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Global Existence and Exponential Decay for a Swelling Porous-Heat System Subject to a Distributed Delay

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Abstract. In this article, we consider a swelling porous-heat system with viscous damping and distributed delay term. It's well-known in the literature that the thermal effect only lacks exponential stability. For that, we need to add another damping mechanism to stabilize exponentially the system. However, we prove, based on the energy method, that the combination of viscous damping and thermal effect provokes an exponential stability in the presence of distributed delay irrespective of the wave speeds of the system.

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1. Introduction

Swelling porous media have attracted many researchers and this is due to its prevalence in a lot of disparate fields including soil science, hydrology, forestry, geotechnical, chemical, mechanical engineering. Among the important researches that have been realized in this area is the study of the asymptotic behavior of the swelling soils that belongs to porous media theory in the case of fluid saturation. The swelling soils are caused by the chemical attraction of water where water molecules are incorporated in the clay structure in between the clay plates separating and destabilizing the mineral structure. Furthermore, the clay's particle has the properties is that it consists of lattice hydrated aluminum and magnesium silicate minerals which form a unit (particle). Thus the clay's particle is a mixture of clay platelets and absorbed water (vicinal water). For a brief descriptions concerning the details historical development/review related to the general theory of the mixtures, we refer the readers to Bedford and Drumheller [2] and Eringen [4].

The basic field equations for the theory of swelling of one-dimensional porous elastic soils are given by

$$\rho_u u_{tt} = T_x + P_1 + F_1, \ \rho_z z_{tt} = H_x - P_2 + F_2, \tag{1}$$

where the constituents u and z represent the displacement of the fluid and elastic solid material. The parameters ρ_u and ρ_z are the densities of each constituent which are assumed to be strictly positive constants. T, H are the partial tensions, F_1, F_2 are the external forces, P_1, P_2 are internal body forces associated with the dependent

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variables u, z. Here we assume that the constitutive equations of partial tensions are given as in [6] by

$$\begin{pmatrix} T \\ H \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} u_x \\ z_x \end{pmatrix},$$

where $\mathcal{M}$ is a positive definite symmetric array, i.e.,

$$\alpha_1 \alpha_3 > \alpha_2^2. \tag{2}$$

Many investigations have been realized regarding the theory of swelling porous elastic soils and among them, we cite the work of Quintanilla [10] when he considered the following problem

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} - \beta_1 T_x + \xi (z_t - u_t) - \mu_z z_{xxt} = 0, \text{ in } (0, L) \times (0, \infty), \\ \rho_u u_{tt} - \mu u_{xx} - a_2 z_{xx} - \beta_2 T_x - \xi (z_t - u_t) = 0, \text{ in } (0, L) \times (0, \infty), \\ c T_t - \beta_1 z_{xt-} \beta_2 u_{xt} - k T_{xx} = 0, \text{ in } (0, L) \times (0, \infty), \end{cases} \tag{3}$$

with the initial conditions

$$u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \; z(x, 0) = z_0(x),$$
$$z_t(x, 0) = z_1(x), \; T(x, 0) = T_0(x), \; x \in (0, L),$$

and homogeneous Dirichlet boundary conditions

$$u(x, t) = z(x, t) = T(x, t) = 0, \; x = 0, L, \; t \in (0, \infty),$$

under the following condition on the constants

$$\beta_1 = \beta_2 = 0, \; a_2^2 < a_1 \xi, \; a_3 > 0,$$

the author established an exponential stability result for the solution of (3) in the isothermal case ($\Delta T = 0$). Furthermore, in the nonisothermal case and $\beta_1, \beta_2 \neq 0$, he showed that the combination of the thermal effects with the elastic effects provokes exponential stability. In [13], Wang and Guo considered a problem of swelling of one-dimensional porous elastic soils given by

$$\begin{cases} \rho_u u_{tt} = \alpha_1 u_{xx} + \alpha_2 z_{xx}, \text{ in } (0, L) \times (0, \infty), \\ \rho_z z_{tt} = \alpha_3 z_{xx} + \alpha_2 u_{xx} + \rho_z \gamma(x) z_t, \text{ in } (0, L) \times (0, \infty), \\ u(0, t) = u_x(L, t) = z(0, t) = z_x(L, t) = 0, \; t \in (0, \infty), \\ u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \; x \in (0, L), \\ z(x, 0) = z_0(x), \; z_t(x, 0) = z_1(x), \; x \in (0, L), \end{cases}$$

where $\gamma(x)$ is an internal viscous damping function satisfying the condition

$$\int_0^L \gamma(x)\, dx > 0,$$

and they proved an exponential stability of the system by using the spectral method. We refer the reader to [3, 7] for some other interesting related results. In [12], the authors considered the following system

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \text{ in } (0, L) \times (0, \infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \gamma(t) g(u_t) = 0, \text{ in } (0, L) \times (0, \infty), \\ u(0, t) = u_x(L, t) = z(0, t) = z_x(L, t) = 0, \; t \in (0, \infty), \\ u(x, 0) = u_0(x), \; u_t(x, 0) = u_1(x), \; x \in (0, L), \\ z(x, 0) = z_0(x), \; z_t(x, 0) = z_1(x), \; x \in (0, L), \end{cases}$$

and under some properties of convex functions they showed that the dissipation given only by the nonlinear damping term $\gamma(t)\,g(u_t)$ is strong enough to provoke an exponential decay rate. In [1], Apalara considered a swelling porous-elastic system with a viscoelastic damping given by

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} = 0, \text{ in } (0,1)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \int_0^t g(t-s)\, u_{xx}(s)\, ds = 0, \text{ in } (0,1)\times(0,\infty), \\ u(0,t) = u(1,t) = z(0,t) = z(1,t) = 0, \ t\in(0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x\in(0,1), \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x), \ x\in(0,1), \end{cases}$$

where g is the kernel (also known as the relaxation function) of the finite memory term. The author established a general decay of the solution irrespective of the wave speeds of the system under some assumptions on the function g. Recently, in [11], the authors considered the following swelling problem in porous elastic soils with fluid saturation, viscous damping and a time delay term

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \xi_1 z_t + \xi_2 z_t(x,t-\tau) = 0, \text{ in } (0,L)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} = 0, \text{ in } (0,L)\times(0,\infty), \\ z(0,t) = z_x(L,t) = u(0,t) = u_x(L,t) = 0, \ t\in(0,\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x\in(0,\ L), \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x), \ x\in(0,\ L). \end{cases}$$

Under the appropriate assumption on the weight of the delay term, they established an exponential decay of the solution.

Motivated by the above mentioned work, in this paper we consider the following swelling porous-heat system with Fourier's type heat conduction and distributed delay. The system is written as

$$\begin{cases} \rho_1 u_{tt} - a_1 u_{xx} - a_2 \varphi_{xx} = 0, \\ \rho_2 \varphi_{tt} - a_3 \varphi_{xx} - a_2 u_{xx} + \gamma\theta_x + \mu_0 \varphi_t + \int_{\tau_1}^{\tau_2} \mu(s)\, \varphi_t(x,t-s)\, ds = 0, \\ \rho_3 \theta_t - \delta\theta_{xx} + \gamma\varphi_{tx} = 0, \end{cases} \tag{4}$$

where $(x,t)\in(0,1)\times(0,+\infty)$, with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x\in(0,1), \\ \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), & x\in(0,1), \\ \theta(x,0) = \theta_0(x), & x\in(0,1), \\ u(0,t) = \varphi(0,t) = 0, & \forall t\geq 0, \\ u_x(1,t) = \varphi_x(1,t) = \theta(1,t) = 0, & \forall t\geq 0, \\ \varphi_t(x,-t) = f_0(x,t), & 0 < t \leq \tau_2. \end{cases} \tag{5}$$

The coefficient μ_0 is positive constant, and $\mu : [\tau_1;\tau_2] \longrightarrow \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers satisfying $0 \leq \tau_1 < \tau_2$. Here, we prove the well-posedness and stability results for problem on the following parameter, under the assumption

$$\mu_0 > \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds. \tag{6}$$

It is well-known concerning swelling problem in porous elastic soils with fluid saturation, that found no exponential stability result when the thermal effect is only considered. For this reason, we added another mechanism damping correspond to the viscous damping in the presence of distributed delay.

Our aims in this paper are: First, by using the semigroup arguments, we show that the system is well-posed. Second, based on the multipliers method, we construct a suitable Lyapunov functional which allows us to estimate the energy of the system, we show that despite of the destructive nature of delays in general, the combination of thermal effect and the viscous damping stabilize exponentially the system under an appropriate assumption on the weight of the delay term and regardless of the wave speeds of the system.

The paper is organized as follows: In section 2, we introduce some assumptions, transformations and well-posedness result. In section 3, we use the energy method to prove our exponential stability result.

2. Well-posedness

In this section, we prove the existence and uniqueness of solutions for (4)-(5). As in [8], we introduce the new variable

$$z(x,\rho,s,t)=\varphi_t\left(x,t-\rho s\right),\ x\in(0,1),\ \rho\in(0,1),\ s\in(\tau_1,\tau_2),\ t>0. \tag{7}$$

Therefore, problem (4) takes the form

$$\begin{cases} \rho_1 u_{tt}-a_1u_{xx}-a_2\varphi_{xx}=0,\\ \rho_2\varphi_{tt}-a_3\varphi_{xx}-a_2u_{xx}+\gamma\theta_x+\mu_0\varphi_t+\int_{\tau_1}^{\tau_2}\mu\left(s\right)z(x,1,s,t)ds=0,\\ \rho_3\theta_t-\delta\theta_{xx}+\gamma\varphi_{tx}=0,\\ sz_t(x,\rho,t,s)+z_\rho(x,\rho,t,s)=0, \end{cases} \tag{8}$$

with the following initial and boundary conditions

$$\begin{cases} u\left(x,0\right)=u_0\left(x\right),\ u_t\left(x,0\right)=u_1\left(x\right), & x\in(0,1),\\ \varphi\left(x,0\right)=\varphi_0\left(x\right),\ \varphi_t\left(x,0\right)=\varphi_1\left(x\right), & x\in(0,1),\\ \theta\left(x,0\right)=\theta_0\left(x\right), & x\in(0,1),\\ z\left(x,\rho,s,0\right)=f_0\left(x,\rho s\right), & (x,\rho,s)\in(0,1)\times(0,1)\times(\tau_1,\tau_2),\\ u\left(0,t\right)=\varphi\left(0,t\right)=0, & \forall t\geq 0,\\ u_x\left(1,t\right)=\varphi_x\left(1,t\right)=\theta\left(1,t\right)=0, & \forall t\geq 0. \end{cases} \tag{9}$$

Introducing the vector function $U=(u,v,\varphi,\psi,\theta,z)^T$, where $v=u_t$ and $\psi=\varphi_t$, system (8)-(9) can be written as

$$\begin{cases} U'\left(t\right)=\mathcal{A}U\left(t\right), \quad t>0,\\ U\left(0\right)=U_0=(u_0,u_1,\varphi_0,\varphi_1,\theta_0,f_0)^T, \end{cases} \tag{10}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U=\begin{pmatrix} v\\ \dfrac{a_1}{\rho_1}u_{xx}+\dfrac{a_2}{\rho_1}\varphi_{xx}\\ \psi\\ \dfrac{a_3}{\rho_2}\varphi_{xx}+\dfrac{a_2}{\rho_2}u_{xx}-\dfrac{\gamma}{\rho_2}\theta_x-\dfrac{\mu_0}{\rho_2}\psi-\dfrac{1}{\rho_2}\int_{\tau_1}^{\tau_2}\mu\left(s\right)z(x,1,s,t)ds\\ \dfrac{\delta}{\rho_3}\theta_{xx}-\dfrac{\gamma}{\rho_3}\psi_x\\ -s^{-1}z_\rho \end{pmatrix}.$$

We consider the following space

$H_*^1\left(0,1\right)=\left\{f\in H^1\left(0,1\right);\ \ f\left(0\right)=0\right\},\ \tilde{H}^1\left(0,1\right)=\left\{f\in H^1\left(0,1\right);\ \ f\left(1\right)=0\right\},$

and

$$\begin{aligned}\mathcal{H} &= H_*^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L^2(0,1)\times L^2(0,1)\\ &\quad\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),\end{aligned}$$

be the Hilbert space equipped with the inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \rho_1\int_0^1 v\tilde{v}dx + a_1\int_0^1 u_x\widetilde{u}_x dx + \rho_2\int_0^1 \psi\tilde{\psi}dx + a_3\int_0^1 \varphi_x\widetilde{\varphi}_x dx\\ &\quad+\rho_3\int_0^1 \theta\widetilde{\theta}dx + a_2\int_0^1 (u_x\widetilde{\varphi}_x+\varphi_x\widetilde{u}_x)\,dx\\ &\quad+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\,|\mu(s)|\,z(x,\rho,s,t)\,\widetilde{z}(x,\rho,s,t)\,dsd\rho dx.\end{aligned}$$

Then, the domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{\begin{array}{c} U\in\mathcal{H}\mid u,\varphi\in H_*^2(0,1),\ v,\psi\in H_*^1(0,1),\ \theta\in H^2(0,1)\cap\tilde{H}^1(0,1)\\ z,z_\rho\in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2))\end{array}\right\},$$

where

$$H_*^2(0,1)=\left\{f\in H^2(0,1);\quad f(0)=f_x(1)=0\right\}.$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We have the following existence and uniqueness result.

Theorem 2.1. *Let $U_0\in\mathcal{H}$. Assume that $a_1a_3\neq a_2^2$ and (6) holds, then problem (10) exists a unique solution $U\in C(\mathbb{R}^+;\mathcal{H})$. Moreover, if $U_0\in D(\mathcal{A})$, then*

$$U\in C\left(\mathbb{R}^+;D(\mathcal{A})\right)\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right).$$

Proof. To obtain the above result, we need to prove that $\mathcal{A}:\,D(\mathcal{A})\to\mathcal{H}$ is a maximal monotone operator. For this purpose, we need the following two steps: $\mathcal{A}$ is dissipative and $Id-\mathcal{A}$ is surjective.

Step 1. $\mathcal{A}$ is dissipative.

For any $U=(u,v,\varphi,\psi,\theta,z)^T\in D(\mathcal{A})$, by using the inner product and integrating by parts, we obtain

$$\begin{aligned}\langle\mathcal{A}U,U\rangle_{\mathcal{H}} &= -\left(\mu_0-\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\right)\int_0^1\psi^2dx-\int_0^1\psi\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,s,t)dsdx\\ &\quad-\delta\int_0^1\theta_x^2dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(s)|\,z^2(x,1,s,t)\,dsdx.\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}&-\int_0^1\psi\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,s,t)dsdx\\ &\leq\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\int_0^1\psi^2dx+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu(s)|\,z^2(x,1,s,t)\,dsdx.\end{aligned}$$

Consequently,

$$\langle\mathcal{A}U,U\rangle_{\mathcal{H}}\leq-\left(\mu_0-\int_{\tau_1}^{\tau_2}|\mu(s)|\,ds\right)\int_0^1\psi^2dx-\delta\int_0^1\theta_x^2dx\leq 0.$$

Hence, $\mathcal{A}$ is a dissipative operator.

Step 2. $Id-\mathcal{A}$ is surjective.

To prove that the operator $Id-\mathcal{A}$ is surjective, that is, for any $F=(f_1,...,f_6)^T\in\mathcal{H}$, there exists $U=(u,u_t,\varphi,\varphi_t,\theta,z)^T\in D(\mathcal{A})$ satisfying

$$(Id-\mathcal{A})\,U=F, \tag{11}$$

which is equivalent to

$$\begin{cases} u-v=f_1, \\ \rho_1 v-a_1u_{xx}-a_2\varphi_{xx}=\rho_1 f_2, \\ \varphi-\psi=f_3, \\ (\rho_2+\mu_0)\,\psi-a_3\varphi_{xx}-a_2u_{xx}+\gamma\theta_x+\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,s,t)ds=\rho_2 f_4, \\ \rho_3\theta-\delta\theta_{xx}+\gamma\psi_x=\rho_3 f_5, \\ sz+z_\rho=sf_6. \end{cases} \tag{12}$$

We note that the fourth equation in (12) with $z(x,0,s,t)=\varphi_t(x,t)$, has a unique solution

$$z(x,\rho,s,t)=\varphi(x)e^{-\rho s}-f_3(x)\,e^{-\rho s}+se^{-\rho s}\int_0^\rho f_4(x,\tau,s)e^{\tau s}d\tau. \tag{13}$$

Clearly, $z,z_\rho\in L^2((0,L)\times(0,1)\times(\tau_1,\tau_2))$.

By (12) and (13), the functions (u,φ,θ) satisfy the equations

$$\begin{cases} \rho_1 u-a_1u_{xx}-a_2\varphi_{xx}=g_1, \\ \eta\varphi-a_3\varphi_{xx}-a_2u_{xx}+\gamma\theta_x=g_2, \\ \rho_3\theta-\delta\theta_{xx}+\gamma\varphi_x=g_3, \end{cases} \tag{14}$$

where

$$\begin{aligned} &\eta=\rho_2+\mu_0+\textstyle\int_{\tau_1}^{\tau_2}\mu(s)\,e^{-s}ds, \\ &g_1=\rho_1 f_1+\rho_1 f_2, \\ &g_2=\eta f_3+\rho_2 f_4-\textstyle\int_{\tau_1}^{\tau_2}se^{-s}\mu(s)\int_0^1 f_4(x,\tau,s)e^{\tau s}d\tau ds, \\ &g_3=\gamma f_{3x}+\rho_3 f_5. \end{aligned}$$

To solve (14), we consider the following variational formulation

$$\mathcal{B}\left((u,\varphi,\theta)^T,(\bar u,\bar\varphi,\bar\theta)^T\right)=\mathcal{G}\left(\bar u,\bar\varphi,\bar\theta\right)^T, \tag{15}$$

where

$$\mathcal{B}:\left[H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1)\right]^2\longrightarrow\mathbb{R},$$

is the bilinear form given by

$$\begin{aligned} &\mathcal{B}\left((u,\varphi,\theta)^T,(\bar u,\bar\varphi,\bar\theta)^T\right) \\ &\quad=\rho_1\int_0^1 u\bar u dx+a_1\int_0^1 u_x\bar u_x dx+a_2\int_0^1\varphi_x\bar u_x dx+\eta\int_0^1\varphi\bar\varphi dx+a_3\int_0^1\varphi_x\bar\varphi_x dx \\ &\qquad+a_2\int_0^1 u_x\bar\varphi_x dx+\gamma\int_0^1\theta_x\bar\varphi dx+\rho_3\int_0^1\theta\bar\theta dx+\delta\int_0^1\theta_x\bar\theta_x dx+\gamma\int_0^1\varphi_x\bar\theta dx, \end{aligned}$$

and

$$\mathcal{G}:\left[H_*^1(0,1)\times H_*^1(0,1)\times L^2(0,1)\right]\longrightarrow\mathbb{R},$$

is the linear form defined by

$$\mathcal{G}\left(\bar u,\bar\varphi,\bar\theta\right)^T=\int_0^1 g_1\bar u dx+\int_0^1 g_2\bar\varphi dx+\int_0^1 g_3\bar\theta dx.$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{G}$ is continuous. Applying the Lax-Milgram theorem, we deduce that problem (15) admits a unique solution $(u,\varphi,\theta) \in H_*^1(0,1) \times H_*^1(0,1) \times L^2(0,1)$ for all $\left(\bar{u},\bar{\varphi},\bar{\theta}\right) \in H_*^1(0,1) \times H_*^1(0,1) \times L^2(0,1)$. The application of the classical regularity theory, it follows from (14) that

$$u,\varphi \in H_*^2(0,1)\,,\ \ \theta \in H^2(0,1) \cap \tilde{H}^1(0,1)\,.$$

Hence, the operator $Id - \mathcal{A}$ is surjective. Consequently, the result of Theorem 2.1 follows from Lumer-Phillips theorem (see [5, 9]). □

3. Exponential stability

In this section, we prove the exponential decay for problem (8)-(9). It will be achieved by using the perturbed energy method. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \rho_2 \varphi_t^2 + a_1 u_x^2 + a_3 \varphi_x^2 + 2a_2 \varphi_x u_x + \rho_3 \theta^2\right] dx \\ &\quad + \frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\, z^2(x,\rho,z,t)\, ds\,d\rho\,dx. \end{aligned} \tag{16}$$

We have the following exponentially stable result.

Theorem 3.1. *Let (u,φ,θ,z) be the solution of (8)-(9). Assume that $a_1 a_3 - a_2^2 > 0$ and (6) holds. Then there exist two positive constants k_0 and k_1, such that*

$$E(t) \le k_0 e^{-k_1 t}, \ \ \forall t \ge 0. \tag{17}$$

To prove our this result, we will state and prove some useful lemmas in advance.

Lemma 3.2. *Let (u,φ,θ,z) be the solution of (8)-(9) and (6) holds. Then, the energy functional, defined by equation (16), satisfies*

$$E'(t) \le -\left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds\right)\int_0^1 \varphi_t^2 dx - \delta \int_0^1 \theta_x^2 dx \le 0. \tag{18}$$

Proof. Multiplying $(8)_1$ by u_t, $(8)_2$ by φ_t and $(8)_3$ by θ, integrating over $(0,1)$ with respect to x, multiplying equation $(8)_4$ by $|\mu(s)|\,z$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$ with respect to x, ρ and s, summing them up, we obtain

$$\begin{aligned} E'(t) &= -\mu_0 \int_0^1 \varphi_t^2 dx - \delta \int_0^1 \theta_x^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,s,t) ds dx \\ &\quad - \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds dx + \frac{1}{2}\int_0^1 \varphi_t^2 \int_{\tau_1}^{\tau_2} |\mu(s)|\, ds dx. \end{aligned} \tag{19}$$

Using Young's inequality, we obtain

$$\begin{aligned} &- \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,s,t) ds dx \\ &\le \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu(s)|\, ds \int_0^1 \varphi_t^2 dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, ds dx. \end{aligned} \tag{20}$$

Substitution of (20) into (19), and using (6) give (18), which concludes the proof. □

Next, in order to construct a Lyapunov functional equivalent to the energy, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.3. *Let (u, φ, θ, z) be the solution of (8)-(9). Then the functional*

$$L_1(t) = -\rho_1 \int_0^1 u u_t dx,$$

satisfies

$$L_1'(t) \leq -\rho_1 \int_0^1 u_t^2 dx + \frac{3a_1}{2} \int_0^1 u_x^2 dx + \frac{a_2^2}{2a_1} \int_0^1 \varphi_x^2 dx. \tag{21}$$

Proof. By differentiating $L_1(t)$ with respect to t, using $(8)_1$ and integrating by parts, we obtain

$$L_1'(t) = -\rho_1 \int_0^1 u_t^2 dx + a_1 \int_0^1 u_x^2 dx + a_2 \int_0^1 \varphi_x u_x dx,$$

then, by Young's inequality, we obtain the result. □

Lemma 3.4. *Let (u, φ, θ, z) be the solution of (8)-(9). Then the functional*

$$L_2(t) = a_1\rho_2 \int_0^1 \varphi\varphi_t dx - a_2\rho_1 \int_0^1 \varphi u_t dx,$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$\begin{aligned} L_2'(t) \quad \leq \quad & -\frac{a}{4} \int_0^1 \varphi_x^2 dx + C_1(\varepsilon_1) \int_0^1 \varphi_t^2 dx + \frac{\gamma^2 a_1^2}{a} \int_0^1 \theta^2 dx \\ & + \varepsilon_1 \int_0^1 u_t^2 dx + \frac{a_1^2 \mu_0}{a} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x, 1, s, t) ds dx, \end{aligned} \tag{22}$$

where

$$a = a_1 a_3 - a_2^2 > 0, \qquad C_1(\varepsilon_1) = a_1\rho_2 + \frac{\mu_0^2 a_1^2}{a} + \frac{a_2^2 \rho_1^2}{4\varepsilon_1}.$$

Proof. By differentiating $L_2(t)$ with respect to t, using the equations $(8)_1$ and $(8)_2$, and integrating by parts, we obtain

$$\begin{aligned} L_2'(t) \quad = \quad & -a \int_0^1 \varphi_x^2 dx + a_1\rho_2 \int_0^1 \varphi_t^2 dx + \gamma a_1 \int_0^1 \theta\varphi_x dx - \mu_0 a_1 \int_0^1 \varphi_t \varphi dx \\ & - a_2\rho_1 \int_0^1 \varphi_t u_t dx - a_1 \int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu(s) z(x, 1, s, t) ds dx, \end{aligned} \tag{23}$$

where $a = a_3 a_1 - a_2^2 > 0$. Using Young's and Poincaré inequalities, and (6), we get for $\varepsilon_1 > 0$,

$$\gamma a_1 \int_0^1 \theta\varphi_x dx \leq \frac{\gamma^2 a_1^2}{a} \int_0^1 \theta^2 dx + \frac{a}{4} \int_0^1 \varphi_x^2 dx, \tag{24}$$

$$-\mu_0 a_1 \int_0^1 \varphi_t \varphi dx \leq \frac{\mu_0^2 a_1^2}{a} \int_0^1 \varphi_t^2 dx + \frac{a}{4} \int_0^1 \varphi_x^2 dx, \tag{25}$$

$$-a_2\rho_1 \int_0^1 \varphi_t u_t dx \leq \frac{a_2^2 \rho_1^2}{4\varepsilon_1} \int_0^1 \varphi_t^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx, \tag{26}$$

$$\begin{aligned}&-a_1\int_0^1 \varphi \int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,s,t)dsdx \\ \leq\ & \frac{a}{4}\int_0^1 \varphi_x^2 dx + \frac{a_1^2\mu_0}{a}\int_0^1\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, dsdx. \qquad (27)\end{aligned}$$

Estimate (22) follows by substituting (24)-(27) into (23). □

Lemma 3.5. *Let (u,φ,θ,z) be the solution of (8)-(9) and (6) holds. Then the functional*

$$L_3(t) = \frac{a_1\rho_2}{a_2}\int_0^1 \varphi_t u dx - \frac{a_3\rho_1}{a_2}\int_0^1 u_t\varphi dx,$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned}L_3'(t) \ \leq\ & -\frac{a_1}{4}\int_0^1 u_x^2 dx + a_3\int_0^1 \varphi_x^2 dx + \frac{a_1\gamma^2}{a_2^2}\int_0^1 \theta^2 dx + \varepsilon_2\int_0^1 u_t^2 dx \\ & + C_2(\varepsilon_2)\, a_3\int_0^1 \varphi_t^2 dx + \frac{a_1\mu_0}{a_2^2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx, \qquad (28)\end{aligned}$$

where

$$C_2(\varepsilon_2) = \frac{a_1\mu_0^2}{a_2^2} + \frac{1}{4\varepsilon_2}\left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)^2.$$

Proof. By differentiating $L_3(t)$ with respect to t, using the equations $(8)_1$ and $(8)_2$, and integrating by parts, we obtain

$$\begin{aligned}L_3'(t) \ =\ & -a_1\int_0^1 u_x^2 dx + a_3\int_0^1 \varphi_x^2 dx + \frac{a_1\gamma}{a_2}\int_0^1 \theta u_x dx - \frac{a_1\mu_0}{a_2}\int_0^1 \varphi_t u dx \\ & + \frac{a_1\rho_2 - a_3\rho_1}{a_2}\int_0^1 \varphi_t u_t dx - \frac{a_1}{a_2}\int_0^1 u\int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,s,t)dsdx. \qquad (29)\end{aligned}$$

Using Young's and Poincaré inequalities, and (6), we get for $\varepsilon_2 > 0$,

$$\frac{a_1\gamma}{a_2}\int_0^1 \theta u_x dx \leq \frac{a_1\gamma^2}{a_2^2}\int_0^1 \theta^2 dx + \frac{a_1}{4}\int_0^1 u_x^2 dx, \qquad (30)$$

$$-\frac{a_1\mu_0}{a_2}\int_0^1 \varphi_t u dx \leq \frac{a_1\mu_0^2}{a_2^2}\int_0^1 \varphi_t^2 dx + \frac{a_1}{4}\int_0^1 u_x^2 dx, \qquad (31)$$

$$\frac{a_1\rho_2 - a_3\rho_1}{a_2}\int_0^1 \varphi_t u_t dx \leq \frac{1}{4\varepsilon_2}\left(\frac{a_1\rho_2 - a_3\rho_1}{a_2}\right)^2\int_0^1 \varphi_t^2 dx + \varepsilon_2\int_0^1 u_t^2 dx, \qquad (32)$$

$$\begin{aligned}&-\frac{a_1}{a_2}\int_0^1 u\int_{\tau_1}^{\tau_2} \mu(s)\, z(x,1,s,t)dsdx \\ \leq\ & \frac{a_1}{4}\int_0^1 u_x^2 dx + \frac{a_1\mu_0}{a_2^2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)\, dsdx. \qquad (33)\end{aligned}$$

Estimate (28) follows by substituting (30)-(33) into (29). □

Lemma 3.6. *Let (u,φ,θ,z) be the solution of (8)-(9) and (6) holds. Then the functional*

$$L_4(t) = \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx,$$

satisfies, for some positive constant n_1, the estimate

$$\begin{aligned} L_4'(t) &\leq -n_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx \\ &\quad -n_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\, z^2(x,1,s,t)dsdx + \mu_0 \int_0^1 \varphi_t^2 dx. \end{aligned} \tag{34}$$

Proof. By differentiating $L_4(t)$ with respect to t, and using the equation $(8)_4$, we obtain

$$\begin{aligned} L_4'(t) &= -2\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\mu(s)|\, z(x,\rho,s,t)z_\rho(x,\rho,s,t)dsd\rho dx \\ &= -\frac{d}{d\rho}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx \\ &\quad -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx \\ &= -\int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)|\,[e^{-s}z^2(x,1,s,t) - z^2(x,0,s,t)]dsdx \\ &\quad -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx. \end{aligned}$$

Using the fact that $z(x,0,s,t) = \varphi_t$ and $e^{-s} \leq e^{-s\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} L_4'(t) &\leq -\int_0^1 \int_{\tau_1}^{\tau_2} e^{-s}\,|\mu(s)|\, z^2(x,1,s,t)\,dsdx + \int_{\tau_1}^{\tau_2} |\mu(s)|\,ds \int_0^1 \varphi_t^2 dx \\ &\quad -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu(s)|\, z^2(x,\rho,s,t)dsd\rho dx. \end{aligned}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$.
Finally, setting $n_1 = e^{-\tau_2}$and recalling (6), we obtain (34). □

Now, we prove our main result in this section.

Proof. (**of Theorem 3.1**) We define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) := NE(t) + \sum_{i=1}^{4} N_i L_i(t),$$

where N, N_1, N_2, N_3, N_4 are positive real numbers to be chosen appropriately later.
By differentiating $\mathcal{L}(t)$, exploiting (18), (21), (22), (28) and (34), and using

$$\int_0^1 \theta^2 dx \leq \int_0^1 \theta_x^2 dx,$$

we arrive at

$$\begin{aligned}
\mathcal{L}'(t) \leq & -[\rho_1 N_1 - \varepsilon_1 N_2 - \varepsilon_2 N_3] \int_0^1 u_t^2 dx - \left[\frac{a_1}{4} N_3 - \frac{3a_1}{2} N_1\right] \int_0^1 u_x^2 dx \\
& - \left[\left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)| ds\right) N - C_1(\varepsilon_1) N_2 - C_2(\varepsilon_2) a_3 N_3 - \mu_0 N_4\right] \int_0^1 \varphi_t^2 dx \\
& - \left[\frac{a}{4} N_2 - \frac{a_2^2}{2a_1} N_1 - a_3 N_3\right] \int_0^1 \varphi_x^2 dx - \left[\delta N - \frac{\gamma^2 a_1^2}{a} N_2 - \frac{a_1 \gamma^2}{a_2^2} N_3\right] \int_0^1 \theta_x^2 dx \\
& - n_1 N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu(s)| z^2(x, \rho, s, t) ds d\rho dx \\
& - \left[n_1 N_4 - \frac{a_1^2 \mu_0}{a} N_2 - \frac{a_1 \mu_0}{a_2^2} N_3\right] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu(s)| z^2(x, 1, s, t) ds dx.
\end{aligned}$$

At this point, taking

$$\varepsilon_1 = \frac{1}{N_2}, \quad \varepsilon_2 = \frac{1}{N_3}.$$

We take N_1 large enough so that $N_1 > \frac{2}{\rho_1}$. After that, we choose N_3 large enough, so that $\frac{a_1}{4} N_3 - \frac{3a_1}{2} N_1 > 0$. Then, we choose N_2 large enough, so that $\frac{a}{4} N_2 - \frac{a_2^2}{2a_1} N_1 - a_3 N_3 > 0$. Next, we choose N_4 large enough, so that $n_1 N_4 - \frac{a_1^2 \mu_0}{a} N_2 - \frac{a_1 \mu_0}{a_2^2} N_3 > 0$.

Finally, we choose N so large such that

$$\left(\mu_0 - \int_{\tau_1}^{\tau_2} |\mu(s)| ds\right) N - C_1(\varepsilon_1) N_2 - C_2(\varepsilon_2) a_3 N_3 - \mu_0 N_4 > 0,$$

$$\delta N - \frac{\gamma^2 a_1^2}{a} N_2 - \frac{a_1 \gamma^2}{a_2^2} N_3 > 0.$$

Consequently, from the above, we deduce that there exist a positive constant c_0 such that

$$\mathcal{L}'(t) \leq -c_0 E(t). \tag{35}$$

On the hand, it is not hard to see that $\mathcal{L}(t) \sim E(t)$, i.e. there exist two positive constants c_1 and c_2 such that

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \quad \forall t \geq 0. \tag{36}$$

Combining (35) and (36), we obtain that

$$\mathcal{L}'(t) \leq -k_1 \mathcal{L}(t), \quad \forall t \geq 0, \tag{37}$$

where $k_1 = \frac{c_0}{c_2}$. A simple integration of (37) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-k_1 t}, \quad \forall t \geq 0.$$

It gives the desired result Theorem 3.1 when combined with the equivalence of $\mathcal{L}(t)$ and $E(t)$. $\square$

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