Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 51(2), 2024, Pages 315–331, DOI: 10.52846/ami.v51i2.1801
ISSN: 1223-6934

Paired Hayman Conjecture of Some Delay-Differential Polynomials That Share a Small Function

Pulak Sahoo and Soniya Sultana

Abstract. In this paper, we investigate the uniqueness problems of $P(f)L(g)$ and $P(g)L(f)$ when they share a nonzero small function $\alpha(z)$ with finite weights. Here $L(h)$ may take the derivatives $h^{(k)}(z)$ or the shift $h(z+c)$ or the difference $h(z+c)-h(z)$ or the delay-differential $h^{(k)}(z+c)$, $k \geq 1$ and c is a nonzero constant and $P(z)$ is a polynomial of degree n. Also, $f(z)$ and $g(z)$ are transcendental meromorphic (or entire) functions and $\alpha(z)$ is a small function with respect to both $f(z)$ and $g(z)$. The results of the paper improve and supplement the recent results of Sahoo and Pal [17].

2020 *Mathematics Subject Classification.* 30D35.
Key words and phrases. Uniqueness, Paired Hayman Conjecture, Delay-Differential polynomials, Weighted Sharing.

1. Introduction, Definitions and Results

Let $\mathbb{C}$ denote the complex plane. By a meromorphic function we mean a meromorphic function in the complex plane $\mathbb{C}$. We adopt the standard notations and fundamental results of Nevanlinna theory as explained in [9, 12, 18]. In addition, $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measure. We say that two meromorphic functions $f(z)$ and $g(z)$ share a small function $a(z)$ CM (counting multiplicities), if $f(z)-a(z)$ and $g(z)-a(z)$ admit the same zeros with same multiplicities. If we do not consider the multiplicities, then we say that $f(z)$ and $g(z)$ share the small function $a(z)$ IM (ignoring multiplicities). The order $\rho(f)$ and hyper-order $\rho_2(f)$ of a meromorphic function f are defined as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \quad and \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

In 2001, Lahiri [10, 11] has introduced the concept of weighted sharing of values as follows:

Definition 1.1. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity

Received August 9, 2023. January 13, 2024.

$m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(>k)$ if and only if it is an a-point of g with multiplicity $n(>k)$, where m is not necessarily equal to n.

We write $f,\ g$ share (a,k) to mean that $f,\ g$ share the value a with weight k. Clearly if $f,\ g$ share (a,k) then $f,\ g$ share (a,p) for any integer $p,\ 0\leq p<k$. Also we note that $f,\ g$ share a value a IM or CM if and only if $f,\ g$ share $(a,0)$ or (a,∞) respectively.

Definition 1.2. [10, 11] We denote by $N(r,a;f|\geq k)$ the counting function of those a-points of f whose multiplicities are not less than k, where each a-point of f is counted according to its multiplicity. $\overline{N}(r,a;f|\geq k)$ is the counting function of those a-points of f whose multiplicities are not less than k, where each a-point of f is counted only once ignoring its multiplicity.

Definition 1.3. [10, 11] We denote by $N_2(r,a;f)$ the sum $\overline{N}(r,a;f)+\overline{N}(r,a;f|\geq 2)$.

In 1959, Hayman [8] proved the following result relating to the zero distribution of a special type of complex differential polynomial.

Theorem A. If $f(z)$ is a transcendental entire function and $n\geq 2$ is a positive integer, then $f^n(z)f'(z)-a$ has infinitely many zeros, a is a nonzero constant.

In 1967, Clunie [5] proved that Theorem A is also true if $n=1$. Analogous result for meromorphic function is known as Hayman Conjecture which is as follows:
Hayman Conjecture. [8] If $f(z)$ is a transcendental meromorphic function and n is a positive integer, then $f^n(z)f'(z)-a$ has infinitely many zeros, where a is a nonzero constant.

It is to be noted that the above conjecture has been proved completely by many researchers. Hayman himself proved the conjecture for $n\geq 3$. Mues [16] proved the conjecture for $n=2$. For $n=1$ it was proved in [3, 4, 19]. In 2007, Laine and Yang [14] proved a result for the zero distribution of a complex difference polynomial. Their result is as follows:

Theorem B. If $f(z)$ is a transcendental entire function of finite order and $n\geq 2$, then $f^n(z)f(z+c)-a$ has infinitely many zeros, where $a,\ c$ are nonzero constants.

In 2020, Laine and Latreuch [13] proved the following result related to delay-differential form of Hayman Conjecture.

Theorem C. Let $f(z)$ be a transcendental meromorphic (resp. entire) function with $\rho_2(f)<1$ and $a(z)$ be a nonzero small function with respect to $f(z)$. If $n\geq k+4$ (resp. $n\geq 3$), then $f^n(z)f^{(k)}(z+c)-a(z)$ has infinitely many zeros, c is a nonzero constant.

Henceforth we denote by $\mathcal{M}$ the class of transcendental meromorphic functions and by $\mathcal{M}'$ the class of transcendental meromorphic functions of hyper-order less than 1. Similarly, we denote by $\mathcal{E}$ the class of transcendental entire functions and by $\mathcal{E}'$ the class of entire functions of hyper-order less than 1.

Recently, Gao and Liu [6] have proved a result relating to paired Hayman Conjecture for complex delay-differential polynomials. The result is as follows:

Theorem D. If one of the following conditions is satisfied:
(1) $L(h)\ =\ h^{(k)}(z)$, $n\geq k+4$ and $h\in\mathcal{M}$ or $n\geq 3$ and $h\in\mathcal{E}$;
(2) $L(h)\ =\ h(z+c)$, $n\geq 4$ and $h\in\mathcal{M}'$ or $n\geq 3$ and $h\in\mathcal{E}'$;

(3) $L(h) = h(z+c) - h(z)$, $n \geq 5$ and $h \in \mathcal{M}'$ or $n \geq 3$ and $h \in \mathcal{E}'$;
(4) $L(h) = h^{(k)}(z+c)$, $n \geq k+4$ and $h \in \mathcal{M}'$ or $n \geq 3$ and $h \in \mathcal{E}'$,
then atleast one of $f^n(z)L(g) - a(z)$ and $g^n(z)L(f) - a(z)$ has infinitely many zeros, where $a(z)$ is a nonzero small function with respect to both $f(z)$ and $g(z)$, $k \geq 1$ and c is a nonzero constant.

Let us define a polynomial $P(z)$ of degree n by

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0, \tag{1.1}$$

where $a_0, a_1, ..., a_{n-1}, a_n(\neq 0)$ are complex constants. Also, let m_1 and m_2 be the number of simple zeros and multiple zeros of $P(z)$ respectively.

Now the following question arises.

Question 1.1. What will happen if f^n is replaced by a polynomial $P(f)$ defined as in (1.1) in Theorem D?

Recently, Sahoo and Pal [17] proved the following result which answer the above question in a positive sense. Their result is as follows:

Theorem E. If one of the following conditions is satisfied:
(i) $L(h) = h^{(k)}(z)$ and $n \geq m_1 + m_2 + k + 3$ and $h \in \mathcal{M}$ or $n \geq m_1 + m_2 + 2$ and $h \in \mathcal{E}$;
(ii) $L(h) = h(z+c)$ and $n \geq m_1 + m_2 + 3$ and $h \in \mathcal{M}'$ or $n \geq m_1 + m_2 + 2$ and $h \in \mathcal{E}'$;
(iii) $L(h) = h(z+c) - h(z)$ and $n \geq m_1 + m_2 + 4$ and $h \in \mathcal{M}'$ or $n \geq m_1 + m_2 + 2$ and $h \in \mathcal{E}'$;
(iv) $L(h) = h^{(k)}(z+c)$ and $n \geq m_1 + m_2 + k + 3$ and $h \in \mathcal{M}'$ or $n \geq m_1 + m_2 + 2$ and $h \in \mathcal{E}'$,
then at least one of $P(f)L(g) - \alpha(z)$ and $P(g)L(f) - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to both $f(z)$ and $g(z)$, $k \geq 1$ and c is a nonzero constant.

In the same paper, the authors proved the following theorems on uniqueness of delay-differential polynomials.

Theorem F. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions. If $P(f)L(g)$ and $P(g)L(f)$ share a nonzero small function $\alpha(z)$ CM and one of the following conditions is satisfied:
(i) $L(h) = h^{(k)}(z)$, $n \geq 2m_1 + 4m_2 + 3k + 12$ and $f, g \in \mathcal{M}$;
(ii) $L(h) = h(z+c)$, $n \geq 2m_1 + 4m_2 + 10$ and $f, g \in \mathcal{M}'$;
(iii) $L(h) = h(z+c) - h(z)$, $n \geq 2m_1 + 4m_2 + 15$ and $f, g \in \mathcal{M}'$;
(iv) $L(h) = h^{(k)}(z+c)$, $n \geq 2m_1 + 4m_2 + 3k + 12$ and $f, g \in \mathcal{M}'$,
then either $P(f)L(g) = P(g)L(f)$ or $P(f)L(g)P(g)L(f) = \alpha^2(z)$.

Theorem G. Let $f(z)$ and $g(z)$ be two transcendental entire functions. If $P(f)L(g)$ and $P(g)L(f)$ share a nonzero small function $\alpha(z)$ CM and one of the following conditions is satisfied:
(i) $L(h) = h^{(k)}(z)$, $n \geq 2m_1 + 4m_2 + 4$ and $f, g \in \mathcal{E}$;
(ii) $L(h) = h(z+c)$, $n \geq 2m_1 + 4m_2 + 4$ and $f, g \in \mathcal{E}'$;
(iii) $L(h) = h(z+c) - h(z)$, $n \geq 2m_1 + 4m_2 + 4$ and $f, g \in \mathcal{E}'$;
(iv) $L(h) = h^{(k)}(z+c)$, $n \geq 2m_1 + 4m_2 + 4$ and $f, g \in \mathcal{E}'$,
then either $P(f)L(g) = P(g)L(f)$ or $P(f)L(g)P(g)L(f) = \alpha^2(z)$.

Now it is natural to ask the following question.

Question 1.2. What will happen if we relax the nature of sharing of small function in Theorems F and G?

Regarding above question we prove the following results:

Theorem 1.1. Let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, $n,\ k$ be two positive integers, and $\alpha(z)$ be a nonzero small function with respect to both $f(z)$ and $g(z)$ and P be a polynomial of degree n defined as in (1.1). If $P(f)L(g)$ and $P(g)L(f)$ share (α, l) ($l \geq 2$, an integer) and one of
(1) $L(h) = h^{(k)}(z)$, $n \geq 3k + 2m_1 + 4m_2 + 12$ and $f,\ g \in \mathcal{M}$;
(2) $L(h) = h(z+c)$, $n \geq 2m_1 + 4m_2 + 10$ and $f,\ g \in \mathcal{M}'$;
(3) $L(h) = h(z+c) - h(z)$, $n \geq 2m_1 + 4m_2 + 15$ and $f,\ g \in \mathcal{M}'$;
(4) $L(h) = h^{(k)}(z+c)$, $n \geq 3k + 2m_1 + 4m_2 + 12$ and $f,\ g \in \mathcal{M}'$,
holds, then either $P(f)L(g) = P(g)L(f)$ or $P(f)L(g)P(g)L(f) = \alpha^2(z)$.

Remark 1.1. Theorem 1.1 improves Theorem F by relaxing the nature of sharing of small function.

Theorem 1.2. Let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, $n,\ k$ be two positive integers, and $\alpha(z)$ be a nonzero small function with respect to both $f(z)$ and $g(z)$ and P be a polynomial of degree n defined as in (1.1). If $P(f)L(g)$ and $P(g)L(f)$ share $(\alpha, 1)$ and one of
(1) $L(h) = h^{(k)}(z)$, $n > \frac{7}{2}k + \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{25}{2}$ and $f,\ g \in \mathcal{M}$;
(2) $L(h) = h(z+c)$, $n > \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{21}{2}$ and $f,\ g \in \mathcal{M}'$;
(3) $L(h) = h(z+c) - h(z)$, $n > \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{33}{2}$ and $f,\ g \in \mathcal{M}'$;
(4) $L(h) = h^{(k)}(z+c)$, $n > \frac{7}{2}k + \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{25}{2}$ and $f,\ g \in \mathcal{M}'$,
holds, then either $P(f)L(g) = P(g)L(f)$ or $P(f)L(g)P(g)L(f) = \alpha^2(z)$.

Theorem 1.3. Let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, $n,\ k$ be two positive integers, and $\alpha(z)$ be a nonzero small function with respect to both $f(z)$ and $g(z)$ and P be a polynomial of degree n defined as in (1.1). If $P(f)L(g)$ and $P(g)L(f)$ share $(\alpha, 0)$ and one of
(1) $L(h) = h^{(k)}(z)$, $n \geq 6k + 5m_1 + 7m_2 + 21$ and $f,\ g \in \mathcal{M}$;
(2) $L(h) = h(z+c)$, $n \geq 5m_1 + 7m_2 + 19$ and $f,\ g \in \mathcal{M}'$;
(3) $L(h) = h(z+c) - h(z)$, $n \geq 5m_1 + 7m_2 + 30$ and $f,\ g \in \mathcal{M}'$;
(4) $L(h) = h^{(k)}(z+c)$, $n \geq 6k + 5m_1 + 7m_2 + 21$ and $f,\ g \in \mathcal{M}'$,
holds, then either $P(f)L(g) = P(g)L(f)$ or $P(f)L(g)P(g)L(f) = \alpha^2(z)$.

For transcendental entire functions f and g we obtain the following corollaries.

Corollary 1.1. Under the same hypothesis as in Theorem 1.1, the same conclusions hold in each of the following cases:
(1) $L(h) = h^{(k)}(z)$, $n \geq 2m_1 + 4m_2 + 4$ and $f,\ g \in \mathcal{E}$;
(2) $L(h) = h(z+c)$, $n \geq 2m_1 + 4m_2 + 4$ and $f,\ g \in \mathcal{E}'$;
(3) $L(h) = h(z+c) - h(z)$, $n \geq 2m_1 + 4m_2 + 4$ and $f,\ g \in \mathcal{E}'$;
(4) $L(h) = h^{(k)}(z+c)$, $n \geq 2m_1 + 4m_2 + 4$ and $f,\ g \in \mathcal{E}'$.

Remark 1.2. Corollary 1.1 improves Theorem G by relaxing the nature of sharing of small function.

Corollary 1.2. Under the same hypothesis as in Theorem 1.2, the same conclusions hold in each of the following cases:
(1) $L(h) = h^{(k)}(z)$, $n > \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{7}{2}$ and $f,\ g \in \mathcal{E}$;
(2) $L(h) = h(z+c)$, $n > \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{7}{2}$ and $f,\ g \in \mathcal{E}'$;
(3) $L(h) = h(z+c) - h(z)$, $n > \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{7}{2}$ and $f,\ g \in \mathcal{E}'$;
(4) $L(h) = h^{(k)}(z+c)$, $n > \frac{5}{2}m_1 + \frac{9}{2}m_2 + \frac{7}{2}$ and $f,\ g \in \mathcal{E}'$.

Corollary 1.3. Under the same hypothesis as in Theorem 1.3, the same conclusions hold in each of the following cases:
(1) $L(h) = h^{(k)}(z)$, $n \geq 5m_1 + 7m_2 + 7$ and $f,\ g \in \mathcal{E}$;
(2) $L(h) = h(z+c)$, $n \geq 5m_1 + 7m_2 + 7$ and $f,\ g \in \mathcal{E}'$;
(3) $L(h) = h(z+c) - h(z)$, $n \geq 5m_1 + 7m_2 + 7$ and $f,\ g \in \mathcal{E}'$;
(4) $L(h) = h^{(k)}(z+c)$, $n \geq 5m_1 + 7m_2 + 7$ and $f,\ g \in \mathcal{E}'$.

2. Lemmas

We consider
$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right),$$
where F and G are nonconstant meromorphic functions defined in the complex plane $\mathbb{C}$.

Now we give the following lemmas which will be needed in the sequal.

Lemma 2.1. [18] Let f be a nonconstant meromorphic function and k be a positive integer. Then
$$N\left(r, \frac{1}{f^{(k)}(z)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + k\overline{N}(r, f(z)) + S(r, f(z)). \tag{2.1}$$

Lemma 2.2. [17]
(1) If $f,\ g \in \mathcal{M}$, then
$$nT(r,f) - (k+1)T(r,g) \leq T(r, P(f)g^{(k)}) + S(r,g) \leq nT(r,f) + (k+1)T(r,g).$$
(2) If $f,\ g \in \mathcal{E}$, then
$$nT(r,f) - T(r,g) \leq T(r, P(f)g^{(k)}) + S(r,g) \leq nT(r,f) + T(r,g).$$

Lemma 2.3. [17] If $f,\ g \in \mathcal{M}'$ or $\mathcal{E}'$, then
$$nT(r,f) - T(r,g) \leq T(r, P(f)g(z+c)) + S(r,g) \leq nT(r,f) + T(r,g).$$

Lemma 2.4. [17]
(1) If $f,\ g \in \mathcal{M}'$ and $g(z+c) - g(z) \not\equiv 0$, then
$$nT(r,f) - 2T(r,g) \leq T(r, P(f)(g(z+c) - g(z))) + S(r,g) \leq nT(r,f) + 2T(r,g).$$
(2) If $f,\ g \in \mathcal{E}'$ and $g(z+c) - g(z) \not\equiv 0$, then
$$nT(r,f) - T(r,g) \leq T(r, P(f)(g(z+c) - g(z))) + S(r,g) \leq nT(r,f) + T(r,g).$$

Lemma 2.5. [17]
(1) If $f,\ g \in \mathcal{M}'$, then
$$nT(r,f) - (k+1)T(r,g) \leq T(r, P(f)g^{(k)}(z+c)) + S(r,g) \leq nT(r,f) + (k+1)T(r,g).$$

(2) If $f,\ g \in \mathcal{E}'$, then

$$nT(r,f) - T(r,g) \leq T(r, P(f)g^{(k)}(z+c)) + S(r,g) \leq nT(r,f) + T(r,g).$$

Lemma 2.6. [15] Suppose that $T : [0,\infty) \to [0,\infty)$ is a non-decreasing continuous function with $\rho_2(T) < 1$ and c is a nonzero real number. If $\delta \in (0, 1-\rho_2(T))$, then

$$T(r+c) \ = \ T(r) + o\left(\frac{T(r)}{r^{\delta}}\right).$$

Lemma 2.7. Let f be a transcendental meromorphic function with $\rho_2(f) < 1$ and c be a nonzero constant. Then the following inequalities hold:
(1) $N(r,0;f(z+c)) \leq N(r,0;f) + S(r,f)$;
(2) $N(r,\infty;f(z+c)) \leq N(r,\infty;f) + S(r,f)$;
(3) $\overline{N}(r,0;f(z+c)) \leq \overline{N}(r,0;f) + S(r,f)$;
(4) $\overline{N}(r,\infty;f(z+c)) \leq \overline{N}(r,\infty;f) + S(r,f)$.

Proof. The lemma can be proved easily by using Lemma 2.6 above. □

Lemma 2.8.
(1) $T\left(r, \frac{1}{h(z+c)-h(z)}\right) \leq 2T(r,h(z)) + S(r,h(z)),\ h \in \mathcal{M}'$
and $T\left(r, \frac{1}{h(z+c)-h(z)}\right) \leq T(r,h(z)) + S(r,h(z)),\ h \in \mathcal{E}'$.
(2) $T\left(r, \frac{1}{h^{(k)}(z+c)}\right) \leq (k+1)T(r,h(z)) + S(r,h(z)),\ h \in \mathcal{M}'$
and $T\left(r, \frac{1}{h^{(k)}(z+c)}\right) \leq T(r,h(z)) + S(r,h(z)),\ h \in \mathcal{E}'$.

Proof. The results can easily be obtained by Lemma 8.3 of [7] and the first fundamental theorem of Nevanlinna. □

Lemma 2.9. Let $f,\ g$ be two nonconstant meromorphic functions, and let α be a nonzero small function with respect to both f and g. If f and g share $(\alpha, 2)$, then one of the following holds:
(1) $T(r,f)+T(r,g) \leq 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\}+S(r,f)+S(r,g)$;
(2) $f \ = g$;
(3) $fg \ = \alpha^2$.

Proof. The proof is exactly similar to the proof of Lemma 2 [1]. □

Lemma 2.10. Let $f,\ g$ be two nonconstant meromorphic functions, and let α be a nonzero small function with respect to both f and g. If f and g share $(\alpha, 1)$ and $H \not\equiv 0$, then

$$\begin{aligned} T(r,f)+T(r,g) \ \leq \ & 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\} \\ & +\frac{1}{2}\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} \\ & +S(r,f)+S(r,g). \end{aligned} \tag{2.2}$$

Proof. Let $F \ = \ \frac{f}{\alpha}$ and $G \ = \ \frac{g}{\alpha}$. If f and g do not share any zero or pole with α, then F and G share $(1,1)$. Now

$$N_2(r,\infty;F) = N_2(r,\infty;f) + N_2(r,\infty;\alpha).$$

Also

$$T(r,f) = T(r,F.\alpha) \leq T(r,F) + S(r,f),$$

$$T(r,g) = T(r,G.\alpha) \leq T(r,G) + S(r,g).$$

Then using Lemma 2.15 of [2] we get the desired result. □

Lemma 2.11. Let f, g be two nonconstant meromorphic functions, and let α be a nonzero small function with respect to both f and g. If f and g share $(\alpha, 0)$ and $H \not\equiv 0$, then

$$\begin{aligned} T(r,f) + T(r,g) \leq\ & 2\{N_2(r,0;f) + N_2(r,0;g) + N_2(r,\infty;f) + N_2(r,\infty;g)\} \\ & +3\{\overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\} \\ & +S(r,f) + S(r,g). \end{aligned} \tag{2.3}$$

Proof. Let $F = \frac{f}{\alpha}$ and $G = \frac{g}{\alpha}$. Then F and G share $(1,0)$. Also $T(r,f) = T(r,F.\alpha) \leq T(r,F) + S(r,f)$ and $T(r,g) = T(r,G.\alpha) \leq T(r,G) + S(r,g)$. Then using Lemma 2.14 of [2] we get the result. □

3. Proof of the Theorems

Proof of Theorem 1.1. Let $F(z) = P(f)L(g)$, $G(z) = P(g)L(f)$. Then F and G share $(\alpha, 2)$. Suppose that (1) of Lemma 2.9 holds. Then

$$\begin{aligned} T(r,F) + T(r,G) \leq\ & 2\{N_2(r,0;F) + N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G)\} \\ & +S(r,F) + S(r,G). \end{aligned} \tag{3.1}$$

Part I. Let $L(h) = h^{(k)}(z)$. Then $F(z) = P(f)g^{(k)}(z)$, $G(z) = P(g)f^{(k)}(z)$. Therefore

$$N_2(r,\infty;F) \leq 2\overline{N}(r,\infty;F) \leq 2\{\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\}; \tag{3.2}$$

$$N_2(r,\infty;G) \leq 2\overline{N}(r,\infty;G) \leq 2\{\overline{N}(r,\infty;g) + \overline{N}(r,\infty;f)\}. \tag{3.3}$$

Again

$$N_2(r,0;P(f)) \leq (m_1 + 2m_2)T(r,f) + S(r,f).$$

Then using Lemma 2.1 we have

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;P(f)) + N(r,0;g^{(k)}) \\ &\leq (m_1 + 2m_2)T(r,f) + N(r,0;g) + k\overline{N}(r,\infty;g) \\ &\quad +S(r,f) + S(r,g) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} N_2(r,0;G) &\leq N_2(r,0;P(g)) + N(r,0;f^{(k)}) \\ &\leq (m_1 + 2m_2)T(r,g) + N(r,0;f) + k\overline{N}(r,\infty;f) \\ &\quad +S(r,f) + S(r,g). \end{aligned} \tag{3.5}$$

Using (1) of Lemma 2.2 and (3.2)-(3.5) in (3.1) we get

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\leq (2k+8)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+2(m_1+2m_2)\\ &\quad \{T(r,f)+T(r,g)\}+2\{N(r,0;f)+N(r,0;g)\}\\ &\quad +S(r,f)+S(r,g)\\ &\leq (2k+2m_1+4m_2+10)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

Thus we obtain

$$\{n-(3k+2m_1+4m_2+11)\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction to the fact that $n\geq 3k+2m_1+4m_2+12$ for $L(h)=h^{(k)}(z)$.

Part II. Let $L(h)=h(z+c)$. Then $F(z)=P(f)g(z+c)$, $G(z)=P(g)f(z+c)$. Using Lemma 2.7 we get

$$\begin{aligned}N_2(r,\infty;F) &\leq 2\overline{N}(r,\infty;f)+N(r,\infty;g(z+c))\\ &\leq 2\overline{N}(r,\infty;f)+N(r,\infty;g)+S(r,g);\end{aligned}\qquad(3.6)$$

$$\begin{aligned}N_2(r,\infty;G) &\leq 2\overline{N}(r,\infty;g)+N(r,\infty;f(z+c))\\ &\leq 2\overline{N}(r,\infty;g)+N(r,\infty;f)+S(r,f);\end{aligned}\qquad(3.7)$$

$$\begin{aligned}N_2(r,0;F) &\leq N_2(r,0;P(f))+N(r,0;g(z+c))\\ &\leq (m_1+2m_2)T(r,f)+N(r,0;g)+S(r,g)+S(r,f);\end{aligned}\qquad(3.8)$$

$$\begin{aligned}N_2(r,0;G) &\leq N_2(r,0;P(g))+N(r,0;f(z+c))\\ &\leq (m_1+2m_2)T(r,g)+N(r,0;f)+S(r,f)+S(r,g).\end{aligned}\qquad(3.9)$$

Using Lemma 2.3 and (3.6)-(3.9) in (3.1) we get

$$\begin{aligned}(n-1)\{T(r,f)+T(r,g)\} &\leq 6\{N(r,\infty;f)+N(r,\infty;g)\}+2\{N(r,0;f)+N(r,0;g)\}\\ &\quad +2(m_1+2m_2)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (2m_1+4m_2+8)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

Hence we have

$$\{n-(2m_1+4m_2+9)\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction to the assumption that $n\geq 2m_1+4m_2+10$ for $L(h)=h(z+c)$.

Part III. Let $L(h)=h(z+c)-h(z)$. Then $F(z)=P(f)(g(z+c)-g(z))$, $G(z)=P(g)(f(z+c)-f(z))$. Using Lemma 2.7 we get

$$\begin{aligned}N_2(r,\infty;F) &\leq 2\overline{N}(r,\infty;f)+N(r,\infty;g(z+c))+N(r,\infty;g)\\ &\leq 2\{\overline{N}(r,\infty;f)+N(r,\infty;g)\}+S(r,g);\end{aligned}\qquad(3.10)$$

$$\begin{aligned}N_2(r,\infty;G) &\leq 2\overline{N}(r,\infty;g)+N(r,\infty;f(z+c))+N(r,\infty;f)\\ &\leq 2\{\overline{N}(r,\infty;g)+N(r,\infty;f)\}+S(r,f).\end{aligned}\qquad(3.11)$$

Using (1) of Lemma 2.8 we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;P(f))+N(r,0;g(z+c)-g(z))+S(r,g)\\ &\leq (m_1+2m_2)T(r,f)+2T(r,g)+S(r,f)+S(r,g); \end{aligned} \tag{3.12}$$

$$\begin{aligned} N_2(r,0;G) &\leq N_2(r,0;P(g))+N(r,0;f(z+c)-f(z))+S(r,f)\\ &\leq (m_1+2m_2)T(r,g)+2T(r,f)+S(r,f)+S(r,g). \end{aligned} \tag{3.13}$$

Using (1) of Lemma 2.4 and (3.10)-(3.13) in (3.1) we get

$$\begin{aligned} (n-2)\{T(r,f)+T(r,g)\} &\leq 8\{N(r,\infty;f)+N(r,\infty;g)\}+(2m_1+4m_2+4)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (2m_1+4m_2+12)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g). \end{aligned}$$

Therefore

$$\{n-(2m_1+4m_2+14)\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction to the assumption that $n\geq 2m_1+4m_2+15$ for $L(h)=h(z+c)-h(z)$.

Part IV. Let $L(h)=h^{(k)}(z+c)$. Then $F(z)=P(f)g^{(k)}(z+c),\ G(z)=P(g)f^{(k)}(z+c)$. Using Lemma 2.7 we obtain

$$\begin{aligned} N_2(r,\infty;F)\leq 2\overline{N}(r,\infty;F) &\leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c))\}\\ &\leq 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+S(r,g); \end{aligned} \tag{3.14}$$

$$\begin{aligned} N_2(r,\infty;G)\leq 2\overline{N}(r,\infty;G) &\leq 2\{\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c))\}\\ &\leq 2\{\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)\}+S(r,f). \end{aligned} \tag{3.15}$$

Using (2) of Lemma 2.8 we obtain

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;P(f))+N(r,0;g^{(k)}(z+c))\\ &\leq (m_1+2m_2)T(r,f)+(k+1)T(r,g)+S(r,f)+S(r,g); \end{aligned} \tag{3.16}$$

$$\begin{aligned} N_2(r,0;G) &\leq N_2(r,0;P(g))+N(r,0;f^{(k)}(z+c))\\ &\leq (m_1+2m_2)T(r,g)+(k+1)T(r,f)+S(r,f)+S(r,g). \end{aligned} \tag{3.17}$$

Using (1) of Lemma 2.5 and (3.14)-(3.17) in (3.1) we get

$$\begin{aligned} (n-k-1)\{T(r,f)+T(r,g)\} &\leq 8\{N(r,\infty;f)+N(r,\infty;g)\}+(2m_1+4m_2+2k+2)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (2m_1+4m_2+2k+10)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g), \end{aligned}$$

i.e. $\{n-(3k+2m_1+4m_2+11)\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$,
a contradiction to the assumption that $n\geq 3k+2m_1+4m_2+12$ for $L(h)=h^{(k)}(z+c)$. Thus we have either $F=G$ or $FG=\alpha^2$. This however means that either $P(f)L(g)=P(g)L(f)$ or $P(f)L(g)P(g)L(f)=\alpha^2$. This proves Theorem 1.1. □

Proof of Theorem 1.2. Let F, G be defined as in Theorem 1.1. Then F and G share $(\alpha, 1)$. Let H be defined as in the beginning of section 2 and $H \not\equiv 0$. Then using Lemma 2.10 we have

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\{N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\} \\ &\quad +\frac{1}{2}\{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\} \\ &\quad +S(r,F)+S(r,G). \end{aligned} \tag{3.18}$$

Part I. Let $L(h)=h^{(k)}(z)$. Then

$$\overline{N}(r,\infty;F)=\overline{N}(r,\infty;P(f)g^{(k)})=\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g); \tag{3.19}$$

$$\overline{N}(r,\infty;G)=\overline{N}(r,\infty;P(g)f^{(k)})=\overline{N}(r,\infty;g)+\overline{N}(r,\infty;f). \tag{3.20}$$

Now

$$\overline{N}(r,0;P(f)) \leq (m_1+m_2)T(r,f)+S(r,f).$$

Using Lemma 2.1 we get

$$\begin{aligned} \overline{N}(r,0;F) &\leq \overline{N}(r,0;P(f))+\overline{N}(r,0;g^{(k)}) \\ &\leq (m_1+m_2)T(r,f)+N(r,0;g)+k\overline{N}(r,\infty;g) \\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{3.21}$$

Similarly,

$$\begin{aligned} \overline{N}(r,0;G) &\leq \overline{N}(r,0;P(g))+\overline{N}(r,0;f^{(k)}) \\ &\leq (m_1+m_2)T(r,g)+N(r,0;f)+k\overline{N}(r,\infty;f) \\ &\quad +S(r,f)+S(r,g). \end{aligned} \tag{3.22}$$

Therefore using (1) of Lemma 2.2, (3.2)-(3.5) and (3.19)-(3.22) in (3.18) we obtain

$$\begin{aligned} &(n-k-1)\{T(r,f)+T(r,g)\} \leq \left(\frac{5}{2}k+9\right)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\} \\ &\quad +\frac{5}{2}\{N(r,0;f)+N(r,0;g)\}+\left(\frac{5}{2}m_1+\frac{9}{2}m_2\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g) \\ &\leq \left(\frac{5}{2}k+\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{23}{2}\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g), \end{aligned}$$

i.e. $\left\{n-\left(\frac{7}{2}k+\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{25}{2}\right)\right\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$.
Since $n>\frac{7}{2}k+\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{25}{2}$, for $L(h)=h^{(k)}(z)$, we arrive at a contradiction.

Part II. Let $L(h)=h(z+c)$. Using Lemma 2.7 we obtain

$$\begin{aligned} \overline{N}(r,\infty;F) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c)) \\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,g); \end{aligned} \tag{3.23}$$

$$\begin{aligned} \overline{N}(r,\infty;G) &\leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c)) \\ &\leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+S(r,f); \end{aligned} \tag{3.24}$$

$$\begin{aligned} \overline{N}(r,0;F) &\leq \overline{N}(r,0;P(f))+\overline{N}(r,0;g(z+c)) \\ &\leq (m_1+m_2)T(r,f)+\overline{N}(r,0;g)+S(r,f)+S(r,g); \end{aligned} \tag{3.25}$$

$$\begin{aligned}\overline{N}(r,0;G) &\leq \overline{N}(r,0;P(g))+\overline{N}(r,0;f(z+c))\\ &\leq (m_1+m_2)T(r,g)+\overline{N}(r,0;f)+S(r,f)+S(r,g). \qquad (3.26)\end{aligned}$$

Therefore using Lemma 2.3, (3.6)-(3.9) and (3.23)-(3.26) in (3.18) we obtain

$$\begin{aligned}&(n-1)\{T(r,f)+T(r,g)\} \leq 7\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\frac{1}{2}(\overline{N}(r,0;f)+\overline{N}(r,0;g))\\ &\quad+\left(\frac{5}{2}m_1+\frac{9}{2}m_2\right)\{T(r,f)+T(r,g)\}+2\{N(r,0;f)+N(r,0;g)\}+S(r,f)+S(r,g)\\ &\leq \left(\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{19}{2}\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),\end{aligned}$$

i.e. $\left\{n-\left(\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{21}{2}\right)\right\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$, contradicts with the fact that $n>\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{21}{2}$, for $L(h)=h(z+c)$.

Part III. Let $L(h)=h(z+c)-h(z)$. Then using Lemma 2.7 we get

$$\begin{aligned}\overline{N}(r,\infty;F) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c))+\overline{N}(r.\infty;g)\\ &\leq \overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+S(r,g); \qquad (3.27)\end{aligned}$$

$$\begin{aligned}\overline{N}(r,\infty;G) &\leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c))+\overline{N}(r.\infty;f)\\ &\leq \overline{N}(r,\infty;g)+2\overline{N}(r,\infty;f)+S(r,f). \qquad (3.28)\end{aligned}$$

From Lemma 2.8 we have

$$\begin{aligned}\overline{N}(r,0;F) &\leq \overline{N}(r,0;P(f))+\overline{N}(r,0;g(z+c)-g(z))\\ &\leq (m_1+m_2)T(r,f)+2T(r,g)+S(r,f)+S(r,g); \qquad (3.29)\end{aligned}$$

$$\begin{aligned}\overline{N}(r,0;G) &\leq \overline{N}(r,0;P(g))+\overline{N}(r,0;f(z+c)-f(z))\\ &\leq (m_1+m_2)T(r,g)+2T(r,f)+S(r,f)+S(r,g). \qquad (3.30)\end{aligned}$$

Therefore using (1) of Lemma 2.4, (3.10)-(3.13) and (3.27)-(3.30) in (3.18) we obtain

$$\begin{aligned}(n-2)\{T(r,f)+T(r,g)\} &\leq \frac{19}{2}\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\left(\frac{5}{2}m_1+\frac{9}{2}m_2+5\right)\\ &\quad\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq \left(\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{29}{2}\right)\{T(r,f)+T(r,g)\}\\ &\quad+S(r,f)+S(r,g),\end{aligned}$$

i.e. $\left\{n-\left(\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{33}{2}\right)\right\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$, contradicts with the fact that $n>\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{33}{2}$, for $L(h)=h(z+c)-h(z)$.

Part IV. Let $L(h)=h^{(k)}(z+c)$. Then using Lemma 2.7 we get

$$\begin{aligned}\overline{N}(r,\infty;F) &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g(z+c))\\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+S(r,g); \qquad (3.31)\end{aligned}$$

$$\begin{aligned}\overline{N}(r,\infty;G) &\leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f(z+c))\\ &\leq \overline{N}(r,\infty;g)+\overline{N}(r,\infty;f)+S(r,f). \qquad (3.32)\end{aligned}$$

Using Lemma 2.8, we get

$$\begin{aligned}\overline{N}(r,0;F) &\leq \overline{N}(r,0;P(f))+\overline{N}(r,0;g^{(k)}(z+c))\\ &\leq (m_1+m_2)T(r,f)+(k+1)T(r,g)+S(r,f)+S(r,g); \quad (3.33)\end{aligned}$$

$$\begin{aligned}\overline{N}(r,0;G) &\leq \overline{N}(r,0;P(g))+\overline{N}(r,0;f^{(k)}(z+c))\\ &\leq (m_1+m_2)T(r,g)+(k+1)T(r,f)+S(r,f)+S(r,g). \quad (3.34)\end{aligned}$$

Therefore using (1) of Lemma 2.5, (3.14)-(3.17) and (3.31)-(3.34) in (3.18) we obtain

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\leq 9\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\ &+\left(\frac{5}{2}k+\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{5}{2}\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq \left(\frac{5}{2}k+\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{23}{2}\right)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),\end{aligned}$$

i.e. $\left\{n-\left(\frac{7}{2}k+\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{25}{2}\right)\right\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$, contradicts with the fact that $n>\frac{7}{2}k+\frac{5}{2}m_1+\frac{9}{2}m_2+\frac{25}{2}$, for $L(h)=h^{(k)}(z+c)$.

Thus we have $H=0$. Then $\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)=\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right)$. Integrating twice, we get

$$F = \frac{(B-1)G-(A+B-1)}{BG-(A+B)} \quad \textit{and} \quad G=\frac{(A+B)F-(A+B-1)}{BF-(B-1)},$$

where $A(\neq 0)$, B are constants. Now we consider the following two cases.

Case 1. Let $B=0$. Then $F=\frac{G-(1-A)}{A}$ and $G=A(F-\frac{A-1}{A})$.

If $A\neq 1$, then $N(r,1-A;G)=N(r,0;F)$ and $N(r,\frac{A-1}{A};F)=N(r,0;G)$. Using Nevanlinna's second fundamental theorem we have

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{A-1}{A};F\right)+\overline{N}(r,\infty;F)+S(r,F)\\ &= \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F)\end{aligned}$$

and

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}(r,1-A;G)+\overline{N}(r,\infty;G)+S(r,G)\\ &= \overline{N}(r,0;G)+\overline{N}(r,0;F)+\overline{N}(r,\infty;G)+S(r,G).\end{aligned}$$

Hence

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &+ S(r,F)+S(r,G). \quad (3.35)\end{aligned}$$

We now consider the following:

Part I. Let $L(h)=h^{(k)}(z)$. Using (1) of Lemma 2.2 and (3.19)-(3.22) in (3.35) we get

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\leq (2k+2)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\ &+2\{N(r,0;f)+N(r,0;g)\}+(2m_1+2m_2)\\ &\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (2k+2m_1+2m_2+4)\{T(r,f)+T(r,g)\}\\ &+S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(3k+2m_1+2m_2+5)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$, a contradiction to the assumption that $n \geq 3k+2m_1+2m_2+6$ for $L(h)=h^{(k)}(z)$.

Part II. Let $L(h)=h(z+c)$. Using Lemma 2.3 and (3.23)-(3.26) in (3.35) we get

$$\begin{aligned}(n-1)\{T(r,f)+T(r,g)\} \leq & 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+2\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &+(2m_1+2m_2)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ \leq&(2m_1+2m_2+4)\{T(r,f)+T(r,g)\}\\ &+S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(2m_1+2m_2+5)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$, a contradiction to the assumption that $n \geq 2m_1+2m_2+6$ for $L(h)=h(z+c)$.

Part III. Let $L(h)=h(z+c)-h(z)$. Using (1) of Lemma 2.4 and (3.27)-(3.30) in (3.35) we get

$$\begin{aligned}(n-2)\{T(r,f)+T(r,g)\} &\leq 3\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+(2m_1+2m_2+4)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (2m_1+2m_2+7)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g).\end{aligned}$$

From this we obtain

$$\{n-(2m_1+2m_2+9)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

a contradiction to the assumption that $n \geq 2m_1+2m_2+10$ for $L(h)=h(z+c)-h(z)$.

Part IV. Let $L(h)=h^{(k)}(z+c)$. Using (1) of Lemma 2.5 and (3.31)-(3.34) in (3.35) we get

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} \leq & 2\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+(2k+2m_1+2m_2+2)\\ &\times\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ \leq& (2k+2m_1+2m_2+4)\{T(r,f)+T(r,g)\}\\ &+S(r,f)+S(r,g).\end{aligned}$$

Hence we obtain

$$\{n-(3k+2m_1+2m_2+5)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

a contradiction with the assumption that $n \geq 3k+2m_1+2m_2+6$ for $L(h)=h^{(k)}(z+c)$.

If $A=1$, then $F=G$, that is $P(f)L(g) = P(g)L(f)$.

Case 2. Let $B \neq 0$. Now we consider the following three subcases.
Subcase 2.1 Assume that $B \neq 1$. Then $N(r,\frac{B-1}{B};F) = N(r,\infty;G)$ and $N(r,\frac{A+B}{B};G) = N(r,\infty;F)$. Using Nevanlinna's second fundamental theorem we obtain

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{B-1}{B};F\right)+\overline{N}(r,\infty;F)+S(r,F)\\ &= \overline{N}(r,0;F)+\overline{N}(r,\infty;G)+\overline{N}(r,\infty;F)+S(r,F)\end{aligned}$$

and

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}\left(r,\frac{A+B}{B};G\right)+\overline{N}(r,\infty;G)+S(r,G)\\ &= \overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+S(r,G). \end{aligned}$$

Therefore

$$\begin{aligned} T(r,F)+T(r,G) &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+2\{\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\}\\ &\quad +S(r,F)+S(r,G). \end{aligned} \tag{3.36}$$

We now discuss the following:

Part I. Let $L(h)=h^{(k)}(z)$. Using (1) of Lemma 2.2 and (3.19)-(3.22) in (3.36) we get

$$\begin{aligned} (n-k-1)\{T(r,f)+T(r,g)\} &\leq (k+4)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\ &\quad +\{N(r,0;f)+N(r,0;g)\}+(m_1+m_2)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (k+m_1+m_2+5)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g), \end{aligned}$$

i.e. $\{n-(2k+m_1+m_2+6)\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$,
a contradiction to the assumption that $n\geq 2k+m_1+m_2+7$ for $L(h)=h^{(k)}(z)$.

Part II. Let $L(h)=h(z+c)$. Using Lemma 2.3 and (3.23)-(3.26) in (3.36) we obtain

$$\begin{aligned} (n-1)\{T(r,f)+T(r,g)\} &\leq 4\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+\overline{N}(r,0;f)+\overline{N}(r,0;g)\\ &\quad +(m_1+m_2)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (m_1+m_2+5)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g), \end{aligned}$$

i.e. $\{n-(m_1+m_2+6)\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$,
a contradiction to the assumption that $n\geq m_1+m_2+7$ for $L(h)=h(z+c)$.

Part III. Let $L(h)=h(z+c)-h(z)$. Using (1) of Lemma 2.4 and (3.27)-(3.30) in (3.36) we obtain

$$\begin{aligned} (n-2)\{T(r,f)+T(r,g)\} &\leq 6\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+(m_1+m_2+2)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (m_1+m_2+8)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g), \end{aligned}$$

i.e. $\{n-(m_1+m_2+10)\}\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g)$,
a contradiction to the assumption that $n\geq m_1+m_2+11$ for $L(h)=h(z+c)-h(z)$.

Part IV. Let $L(h) = h^{(k)}(z+c)$. Using (1) of Lemma 2.5 and (3.31)-(3.34) in (3.36) we get

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\leq 4\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+(k+m_1+m_2+1)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (k+m_1+m_2+5)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(2k+m_1+m_2+6)\}\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g)$, a contradiction to the assumption that $n \geq 2k+m_1+m_2+7$ for $L(h) = h^{(k)}(z+c)$.

Subcase 2.2 Assume that $B = 1,\ A \neq -1$. Then $F = -\frac{A}{G-(A+1)}$ and $G = \frac{(A+1)F-A}{F}$. Hence $N(r,0;F) = N(r,A+1;G)$ and $N(r,0;G) = N(r,\frac{A}{A+1};F)$. Using Nevanlinna's second fundamental theorem we have

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}\left(r,\frac{A}{A+1};F\right)+\overline{N}(r,\infty;F)+S(r,F)\\ &= \overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+S(r,F)\end{aligned}$$

and

$$\begin{aligned}T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}(r,A+1;G)+\overline{N}(r,\infty;G)+S(r,G)\\ &= \overline{N}(r,0;G)+\overline{N}(r,0;F)+\overline{N}(r,\infty;G)+S(r,G).\end{aligned}$$

Thus

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\\ &\quad +S(r,F)+S(r,G).\end{aligned}$$

Now proceeding similarly as in Subcase 2.1, we reach at a contradiction.

Subcase 2.3 Let $B = 1,\ A = -1$. Then $FG = 1$, and hence $P(f)L(g)P(g)L(f) = \alpha^2(z)$. This proves the theorem.

Proof of Theorem 1.3. Let F, G be defined as in Theorem 1.1. Then F and G share $(\alpha,0)$. Assume that $H \not\equiv 0$. Therefore by Lemma 2.11 we have

$$\begin{aligned}T(r,F)+T(r,G) &\leq 2\{N_2(r,0;F)+N_2(r,0;G)+N_2(r,\infty;F)+N_2(r,\infty;G)\}\\ &\quad +3\{\overline{N}(r,0;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)\}\\ &\quad +S(r,F)+S(r,G). \qquad (3.37)\end{aligned}$$

Part I. Let $L(h) = h^{(k)}(z)$. Using (1) of Lemma 2.2, (3.2)-(3.5) and (3.19)-(3.22) in (3.37) we obtain

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\leq (5k+14)\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}\\ &\quad +5\{N(r,0;f)+N(r,0;g)\}+(5m_1+7m_2)\\ &\quad \times\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\leq (5k+5m_1+7m_2+19)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(6k+5m_1+7m_2+20)\}\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$, a contradiction to the assumption that $n \ge 6k+5m_1+7m_2+21$ for $L(h)=h^{(k)}(z)$.

Part II. Let $L(h)=h(z+c)$. Using Lemma 2.3, (3.6)-(3.9) and (3.23)-(3.26) in (3.37) we get

$$\begin{aligned}(n-1)\{T(r,f)+T(r,g)\} &\le 12\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+3\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\}\\ &\quad +2\{N(r,0;f)+N(r,0;g)\}+(5m_1+7m_2)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g)\\ &\le (5m_1+7m_2+17)\{T(r,f)+T(r,g)\}+S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(5m_1+7m_2+18)\}\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$, a contradiction to the assumption that $n \ge 5m_1+7m_2+19$ for $L(h)=h(z+c)$.

Part III. Let $L(h)=h(z+c)-h(z)$. Using (1) of Lemma 2.4, (3.10)-(3.13) and (3.27)-(3.30) in (3.37) we obtain

$$\begin{aligned}(n-2)\{T(r,f)+T(r,g)\} &\le 17\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+(5m_1+7m_2+10)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\le (5m_1+7m_2+27)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(5m_1+7m_2+29)\}\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$, a contradiction to the assumption that $n \ge 5m_1+7m_2+30$ for $L(h)=h(z+c)-h(z)$.

Part IV. Let $L(h)=h^{(k)}(z+c)$. Using (1) of Lemma 2.5, (3.14)-(3.17) and (3.31)-(3.34) in (3.37) we get

$$\begin{aligned}(n-k-1)\{T(r,f)+T(r,g)\} &\le 14\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+(5k+5m_1+7m_2+5)\\ &\quad \{T(r,f)+T(r,g)\}+S(r,f)+S(r,g)\\ &\le (5k+5m_1+7m_2+19)\{T(r,f)+T(r,g)\}\\ &\quad +S(r,f)+S(r,g),\end{aligned}$$

i.e. $\{n-(6k+5m_1+7m_2+20)\}\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g)$, a contradiction to the assumption that $n \ge 6k+5m_1+7m_2+21$ for $L(h)=h^{(k)}(z+c)$. Therefore $H=0$. Rest of the proof is similar to that of the case $H=0$ in Theorem 1.2. This proves the theorem. □

Proof of Corollary 1.1. Since f and g are entire functions, $L(h)$ is also an entire function. Therefore F and G are also entire functions. Hence

$$N(r,\infty;f)=0,\ N(r,\infty;g)=0,\ N(r,\infty;F)=0 \quad \textit{and} \quad N(r,\infty;G)=0. \tag{3.38}$$

Then the proof follows from the proof of Theorem 1.1. □

Proof of Corollary 1.2. Assume that $H \not\equiv 0$. As f and g are entire, using (3.38), we obtain from Lemma 2.10 that

$$\begin{aligned}T(r,F)+T(r,G) &\le 2\{N_2(r,0;F)+N_2(r,0;G)\}+\frac{1}{2}\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\}\\ &\quad +S(r,F)+S(r,G).\end{aligned}$$

Now the proof follows from the proof of Theorem 1.2. □

Proof of Corollary 1.3. Assume that $H \not\equiv 0$. As f and g are entire functions, using (3.38), we obtain from Lemma 2.11 that

$$T(r,F)+T(r,G) \leq 2\{N_2(r,0;F)+N_2(r,0;G)\}+3\{\overline{N}(r,0;F)+\overline{N}(r,0;G)\} +S(r,F)+S(r,G).$$

Now the proof follows from the proof of Theorem 1.3. □

References

[1] T.T.H. An, N.V Phuong, A lemma about meromorphic functions sharing a small function, *Comput. Methods Funct. Theory* **22** (2022), no. 2, 277-286. https://doi.org/10.1007/340315-021-00388-3.
[2] A. Banerjee, Meromorphic functions sharing one value, *Int. J. Math. Math. Sci.* **22** (2005), 3587-3598.
[3] W. Bergweiler, A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, *Rev. Mat. Iberoamericana* **11** (1995), 355-373.
[4] H.H. Chen, M.L. Fang, The value distribution of $f^n f'$, *Sci. China Ser. A* **38** (1995), 789-798.
[5] J. Clunie, On a result of Hayman, *J. London Math. Soc.* **42** (1967), 389-392.
[6] Y. Gao, K. Liu, Paired Hayman conjecture and uniqueness of complex delay-differential polynomials, *Bull. Korean Math. Soc.* **59** (2022), 155-166.
[7] R. Halburd, R. Korhonen, K. Tohge, Holomorphic curves with shift-invariant hyperplane preimages, *Trans. Amer. Math. Soc.* **366** (2014), 4267-4298.
[8] W.K. Hayman, Picard values of meromorphic functions and their derivatives, *Ann. of Math.* **70** (1959), 9-42.
[9] W.K. Hayman, *Meromorphic Functions*, Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.
[10] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, *Nagoya Math. J.* **161** (2001), 193-206.
[11] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, *Complex Variables Theory Appl.* **46** (2001), 241-253.
[12] I. Laine, *Nevanlinna theory and complex differential equations*, De Gruyter Studies in Mathematics. 15, Walter De Gruyter and Co., Berlin, 1993.
[13] I. Laine, Z. Latreuch, Zero distribution of some delay-differential polynomials, *Bull. Korean Math. Soc.* **57** (2020), 1541-1565.
[14] I. Laine, C.C. Yang, Value distribution of difference polynomials, P*roc. Japan Acad. Ser. A Math. Sci.* **83** (2007), 148-151.
[15] K. Liu, I. Laine, L.Z. Yang, *Complex Delay-Differential Equations*, Berlin, Boston: De Gruyter, 2021.
[16] E. Mues, Über ein Problem von Hayman, *Math. Z.* **164** (1979), 239-259.
[17] P. Sahoo, S. Pal, Hayman conjecture and uniqueness of some delay-differential polynomials sharing a small function, *Acta Univ. Sapientiae Math.* (Accepted for publication).
[18] C.C. Yang, H.X. Yi, *Uniqueness theory of meromorphic functions*, Mathematics and its Applications 557, Kluwer Academic Publishers Group, Dordrecht, 2003.
[19] L. Zalcman, *On some problems of Hayman*, Bar-Ilan University, 1995.

(Pulak Sahoo) Department of Mathematics, University of Kalyani, West Bengal-741235, India
E-mail address: sahoopulak1@gmail.com

(Soniya Sultana) Department of Mathematics, Berhampore Girls' College, West Bengal-742101, India
E-mail address: soniyasultana3@gmail.com