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Numerical analysis of a viscoelastic contact problem

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Abstract. We consider a mathematical model which describes the frictional contact between a viscoelastic body and an obstacle, the so-called foundation. The process is quasistatic and the behavior of the material is modeled with a constitutive law with long memory. The contact is bilateral and the friction is modeled with Tresca's law. The existence of a unique weak solution to the model was proved in [15]. Here we describe a fully discrete scheme for the problem, implement it in a computer code and provide numerical results in the study of a two-dimensional test problem.

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1. Introduction

Contact phenomena involving deformable bodies abound in industry and everyday life. For this reason, considerable progress has been made in their modelling and analysis, and the engineering literature concerning this topic is rather extensive. An early attempt to the study of frictional contact problems within the framework of variational inequalities was made in [5]. Comprehensive references on analysis and numerical approximation of contact problems include [7, 8] and, more recently, [6]. Mathematical, mechanical and numerical state of the art on Contact Mechanics can be found in the proceedings [9, 12] and in the special issue [14].

The present paper is devoted to numerical analysis of a problem of bilateral frictional contact. The process is quasistatic and the friction is modelled with the well known Tresca's law in which the friction bound is given. The behavior of the material is described with a linear viscoelastic constitutive law with long memory of the form

$$\sigma_{ij}(t) = \mathcal{A}_{ijkl}\varepsilon_{kl}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}_{ijkl}(t-s)\varepsilon_{kl}(\boldsymbol{u}(s))ds,$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the stress tensor, $\boldsymbol{u} = (u_i)$ is the displacement field, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ denotes the linearized strain tensor and $\mathcal{A} = (\mathcal{A}_{ijkl})$, $\mathcal{B} = (\mathcal{B}_{ijkl})$ are given fourth order tensors. Details concerning such kind of constitutive laws can be found in [4, 5, 11, 16], for instance. The variational analysis of the model was provided in [15]. There, the unique solvability of the problem was proved by using an abstract existence and uniqueness result for a class of evolutionary variational inequalities involving a Volterra-type integral term. In the present paper we describe a fully discrete scheme for the problem, involving finite difference discretization in time and finite element discretization in space, then we implement it in a computer code and provide numerical simulations.

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The paper is organized as follows. In Section 2 we present the contact problem and recall the result obtained in [15] concerning its unique solvability. In Section 3 we describe the fully discrete approximations of the model and state error estimates results. Our main interest lies in Section 4 where we present numerical simulations in the study of a two-dimensional test problem.

2. The model and its well-posedness

The physical setting is the following. A viscoelastic body occupies a regular domain Ω of $\mathbb{R}^d$ $(d = 2, 3)$ with boundary Γ partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that *meas* $(\Gamma_1) > 0$. We are interested in the evolution process of the mechanical state of the body in the time interval $[0, T]$ with $T > 0$. The body is clamped on Γ_1 and so the displacement field vanishes there. Surface tractions of density $\boldsymbol{f}_2$ act on Γ_2 and volume forces of density $\boldsymbol{f}_0$ act in Ω. We assume that the forces and tractions change slowly in time so that the acceleration of the system is negligible. On Γ_3 the body is in bilateral frictional contact with a rigid obstacle, the so-called foundation, and friction is modelled with Tresca's law. Under these assumptions, the classical formulation of the mechanical problem is the following.

Problem P**.** *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}_d$ *such that, for all* $t \in [0, T]$*,*

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) + \int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds \qquad \text{in} \quad \Omega, \tag{1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in} \quad \Omega, \tag{2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on} \quad \Gamma_1, \tag{3}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on} \quad \Gamma_2, \tag{4}$$

$$\left\{ \begin{array}{l} u_\nu(t) = 0, \ |\boldsymbol{\sigma}_\tau(t)| \leq g, \\ |\boldsymbol{\sigma}_\tau(t)| < g \ \Rightarrow \ \dot{\boldsymbol{u}}_\tau(t) = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau(t)| = g \ \Rightarrow \ \exists\, \lambda \geq 0 \text{ s.t. } \boldsymbol{\sigma}_\tau(t) = -\lambda \dot{\boldsymbol{u}}_\tau(t) \end{array} \right. \qquad \text{on} \quad \Gamma_3, \tag{5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{on} \quad \Omega. \tag{6}$$

Here and below $\boldsymbol{\nu}$ denote the unit outer normal on Γ, the subscripts ν and τ represent the *normal* and *tangential* components of vectors or tensors, respectively, and the dot above indicates the derivative with respect to the time; $\mathbb{S}_d$ is the space of second order symmetric tensors on $\mathbb{R}^d$, while "$\cdot$" and $|\cdot|$ represent the inner product and the Euclidean norm on $\mathbb{S}_d$ and $\mathbb{R}^d$, respectively; $\boldsymbol{\varepsilon}$ and Div are the *deformation* and *divergence* operators, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}),$$

where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the spatial variable; finally, the indices i, j, k and l run between 1 and d, and the summation convention over repeated indices is adopted.

Equation (1) is the viscoelastic constitutive law where $\mathcal{A} = (\mathcal{A}_{ijkl})$ represents the fourth order tensor of elastic coefficients and $\mathcal{B} = (\mathcal{B}_{ijkl})$ is the relaxation tensor. Equation (2) represents the equilibrium equation. Relations (3) and (4) are the displacement and traction boundary conditions, respectively, in which $\boldsymbol{\sigma\nu}$ is the Cauchy stress vector. Conditions (5) represent frictional contact, where u_ν denotes the normal displacement, $\boldsymbol{\sigma}_\tau$ represents the tangential stress and $\dot{\boldsymbol{u}}_\tau$ is the tangential velocity.

Equality $u_\nu(t) = 0$ on Γ_3 shows that there is no loss of the contact during the process, that is, the contact is bilateral. The rest of conditions in (5) represent Tresca's law of dry friction where $g \geq 0$ is the friction bound function, i.e. the magnitude of the limiting friction traction at which slip begins. The inequality in (5) holds in the stick zone and the equality holds in the slip zone. Contact problems with Tresca's friction law can be found in [5, 10], and more recently in [1, 2, 6] (see references therein for further details). Finally, (6) is the initial condition in which the initial displacement $\boldsymbol{u}_0$ is given.

We turn now to the variational formulation of Problem P. To this end we use the spaces

$$Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\,\},$$
$$V = \{\boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\},$$

which are real Hilbert spaces with the inner products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \quad (\boldsymbol{u}, \boldsymbol{v})_V = \int_\Omega \varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u})\,dx$$

and the associated norms denoted $\|\cdot\|_Q$ and $\|\cdot\|_V$. We also use the space

$$\mathbf{Q}_\infty = \{\, \boldsymbol{\xi} = (\xi_{ijkl}) \,|\, \xi_{ijkl} = \xi_{jikl} = \xi_{klij} \in L^\infty(\Omega)\,\},$$

which is Banach with the norm

$$\|\boldsymbol{\xi}\|_{\mathbf{Q}_\infty} = \max_{0 \leq i,j,k,l \leq d} \|\xi_{ijkl}\|_{L^\infty(\Omega)}.$$

Also, for any real Banach space X we employ the usual notation for the spaces $C([0,T];X)$, $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$ and $k = 1, 2, \ldots$.

In the study of the mechanical problem P we assume that the elasticity and relaxation tensors satisfy

$$\mathcal{A} \in \mathbf{Q}_\infty, \tag{7}$$
$$\exists\, m > 0 \text{ such that } \ \mathcal{A}\boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq m|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{S}_d, \text{ a.e. in } \Omega, \tag{8}$$
$$\mathcal{B} \in W^{1,2}(0,T;\mathbf{Q}_\infty). \tag{9}$$

We also assume that the force and traction densities satisfy

$$\boldsymbol{f}_0 \in W^{1,2}(0,T;L^2(\Omega)^d), \qquad \boldsymbol{f}_2 \in W^{1,2}(0,T;L^2(\Gamma_2)^d), \tag{10}$$

and the friction bound function is such that

$$g \in L^\infty(\Omega), \qquad g \geq 0 \quad \text{a.e. on} \quad \Gamma_3. \tag{11}$$

Finally, we assume that the initial data satisfies

$$\boldsymbol{u}_0 \in V, \tag{12}$$
$$a(\boldsymbol{u}_0, v) + j(\boldsymbol{v}) \geq (\boldsymbol{f}(0), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V, \tag{13}$$

where the bilinear form $a : V \times V \to \mathbb{R}$, the function $\boldsymbol{f} : [0,T] \to V$ and the functional $j : V \to \mathbb{R}_+$ are defined by

$$a(\boldsymbol{v}, \boldsymbol{w}) = (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \quad \forall\ \boldsymbol{v}, \boldsymbol{w} \in V, \tag{14}$$
$$(\boldsymbol{f}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t) \cdot \boldsymbol{v}\,d\boldsymbol{x} + \int_{\Gamma_2} \boldsymbol{f}_2(t) \cdot \boldsymbol{v}\,da \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T], \tag{15}$$
$$j(\boldsymbol{v}) = \int_{\Gamma_3} g\,|\boldsymbol{v}_\tau|\,da \qquad \forall \boldsymbol{v} \in V. \tag{16}$$

Proceeding in a standard way and using the notation (14)–(16) we obtain the following variational formulation of the contact problem (1)–(6), in terms of displacement.

Problem P_V**.** *Find the displacement field* $\boldsymbol{u} : [0,T] \to V$ *such that*

$$a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + (\int_0^t \mathcal{B}(t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(s))ds, \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \tag{17}$$

$$+ j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{18}$$

The well-posedness of the Problem P_V was proved in [15] and may be stated as follows.

Theorem 2.1. *Assume that* (7)–(13) *hold. Then Problem* P_V *has a unique solution* $\boldsymbol{u} \in W^{1,2}(0,T;V)$.

In the rest of the paper we assume that conditions stated in Theorem 3.1 hold and we denote by $\boldsymbol{u} \in W^{1,2}(0,T;V)$ the solution of Problem P_V. Let $\boldsymbol{\sigma} \in W^{1,2}(0,T;Q)$ be the stress field given by (1). A pair of functions $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ which satisfies (1), (17) and (18) is called a *weak solution* of the Problem P. We conclude that, under the assumptions (7)–(13), the contact problem P has a unique weak solution with regularity $\boldsymbol{u} \in W^{1,2}(0,T;V)$, $\boldsymbol{\sigma} \in W^{1,2}(0,T;Q)$.

3. Fully discrete approximation

We now consider a family of fully discrete schemes to approximate Problem P_V. We assume that Ω is a polyhedron. Let $\mathcal{T}_h$ be a finite element triangulation of $\overline{\Gamma}$ composed by d-simplex, compatible to the boundary decomposition $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, i.e., any point where the boundary condition type changes is a vertex of the triangulation. We denote by $h > 0$ the maximum diameter of triangles of $\mathcal{T}^h$ and we introduce the following finite element space associated to $\mathcal{T}^h$:

$$Q^h = \{\boldsymbol{\tau}^h \in Q : \boldsymbol{\tau}^h{}_{|T^h} \in [P^0(T^h)]_s^{d\times d}, \ \forall\, T^h \in \mathcal{T}^h\},$$

$$V^h = \{\boldsymbol{v}^h = (v_i^h) \in [C(\overline{\Omega})]^d, \boldsymbol{v}^h_{|T^h} \in [P^1(T^h)]^d \ \forall\, T^h \in \mathcal{T}^h, \ \boldsymbol{v}^h = \boldsymbol{0} \text{ on } \overline{\Gamma}_1, \ \boldsymbol{v}_\nu^h = 0 \text{ on } \overline{\Gamma}_3\}.$$

Here $P^m(T^h)$ is the space of polynomials of degree less or equal to m on d variables. Also, we denote by $\mathcal{P}^h : V \to V^h$ the operator given by

$$(\mathcal{P}^h\boldsymbol{v}, \boldsymbol{v}^h)_V = a(\boldsymbol{v}, \boldsymbol{v}^h) \quad \forall\, \boldsymbol{v} \in V, \ \boldsymbol{v}^h \in V^h.$$

In addition to the finite-dimensional subspace V^h, we need a partition of the time interval: $[0,T] = \cup_{n=1}^N [t_{n-1}, t_n]$ with $0 = t_0 < t_1 < \cdots < t_N = T$. We denote by $k_n = t_n - t_{n-1}$ the length of the sub-interval $[t_{n-1}, t_n]$ and let $k = \max_n k_n$ be the maximal step-size. Since $\boldsymbol{u} \in W^{1,2}(0,T;V)$ and $\boldsymbol{f} \in W^{1,2}(0,T;V)$, the pointwise values $\boldsymbol{u}_n = \boldsymbol{u}(t_n)$ and $\boldsymbol{f}_n = \boldsymbol{f}(t_n)$ $(0 \leq n \leq N)$ are well-defined. Also, since $\mathcal{B} \in W^{1,2}(0,T;\mathbf{Q}_\infty)$, the pointwise values $\mathcal{B}^{n,j} = \mathcal{B}(t_n - t_j)$, $0 \leq j \leq n \leq N$, are well-defined. Note that, in particular, $\mathcal{B}^{n,n} = \mathcal{B}_0 = \mathcal{B}(0)$.

Below, the symbol $\Delta\boldsymbol{u}_n$ represents the backward difference $\boldsymbol{u}_n - \boldsymbol{u}_{n-1}$, while $\delta_n\boldsymbol{u}_n = \Delta\boldsymbol{u}_n/k_n$ denotes the backward divided difference. And, for each time step t_n, the constants $\alpha_j^n > 0$ $(0 \leq j \leq n)$ denote the weights of the composed trapezoidal quadrature formula of $n+1$ points in $[0, t_n]$. Finally, note that in this section no summation is considered over the repeated indices n and j.

With the notation above, a family of fully discrete approximation schemes to Problem P_V is the following.

Problem P_V^{hk}**.** *Find* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ *such that* $\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h = \mathcal{P}^h \boldsymbol{u}_0$ *and*

$$a(\boldsymbol{u}_n^{hk}, \boldsymbol{v}^h - \delta_n \boldsymbol{u}_n^{hk}) + (\sum_{j=0}^{n} \alpha_j^n \mathcal{B}^{n,j} \boldsymbol{\varepsilon}(\boldsymbol{u}_j^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{v}^h - \delta_n \boldsymbol{u}_n^{hk}))_Q$$
$$+ j(\boldsymbol{v}^h) - j(\delta_n \boldsymbol{u}_n^{hk}) \geq (\boldsymbol{f}_n, \boldsymbol{v}^h - \delta_n \boldsymbol{u}_n^{hk})_V, \quad n = 1, \ldots, N.$$

Suppose that the assumptions in Theorem 2.1 hold and, in addition, assume that

$$k < \frac{2m}{\|\mathcal{B}_0\|_{\mathbf{Q}_\infty}}, \qquad \alpha_n^n \leq \frac{k}{2}. \tag{19}$$

Then, by using arguments similar as in those used in [13] we deduce that there exists a unique solution $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ of the Problem P_V^{hk}.

Next, we assume that the solution of the problem P_V satisfies $\boldsymbol{u} \in W^{2,\infty}(0,T;V)$, the relaxation tensor verifies $\mathcal{B} \in W^{1,\infty}(0,T;\mathbf{Q}_\infty)$, $\mathcal{B}$ and $\dot{\mathcal{B}}$ are Lipschitz continuous functions on $[0,T]$ with values to $\mathbf{Q}_\infty$ and (19) holds. Under these assumptions, proceeding as in [6, 13] we can show that, for k sufficiently small, the following error estimate holds:

$$\max_{1\leq n\leq N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V^2 \leq d_N \big(k^2 + Nk^3 + Nk^4 + Nk^5 + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^{hk}\|_V^2 \tag{20}$$
$$+ Nk \max_{1\leq n\leq N} \big\{ \inf_{\boldsymbol{v}^h\in V^h} \{\|\dot{\boldsymbol{u}}(t_n) - \boldsymbol{v}^h\|_V + \|\dot{\boldsymbol{u}}(t_n) - \boldsymbol{v}^h\|_V^2\}\big\}\big).$$

Here $d_N = c(1 + c(N+1)(N^2k^3 + k)e^{2c(N+1)(N^2k^3+k)})$ and c is a positive constant which depends on $\mathcal{A}, \mathcal{B}, \boldsymbol{f}, g$ and $\boldsymbol{u}$ but is independent on the discretization parameters h and k.

Inequality (20) is the basis for error estimates. For example if we assume that

$$\dot{\boldsymbol{u}} \in C([0,T];[H^2(\Omega)]^d), \;\; \boldsymbol{u}_0 \in [H^2(\Omega)]^d,$$

and the partition of $[0,T]$ is uniform, then we obtain

$$\max_{1\leq n\leq N} \|\boldsymbol{u}(t_n) - \boldsymbol{u}_n^{hk}\|_V \leq c(k + h^{\frac{1}{2}}).$$

Moreover, if $\dot{\boldsymbol{u}}_\tau \in C([0,T];[H^2(\Gamma_3)]^d)$, then it can be shown that

$$\max_{1\leq n\leq N} \{\|\boldsymbol{u}(t_n) - \boldsymbol{u}_n^{hk}\|_V\} \leq c(k + h),$$

which shows a linear convergence with respect to the parameters h and k.

4. Numerical simulations

To show the performance of the numerical method described in the previous section we have developed a FORTRAN-based software to solve variational inequalities of the second kind involving the functional $j : V \to \mathbb{R}$ given by (16). It combines a fixed point strategy with a method of duality-penalization based on the algorithm in [3]. We tested it in a number of problems and we present below two numerical simulations obtained in the study of a two-dimensional test problem.

The physical setting is presented in Figure 1.

For both simulations we considered a homogenous isotropic material with the modulus of Young $E = 10^6\ N/m^2$ and the coefficient of Poisson $\kappa = 0.3$. The same values were considered for the components of the relaxation tensor $\mathcal{B}$, and the rest

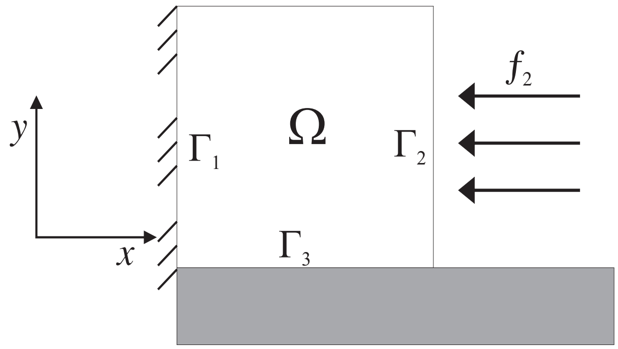

FIGURE 1. Two-dimensional contact problem with friction.

of the data were the following: $T = 0.1$ *sec.*, $k = 1 \times 10^{-3}$ *sec.*, $\boldsymbol{f}_0 = (0, 0)\ N/m^3$, $\boldsymbol{f}_2 = (-10^3, 0)\ N/m^2$ and $\boldsymbol{u}_0 = (0, 0)\ m$.

For the first simulation, we considered a *high* value for the friction bound, $g = 1 \times 10^6\ N/m^2$, while for the second one we took a *low* value, $g = 1\ N/m^2$. The numerical results are shown in Figures 2 and 3, respectively, where the deformations are amplified with 5000% for the sake of a better visual analysis.

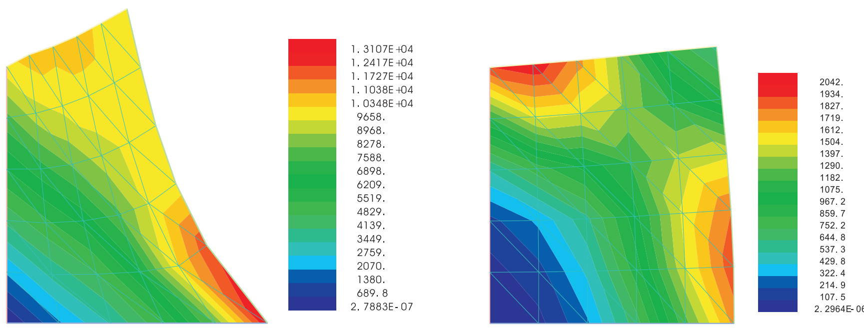

FIGURE 2. Deformed configuration and stress distribution in Von Mises norm for $t = 1 \times 10^{-2}$ *sec.* (left) and $t = 0.1$ *sec.* (right). Friction bound $g = 1 \times 10^6\ N/m^2$.

In Figure 2-left, we observe that, at $t = 1 \times 10^{-2}$ *sec.*, the nodes close to contact zone are subjected to displacements which are smaller than those the nodes situated far from the contact zone are subjected to. The stresses on the nodes close to contact zone are important, as well. Nevertheless, in Figure 3-left we observe that the horizontal displacements are quite similar either the node is far or near to the contact zone. This is due to the low value of the friction bound g. On the other hand, in Figure 2-right and Figure 3-right, we observe that for $t = T = 0.1$ *sec.*, the deformable body has recovered most of its original shape as a collateral effect of the memory.

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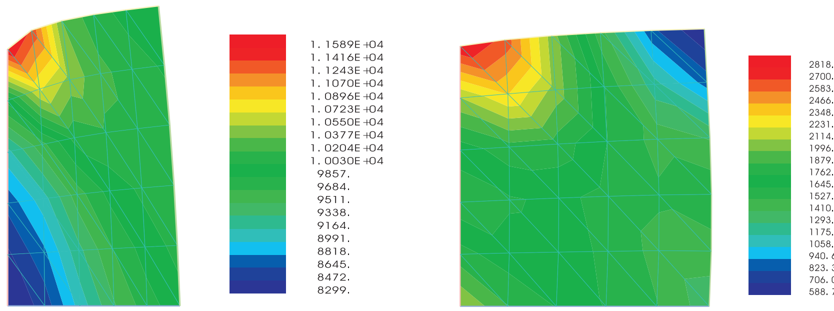

FIGURE 3. Deformed configuration and stress distribution in Von Mises norm for $t = 1 \times 10^{-2}$ *sec.* (left) and $t = 0.1$ *sec.* (right). Friction bound $g = 1\ N/m^2$.

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