Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 31(2), 2004, Pages 91–97
ISSN: 1223-6934

On the valuation of American options

Lori Badea

Abstract. We first give theoretical results obtained in [1] and [2] concerning the existence and uniqueness of a weak solution of the valuation of American options. In our approach, the problem, of free boundary type, has a bounded domain and is described by a variational inequality. The discretized form of this variational inequality is obtained using the finite element method at each time step of the backward Euler method. We find the finite element solution by two methods, the relaxation method and the Brennan–Schwartz direct method. The main goal of this paper is to illustrate by numerical examples that our finite element method over a bounded domain is an effective method. Indeed, the numerical results we have obtained are in a very good agreement with those given in [13].

2000 Mathematics Subject Classification. 65N30, 35R35, 65K10, 60G40, 35K55.
Key words and phrases. option pricing, finite element method, free-boundary value problems, variational inequalities.

1. Introduction

It has shown in [14] and [15] that the value of the live American call is governed by a problem having the parabolic equation

$$\frac{\partial w}{\partial t} - \frac{\sigma^2}{2}x^2\frac{\partial^2 w}{\partial x^2} - (r-\delta)x\frac{\partial w}{\partial x} + rw = 0 \text{ in } D \tag{1.1}$$

with the initial condition

$$w(x,0) = w_0(x) \text{ for } 0 < x < s(0) \tag{1.2}$$

and the boundary conditions

$$\begin{array}{l} w(0,t) = 0 \\ w(s(t),t) = w_0(s(t)) \\ \frac{\partial w}{\partial x}(s(t),t) = 1 \end{array} \tag{1.3}$$

for all $0 < t \le T$. Above, T is the time at which the American call expires, $x = s(t)$, for $0 < t \le T$, is the optimal exercise curve representing the asset price above which American calls are exercised optimally, $D = \{(x,t) : 0 < x < s(t),\, 0 < t \le T\}$ and $w_0(x) = \max(x - Z, 0)$. The curve $x = s(t)$, $0 < t \le T$, is not previously known, and (1.1)–(1.3) is a free boundary problem.

In some analytical approaches, closed form solutions for both the price function and the optimal exercise curve, have given in [9], [11] and [13]. Using variational inequalities, the existence and the uniqueness of the solution are studied in [5] and [12]. Concerning the numerical methods, schemes of finite difference type are used in [6], [7], [8] and [4].

By the transformation $y = \log(x)$, problem (1.1)–(1.3) can be replaced by another one in which the coefficients of the equation are constant, and, to our knowledge, this

Received: September 20, 2004.

is the way in which this problem was approached so far. The above transformation leads to a problem with an unbounded domain which, in practical applications, have to be approached by a bounded one. This is the reason for which a new weak formulation of problem (1.1)–(1.3) has been given in [1] and [2]. In this formulation, the domain of the problem is $(0, S) \times (0, T)$, with an S satisfying only $\max_{t \in [0,T]} s(t) \leq S$.

The main goal of this paper is to show by numerical examples that the finite element method applied to this weak formulation gives very good results. In the next section we give the main results in [1] and [2]. In Section 3, we define a discretized problem of the weak formulation of the problem, which is obtained by a backward Euler method in time and uses the finite element method at each time step. Numerical applications are given in Section 4. In this section, we first apply the relaxation method to solve the finite element problem. Our numerical results match very well with those given in [13]. Then, we apply the Brennan–Schwartz direct method (given in [6] and also used in [12]) to the solution of our discretized problem. In a future paper we shall prove that the Brennan–Schwartz direct method can be applied to our problem, ie. our problem satisfy the hypotheses asked by this method, and we shall give an error estimation between the exact solution and its finite element approximation.

2. Weak formulation of the problem

In [1], it is shown that there exists a positive constant S_0 such that $s(t) \leq S_0$, $t \in [0, T]$, and the following space has been introduced

$$W(0,T) = \{v : \quad v \in L^2(0,T; H_0^1(0,S)), x^{-1}\partial_t v \in L^2(0,T; L^2(0,S))\}$$

for an $S \geq S_0$. Writing $u(x,t) = w(x,t) - w_0(x)$, we get from (1.3) that $u(s(t),t) = 0$ for any $t \in [0,T]$, and we can extend $u(x,t)$ with 0 in $(0,S) \times (0,T) \backslash D$. Over the domain $(0,S) \times (0,T)$, the following weak formulation of problem (1.1)–(1.3) has been considered for the valuation of American calls,

Find $u \in W(0,T)$ such that it satisfies a.e. in $(0,T)$ the equation

$$(x^{-1}\partial_t u, x^{-1}v) + a(u,v) + (qH(u), v)\frac{\sigma^2}{2}v(Z) \quad \forall v \in H_0^1(0,S) \tag{2.4}$$

and the initial condition

$$u(x,0) = 0. \tag{2.5}$$

In the above equation, H is the Heaviside function, and we have written

$$a(u,v) = \frac{\sigma^2}{2}(\partial_x u, \partial_x v) - (r - \delta)(\partial_x u, x^{-1}v) + r(x^{-1}u, x^{-1}v)$$

and

$$q(x) = (\delta x^{-1} - rZx^{-2})H(x - Z), \quad x \in (0,S).$$

In the following we shall also write $\tilde{Z} = Z$ when $r < \delta$ and $\tilde{Z} = rZ/\delta$ when $r \geq \delta$.

We can summarize the main results in [1] and [2] as the following theorem.

Theorem 2.1. *Problem (2.4) and (2.5) has a unique solution $u = u(x,t)$ and it satisfies*

$$x^{-1}u \in L^\infty(0,T; L^2(0,S)), \tag{2.6}$$

$$\partial_x u \in L^\infty(0,T; L^2(0,S)), \tag{2.7}$$

$$x^{-1}\partial_t u \in L^2(0,T; L^2(0,S)). \tag{2.8}$$

Moreover, $u \geq 0$ and $\partial_t u \geq 0$ in $(0, S) \times (0, T)$. If S is large enough, ie. $S \geq S_0$, for any $t \in (0, T)$ there exists $\tilde{Z} < s(t) < S$ such that $u(x, t) > 0$ for $x \in (0, s(t))$ and $u(x, t) = 0$ for $x \in [s(t), S]$, and the values of $u(x, t)$ for $t \in (0, T)$ and $x \in (0, s(t))$ do not depend on the value of S.

Now, let

$$K = \{v \in H_0^1(0, S) \ : \ v \geq 0 \ in \ (0, S)\}$$

and u be the solution of problem (2.4)–(2.5). Since $u(t, x) \in K$ for any $t \in (0, T)$, if we take a $v \in K$, we get $H(u)(v - u) \leq v - u$ in $(0, S)$. Taking into account that $q(x) \geq 0$ in $(\tilde{Z}, S)$ and since $u(x, t) > 0$ for any $(x, t) \in (0, \tilde{Z}) \times (0, T)$, we get that

$$\int_Z^S q(x) H(u)(v - u) \leq \int_Z^S q(x)(v - u),$$

for any $v \in K$. Consequently, a.e. in $(0, T)$, u will be solution of the following inequality

$$\begin{aligned} &(x^{-2}\partial_t u, v - u) + a(u, v - u) + \textstyle\int_Z^S q(x)(v - u) \geq \\ &\tfrac{\sigma^2}{2}(v(Z) - u(Z)) \quad \forall v \in K. \end{aligned} \tag{2.9}$$

Since inequality (2.9) has a unique solution, we have obtained in this way,

Theorem 2.2. *Problem (2.4)–(2.5) is equivalent with the problem of finding $u \in L^2(0, T; K)$ with $x^{-1}\partial_t u \in L^2(0, T; L^2(0, S))$ which satisfies inequality (2.9), a.e. in $(0, T)$, and the initial condition (2.5).*

3. Discretized problem

Most numerical methods ask a symmetric bilinear form in inequality (2.9). To this end, if we write

$$w_s = x^p w \tag{3.10}$$

with $p = \frac{r - \delta}{\sigma^2}$, it is easy to see that problem (1.1)–(1.3) can be written as

$$\frac{\partial w_s}{\partial t} - \frac{\sigma^2}{2} x^2 \frac{\partial^2 w_s}{\partial x^2} + \frac{1}{2}(p^2\sigma^2 + r + \delta) w_s = 0 \text{ in } D \tag{3.11}$$

with the initial condition

$$w_s(x, 0) = x^p w_0(x) \text{ for } 0 < x < s(0) \tag{3.12}$$

and the boundary conditions

$$\begin{aligned} &w_s(0, t) = 0 \\ &w_s(s(t), t) = s(t)^p w_0(s(t)) \\ &\tfrac{\partial w_s}{\partial x}(s(t), t) = \tfrac{d[x^p(x - Z)]}{dx}(s(t)) \end{aligned} \tag{3.13}$$

for all $0 < t \leq T$. Now, writing $u_s(x, t) = w_s(x, t) - x^p w_0(x)$, in a similar way to that in the previous section, we prove that the solution of problem (3.11)–(3.13) can be be found using the solution $u_s(x, t) \in K$, for almost all $t \in (0, T)$, of the problem with the initial condition

$$u_s(x, 0) = 0 \tag{3.14}$$

and satisfying the variational inequality

$$\begin{aligned} &(x^{-2}\partial_t u_s, v - u_s) + b(w, v - u_s) + \textstyle\int_Z^S x^p q(x)(v - u_s) \geq \\ &\tfrac{\sigma^2}{2} Z^p (v(Z) - u_s(Z)) \quad \forall v \in K, \end{aligned} \tag{3.15}$$

where

$$b(u,v) = \frac{\sigma^2}{2}(\partial_x u, \partial_x v) + \frac{1}{2}(p^2\sigma^2 + r + \delta)(x^{-1}u, x^{-1}v).$$

This time, the bilinear form b is symmetric.

Now, for an $n \in \mathbf{N}$, we consider a uniform partition $0 = t_0 < t_1 < \cdots < t_n = T$, $t_i = ik$ and $k = T/n$, of the interval $[0,T]$. Let V_h be the linear finite element space obtained from a uniform partition of the interval $[0,S]$, $0 = x_0 < x_1 < \cdots < x_m = S$, with $x_j = jh$ and $h = S/m$, $m \in \mathbf{N}$. Also, we define

$$K_h = \{v_h \in V_h : v_h(x_j) \geq 0,\ j = 0, \cdots, m\}$$

We notice that since V_h is the linear finite element space, then $K_h = V_h \cap K$. We consider the following discretization of problem (3.14)–(3.15).

Find $w^i_{hk} \in K_h,\ i = 1, \ldots, n$ *satisfying*

$$\begin{gathered}(x^{-2}\partial_k w^i_{hk}, v_h - w^i_{hk}) + b(w^i_{hk}, v_h - w^i_{hk}) + (x^p q(x), v_h - w^i_{hk}) \geq \\ \tfrac{\sigma^2}{2} Z^p(v_h(Z) - w^i_{hk}(Z)) \quad \forall v_h \in K_h.\end{gathered} \tag{3.16}$$

with the initial condition

$$w^0_{hk} = 0. \tag{3.17}$$

We have denoted above by $\partial_k w^i_{hk}$ the difference $(w^i_{hk} - w^{i-1}_{hk})/k$.

4. Numerical examples

First, we solve variational inequalities (3.16) by the relaxation method (see [10]). This method is the simplest variant of the Schwarz domain decomposition method ([3], for instance). In our numerical tests, we have taken over the examples given in [13], and the results are plotted in Fig 4.1. The iteration was stopped when the relative error between the solutions of two successive iterations was lesser then 0.1E-05. In these examples, we have taken $h = 0.25$, $k = 0.0025$, $S = 200$ and $Z = 100$. The constants in problem (3.16) are given in Table 4.1. We also give in this table the average number (over those 400 time steps) of necessary iterations to obtain the imposed relative error. We notice that our results match very well with the results given in [13].

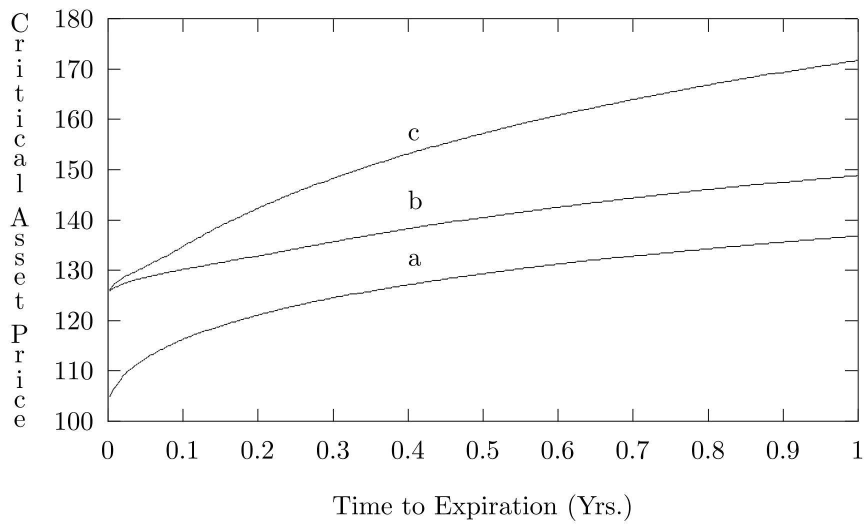

Fig. 4.1. Critical asset price as a function of time to expiration

Curve	σ	r	δ	Average nb. of iter.
a	.2	.08	.08	134.81
b	.2	.10	.08	155.38
c	.3	.10	.08	398.48

Table 4.1

In the following we show numerical results obtained by the Brennan-Schwartz direct method given in [6] and also used in [12]. In Fig. 4.2 we have plotted six critical assets prices depending on the time to expiration for the following values of the constants in problem (3.16)

Curve	σ	r	δ
a	.2	.08	.12
b	.2	.08	.10
c	.2	.08	.08
d	.2	.10	.08
e	.3	.10	.08
f	.2	.12	.08

Table 4.2

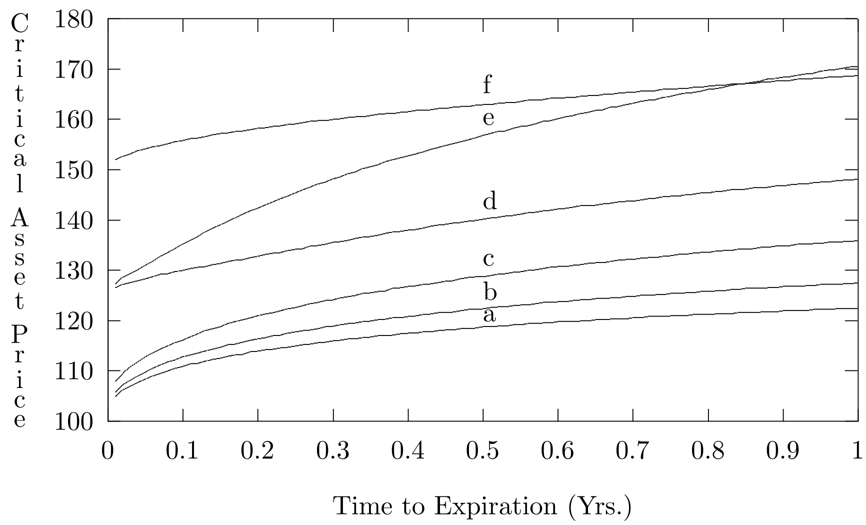

Fig. 4.2. Critical asset price as a function of time to expiration

In these examples we have taken $h = 0.2$, $k = 0.01$, $S = 200$ and $Z = 100$. The curves c, d and e are the examples a, b and c, respectively, which has been used to illustrate the relaxation method, and we see that the results obtained by the two methods are identical.

References

[1] L. Badea and J. Wang, A new formulation for the valuation of American options, I: Solution uniqueness, in *Analysis and Scientific Computing*, Eun-Jae Park and Jongwoo Lee (Eds.), Proceedings of the 19th Daewoo Workshop in Pure Mathematics, Volume 19, Part II, 1999, pp. 3-16.

[2] L. Badea and J. Wang, A new formulation for the valuation of American options, II: Solution existence, in *Analysis and Scientific Computing*, Eun-Jae Park and Jongwoo Lee (Eds.), Proceedings of the 19th Daewoo Workshop in Pure Mathematics, Volume 19, Part II, 1999, pp. 17-33.

[3] L. Badea, On the Schwarz alternating method with more than two subdomains for nonlinear monotone problems, *SIAM J. Numer. Anal.*, **28**, 1, 1991, pp. 179-204.

[4] G. Barone-Adesi and R. Whaley, Efficient analytic approximation of American option value, *Journal of Finance*, 42, 1987, pp. 301–320.

[5] A. Bensoussan and J. L. Lions, *Application des Inequations Variationnelles en Contrôle Stochastique*, Dunod, Paris, 1978.

[6] M. J. Brennan and E. S. Schwartz, The valuation of American put options, *Journal of Finances*, 32, 1977, pp. 449-462.

[7] M. J. Brennan and E. S. Schwartz, Finite difference methods and jump processes arising in the pricing of contingent claims: a synthesis, *Journal of Financial and Quantitative Analysis*, 13, 1978, pp. 461–474.

[8] M. J. Brenner, G. R. Courtadon, and M. G. Subrahmanyam, Options on the spot and options on futures, *Journal of Finance*, 40, 1985, pp. 1303–1318.

[9] P. Carr R. Jarrow and R. Myneni, Alternative characterizations of American put options, *Mathematical Finance*, 2, 1992, pp. 87–106.

[10] J. Céa and R. Glowinski, Sur des méthodes d'optimisation par relaxation, *RAIRO*, 3, 1973, pp. 5-32.

[11] S. D. Jacka, Optimal stopping and the American put, *J. Math. Fin.*, 1, 1991, pp. 1–14.

[12] P. Jaillet, D. Lamberton and B. Lapeyre, Variational inequalities and the pricing of American options, *Acta Appl. Math.*, 21, 1990, pp. 263-289.
[13] In Joo Kim, The analytic valuation of American options, *The Review of Financial Studies*, 3, 1990, pp. 547-572.
[14] R. C. Merton, Theory of rational option pricing, *Bell Journal of Economics and Management Science*, 4, 1973, pp. 141–183.
[15] H. P. McKean, Jr., Appendix: a free boundary problem for the heat equation arising from a problem in mathematical economics, *Industrial Management Review*, 6, 1965, pp. 32–39.

(Lori Badea) Institute of Mathematics of the Romanian Academy
P.O. Box 1-764, 014700 Bucharest, Romania
E-mail address: lori.badea@imar.ro