NNS V is(TSC-NN)-complete.*
(iv) *There exists an increasing triple sequence $\mathfrak{P} = (p_{nmr})$ of natural numbers such that $d(\mathfrak{P}) = 1$ and the triple subsequence $y_{p_{nmr}}$ is a TSCa-NN.*

Proof. The proof is followed directly by Theorem 3.10, 4.2 and 4.4. □

5. Conclusion

The objective of the work is to generalize the statistical convergence of the triple sequence in neutrosophic norm linear space. Along with the established structural unique characteristics of NNSs, examples are offered. Additionally, statistical Cauchy triple sequences and statistically triple completeness for the neutrosophic norm were defined.

The work's main goal is to generalize the triple sequences' statistical convergence in neutrosophic norm linear space. Examples are provided in addition to the well-established structurally distinctive properties of NNSs. Also defined were statistically triple completeness and Cauchy triple sequence for the neutrosophic norm.

References

[1] M. Ali, I. Deli, F. Smarandache, The theory of neutrosophic cubic sets and their applications in pattern recognition, *Journal of Intelligent & Fuzzy Systems* **30** (2016), no. 4, 1957–1963.

[2] K.T. Atanassov, *On intuitionistic fuzzy sets theory*, vol. 283, Springer, 2012.

[3] T. Bera, N.K. Mahapatra, Neutrosophic soft normed linear spaces, *Neutrosohic Sets and Systems* **23** (2018), no. 1, 1–6.
[4] A. Esi, B. Hazarika, Lacunary summable sequence spaces of fuzzy numbers defined by ideal convergence and an Orlicz function, *Afrika Matematika* **25** (2014), 331–343.
[5] H. Fast, Sur la convergence statistique, *Colloquium Mathematicae* **2** (1951), no. 3-4, 241–244.
[6] A. George, P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems* **64** (1994), no. 3, 395–399.
[7] A. George, P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems* **90** (1997), no. 3, 365–368.
[8] C. Granados, A. Dhital, Statistical convergence of double sequences in neutrosophic normed spaces, *Neutrosophic Sets and Systems* **42** (2021), 333–344.
[9] M. Inuiguchi, H. Ichihashi, H. Tanaka, Fuzzy programming: A survey of recent developments. Stochastic versus fuzzy approaches to multiobjective mathematical programming under uncertainty, In: (R. Slowinski and J. Teglem (Eds.)) *Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming under Uncertainty*, Kluwer Academic Publishers, Dordrecht, 1990.
[10] O. Kaleva, S. Seikkala, On fuzzy metric spaces, *Fuzzy Sets and Systems* **12** (1984), no. 3, 215–229.
[11] M. Kirişçi and N. Şimşek, Neutrosophic normed spaces and statistical convergence, *The Journal of Analysis* **28** (2020), 1059–1073.
[12] J. Madore, Fuzzy physics, *Annals of Physics* **219** (1992), no. 1, 187–198.
[13] F. Móricz, Tauberian theorems for Cesaro summable double sequences, *Studia Mathematica* **110** (1994), no. 1, 83–96.
[14] M. Mursaleen, S.A. Mohiuddine, Statistical convergence of double sequences, In: *Convergence Methods for Double Sequences and Applications*, Springer, 2014 , 117–132.
[15] S. Pandit, A. Ahmad, On asymptotically I-equivalent sequences in intuitionistic fuzzy normed spaces, *NeuroQuantology* **20** (2022), no. 19, 502–512.
[16] S. Pandit, A. Ahmad, A study on statistical convergence of triple sequences in intuitionistic fuzzy normed space, *Sahand Communications in Mathematical Analysis* **19** (2022), no. 3, 1–12.
[17] S. Pandit, A. Ahmad, A. Esi, On intuitionistic fuzzy metric space and ideal convergence of triple sequence space, *Sahand Communications in Mathematical Analysis* **20** (2023), no. 1, 35–44.
[18] J.H. Park, Intuitionistic fuzzy metric spaces, *Chaos, Solitons & Fractals* **22** (2004), no. 5, 1039–1046.
[19] A.U. Rahman, M. Saeed, A. Dhital, *Decision making application based on neutrosophic parameterized hypersoft set theory*, Infinite Study (2021).
[20] A. Şahiner, M. Gürdal, F.K. Düden, Triple sequences and their statistical convergence, *Selcuk Journal of Applied Mathematics* **8** (2007), no. 2, 49–55.
[21] F. Smarandache, Neutrosophic set—a generalization of the intuitionistic fuzzy set, *International Journal of Pure and Applied Mathematics* **24** (2005), no. 3, 287.
[22] L.A. Zadeh, Fuzzy sets, *Information and Control* **8** (1965), no. 3, 338–353.
[23] L.A. Zadeh, Similarity relations and fuzzy orderings, *Information Sciences* **3** (1971), no. 2, 177-200.

(H. Shankar) Department of Mathematics, National Institute of Technology Patna, Patna 800005, India
E-mail address: haris.ph21.ma@nitp.ac.in

(A. Ahmad) Department of Mathematics, National Institute of Technology Patna, Patna 800005, India
E-mail address: ayaz@nitp.ac.in

(A. Esi) Basic Eng.Sci.(Math.Sect.) Malatya Turgut Ozal University, Malatya 44040, Turkey
E-mail address: aesi23@hotmail.com