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Existence results for a class of perturbed impulsive Hamiltonian systems

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Abstract. In this paper, we study the multiplicity results for a second-order Hamiltonian system generated by impulsive effects. Our approach is based on variational methods and critical point theory.

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1. Introduction

The aim of this paper is to investigate the multiplicity of solutions for the impulsive Hamiltonian system

$$\begin{cases} -u^{''}(t) - q(t)u^{'}(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) + \mu \nabla G(t, u(t)) + \nabla H(u(t)), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a.e.\ t \in [0, T], \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)),\ i = 1, 2, ..., N, j = 1, 2, ..., p, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{1}$$

where $u = (u_1, u_2, ..., u_N)^{\mathfrak{T}}$ (transpose), $N \geq 1$, $p > 1$, $T > 0$, $\lambda > 0$ and $\mu \geq 0$ are parameters, $q \in L^1([0, T]; \mathbb{R})$, $Q(t) = \int_0^t q(s)ds$ for all $t \in [0, T]$, $Q(T) = 0$, $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$, $A : [0, T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0, T]$ to the set of $N \times N$ symmetric matrices, and $\triangle(u_i'(t_j)) = u_i'(t_j^+) - u_i'(t_j^-) = \lim_{t\to t_j^+} u_i'(t) - \lim_{t\to t_j^-} u_i'(t)$. Here, each $I_{ij} : \mathbb{R} \to \mathbb{R}$ satisfies

$$|I_{ij}(s)| \leq L_{ij}|s|,$$

for every $s \in \mathbb{R}$ and $i = 1, 2, ..., N$, $j = 1, 2, ..., p$; $F, G : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to t for all $u \in \mathbb{R}^N$, continuously differentiable in u for almost every $t \in [0, T]$, and satisfy the summability condition

$$\sup_{|x|\leq\alpha} \max\{|F(., x)|, |\nabla F(., x)|, |G(., x)|, |\nabla G(., x)|\} \in L^1([0, T]), \tag{2}$$

for any $\alpha > 0$; $F(t, \mathbf{0}) = G(t, \mathbf{0}) = 0$ for all $t \in [0, T]$, where $\mathbf{0} = (0, ..., 0)$; and $H : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function for which there is a constant $L > 0$ such that

$$|H(x)| \leq L|x|^2,$$

for every $x \in \mathbb{R}^N$. Note that if $\nabla F, \nabla G : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ are continuous, then clearly condition (2) is satisfied.

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As a special case of dynamical systems, Hamiltonian systems are very important in the study of areas such as fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, and many others. It is now recognized that the theory of Hamiltonian systems is a natural framework for modeling many natural phenomena. For background, theory, and applications of Hamiltonian systems, we refer the reader to [16, 27, 29, 34]. Inspired by the monographs [27, 30], the existence and multiplicity of periodic solutions for Hamiltonian systems using variational methods have been investigated in many papers (see, for example, [2, 5, 7, 8, 11, 13, 14, 15, 17, 18, 21, 22, 26, 35, 36, 37, 39, 40, 41, 42, 43, 44, 46, 47] and the references contained therein). For example, in [36], Tang and Wu obtained existence theorems for periodic solutions of a class of unbounded, non-autonomous, non-convex, sub-quadratic, second-order Hamiltonian systems by using minimax methods in critical point theory. Cordaro [14] established a multiplicity result for an eigenvalue problem related to second-order Hamiltonian systems, and proved the existence of an open interval of positive eigenvalues in which the problem admits three distinct periodic solutions. Faraci [18] studied multiplicity of solutions of a second-order non-autonomous system. He and Wu [22] showed the existence of nontrivial T-periodic solutions to second-order Hamiltonian systems using a mountain pass theorem and a local linking theorem, while Zhang and Tang [43] obtained some new results on T-periodic solutions for the same second-order Hamiltonian systems under weaker assumptions thus generalizing the corresponding results in [22].

In [7], Bonanno and Livrea proved the existence of infinitely many periodic solutions for a class of second-order Hamiltonian systems assuming an oscillating behavior of the nonlinear term. Moreover, they obtained multiplicity of periodic solutions for the system with a coercive potential and also did so in the non-coercive case. Gu and An [21] and Zhang and Liu [42] used a variant of the fountain theorem to show the existence of infinitely many periodic solutions of a class of super-quadratic non-autonomous second-order Hamiltonian systems. Zhang and Zhou [47] studied a class of non-autonomous second-order Hamiltonian system and obtained new existence theorems by the least action principle.

On the other hand, impulsive differential equation is one of the main tools to study the dynamics of processes in which sudden changes occur. In the last few years, variational methods have been used to determine the existence of solutions for impulsive differential equations possessing variational structures under certain boundary conditions (see, for instance, [3, 4, 19, 25, 38, 45] and the references therein for detailed discussions).

Recently, problems of second-order impulsive Hamiltonian systems have been studied by a number of authors. For the background, theory and applications of impulsive Hamiltonian systems, we refer the interest readers to [12, 32, 33, 43] and the references therein. For example, Zhou and Li in [43] by means of some critical point theorems, established some sufficient conditions for the existence of solutions for the second-order Hamiltonian systems with impulsive effects. Sun et al. in [32] based on variational methods, studied the existence of infinitely many solutions for a class of second-order impulsive Hamiltonian systems. Chen and He in [12] by using a variational method and some critical points theorems of Ricceri, studied the existence of three solutions for second-order impulsive Hamiltonian systems.

In [20, 24] the authors, studied the existence of infinitely many periodic solutions and three solutions for second-order impulsive Hamiltonian systems of problem

$$\begin{cases} -u^{''}(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) + \mu \nabla G(t, u(t)) + \nabla H(u(t)), \ a.e. \ t \in (0, T), \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)), \ i = 1, 2, ..., N, j = 1, 2, ..., p, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

Also, in [23] the authors, by using a version of Ricceri's variational principle [31], employed a critical point theorem for differentiable functionals of problem

$$\begin{cases} -u^{''}(t) - q(t)u^{'}(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) + \nabla H(u(t)), \ a.e. \ t \in (0, T), \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)), \ i = 1, 2, ..., N, j = 1, 2, ..., p, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases}$$

The paper organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 proofs to our abstract results. Finally, Section 4 is devoted to some concrete applications.

2. Preliminaries

Our main tools are the following theorems.

Theorem 2.1. [31, Theorem 2.5] *Let X be a real Banach space, $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous, For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty, r))} \frac{\left(\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_X \Phi$ and every $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional*

$$I_\lambda = \Phi - \lambda \Psi,$$

to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) *If $\gamma < +\infty$, then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  (1) *I_λ possesses a global minimum, or*
  (2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that* $\lim_{n \to +\infty} \Phi(u_n) = +\infty$.

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*
  (1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
  (2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ, that converges weakly to a global minimum of Φ.*

For a given non-empty set X and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the following functions

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(r_1, r_2)} \frac{\sup_{u \in \Phi^{-1}(r_1, r_2)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},$$

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(r_1, r_2)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{\Phi(v) - r_1},$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$.

Theorem 2.2. [6, Theorem 5.1] *Let X be a real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2).$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in (\frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)})$ there is $u_{0,\lambda} \in \Phi^{-1}(r_1, r_2)$ such that $I_\lambda(u_{0,\lambda}) \leq I_\lambda(u)$ $\forall u \in \Phi^{-1}(r_1, r_2)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.3. [9, Theorem 2.6] *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.*
Assume that there exist $r > 0$ and $\overline{v} \in X$, with $r < \Phi(\overline{v})$ such that

(*d*) $\dfrac{\sup_{\Phi(u) \leq r} \Psi(u)}{r} < \dfrac{\Psi(\overline{v})}{\Phi(\overline{v})}$,

(*e*) *for each* $\lambda \in \Lambda_r := \left] \dfrac{\Phi(\overline{v})}{\Psi(\overline{v})}, \dfrac{r}{\sup_{\Phi(u) \leq r} \Psi(u)} \right[$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

We assume that A satisfies the following conditions:
(B_1) $A(t) = (a_{ij}(t))$ is a symmetric matrix with $a_{ij} \in L^\infty[0, T]$ for any $t \in [0, T]$, $i, j = 1, ..., N$,
(B_2) there exists $\ell > 0$ such that $(A(t)x, x) \geq \ell |x|^2$ for any $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$ where $(.,.)$ denotes the inner product in $\mathbb{R}^N$.
Here, we recall some basic concepts that will be used in what follows. Let

$$E = \left\{ u : [0, T] \to \mathbb{R}^N : u \text{ is absolutely continuous}, u(0) = u(T), u' \in L^2([0, T], \mathbb{R}^N) \right\},$$

with the inner product

$$\langle u, v \rangle_E = \int_0^T [(u^{'}(t), v^{'}(t)) + (u(t), v(t))] dt$$

for all $u, v \in E$ where $(.,.)$ denotes the inner product in $\mathbb{R}^N$. The corresponding norm is defined by

$$\|u\|_E = \left(\int_0^T \left(|u'(t)|^2 + |u(t)|^2 \right) dt \right)^{\frac{1}{2}}, \forall u \in E.$$

For every $u, v \in E$, we define

$$\langle u, v \rangle = \int_0^T e^{Q(t)} [(u^{'}(t), v^{'}(t)) + (A(t)u(t), v(t))] dt,$$

and we observe that, by the assumptions (B_1) and (B_2), it defines an inner product in E. Then is a separable and reflexive Banach space with the norm

$$\|u\| = \langle u, u\rangle^{\frac{1}{2}} \quad \forall u \in E.$$

Obviously, E is an uniformly convex Banach space.
A simple computation shows that $(A(t)x, x) = \sum_{i,j=1}^{N} a_{ij}(t)x_i x_j \leq \sum_{i,j=1}^{N} \|a_{ij}\|_\infty |x|^2$ for $t \in [0, T]$ and $x \in \mathbb{R}^N$, and this along with (B_2) implies

$$\sqrt{m}\|u\|_E \leq \|u\| \leq \sqrt{M}\|u\|_E, \tag{3}$$

therefore, the norm $\|.\|$ is equivalent to the norm $\|.\|_E$. Since $(E, \|.\|)$ is compactly embedded in $C([0, T], \mathbb{R}^N)$ (see [28]), there exists a positive constants C such that

$$\|u\|_\infty \leq C\|u\| \tag{4}$$

where

$$\|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

Let $Q_1 \leq Q(t) \leq Q_2$ for all $t \in [0, T]$. Now we want to introduce the definition of the weak solution for the system (1). Since $q \in L^1([0, T]; \mathbb{R}^N)$, we have $Q^{'}(t) = q(t)$ for a.e. $t \in [0, T]$. Multiplying both sides of

$$-u^{''}(t)-q(t)u^{'}(t)+A(t)u(t) = \lambda\nabla F(t, u(t))+\mu\nabla G(t, u(t))+\nabla H(u(t)), \quad a.e.\ t \in [0, T]$$

by $e^{Q(t)}$, we obtain

$$\begin{aligned} &- e^{Q(t)}u^{''}(t) - e^{Q(t)}q(t)u^{'}(t) + e^{Q(t)}A(t)u(t) \\ &\quad = e^{Q(t)}\lambda\nabla F(t, u(t)) + e^{Q(t)}\mu\nabla G(t, u(t)) + e^{Q(t)}\nabla H(u(t)), \quad a.e.\ t \in [0, T]. \end{aligned} \tag{5}$$

Since $u^{'}$ is the classical derivative of u a.e. on $[0, T]$(see Remarks [28]), from (5) we obtain

$$\begin{aligned} [-e^{Q(t)}u^{'}(t)]^{'} = e^{Q(t)}(-A(t)u(t) + \lambda\nabla F(t, u(t)) + \mu\nabla G(t, u(t)) + \nabla H(u(t))), \\ a.e.\ t \in [0, T] \end{aligned} \tag{6}$$

Now multiplying (6) by $v \in E$ and integrating between 0 and T, we have

$$\begin{aligned} &\int_0^T ([-e^{Q(t)}u^{'}(t)]^{'}, v(t))dt \\ &= \int_0^T e^{Q(t)}(-A(t)u(t) + \lambda\nabla F(t, u(t)) + \mu\nabla G(t, u(t)) + \nabla H(u(t))), v(t))dt. \end{aligned} \tag{7}$$

As calculated in [1] , the first term of above equation is

$$\begin{aligned} \int_0^T ([-e^{Q(t)}u^{'}(t)]^{'}, v(t))dt &= \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} ([-e^{Q(t)}u^{'}(t)]^{'}, v(t))dt \\ &= \sum_{j=1}^{p}\sum_{i=1}^{N} e^{Q(t_j)} I_{ij}(u_i(t_j))v_i(t_j) + \int_0^T e^{Q(t)}(u^{'}(t), v^{'}(t))dt, \end{aligned}$$

which combined with (7) yields

$$\int_0^T e^{Q(t)}[(u^{'}(t), v^{'}(t)) + (A(t)u(t), v(t))]dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{Q(t_j)} I_{ij}(u_i(t_j))v_i(t_j)$$
$$- \lambda \int_0^T e^{Q(t)}(\nabla F(t, u(t)), v(t))dt - \mu \int_0^T e^{Q(t)}(\nabla G(t, u(t)), v(t))dt$$
$$- \int_0^T e^{Q(t)}(\nabla H(u(t)), v(t))dt = 0. \tag{8}$$

So, a function $u : [0, T] \to \mathbb{R}^N$ is a weak solution to the system (1) if $u \in E$ and for every $v \in E$, (8) holds.

$$\int_0^T \Big[\big(u'(t), v'(t)\big) + \big(A(t)u(t), v(t)\big)\Big]dt + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j)$$
$$-\lambda \int_0^T \big(\nabla F(t, u(t)), v(t)\big)dt - \mu \int_0^T \big(\nabla G(t, u(t)), v(t)\big)dt = 0,$$

for all $v \in E$.

3. Main results

We use the following notations

$$k := C^2(2LT + \sum_{j=1}^{p}\sum_{i=1}^{N} L_{ij}) < 1,$$

$$D := \frac{(T - t_p)^2}{t_1 t_p^2} + \frac{t_1}{3t_p^2}(t_p^2 + t_pT + T^2) + (t_p - t_1) + \frac{T - t_p}{t_p^2} + \frac{1}{3t_p^2}(T^3 - t_p^3) > 0,$$

$$f_\infty := \liminf_{\xi \to +\infty} \frac{\int_0^T e^{Q(t)} \max_{|v|<\xi} F(t, v(t))dt}{\xi^2},$$

$$G_\infty := \lim_{\xi \to +\infty} \frac{\int_0^T e^{Q(t)} \max_{|v|<\xi} G(t, v(t))dt}{\xi^2},$$

$$B_\infty := \limsup_{\xi \to +\infty} \frac{\int_{t_1}^{t_p} e^{Q(t)} F(t, \xi\varepsilon)dt}{\xi^2},$$

$$\lambda_1 := \frac{MD(1 + k)e^{Q_2}}{2B},$$

and

$$\lambda_2 := \frac{(1 - k)e^{Q_1}}{2C^2 f_\infty},$$

where $\varepsilon = (1, 0, ..., 0)$. We now formulate our main result as follows.

Theorem 3.1. *Assume that*

(A_1) $\int_{t_1}^{t_p} e^{Q(t)} F(t, d_n\varepsilon)dt \geq 0$ *for every* $t \in [0, T]$;

(A_2) $(\frac{1+k}{1-k})MDC^2 e^{Q_2 - Q_1} f_\infty < B_\infty$.

Then, for each $\lambda \in (\lambda_1, \lambda_2)$ *and for every arbitrary non-negative function* $G(t,x) : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ *that is measurable with respect to* t *for all* $x \in \mathbb{R}^N$*, continuously differentiable in* x *for almost every* $t \in [0,T]$*, and satisfies*

$$G_\infty < +\infty, \tag{9}$$

if we put

$$\mu_{G,\lambda} := \frac{(1-k)e^{Q_1} - 2\lambda C^2 f_\infty}{2C^2 G_\infty},$$

where $\mu_{G,\lambda} = +\infty$ *when* $G_\infty = 0$*, the problem* (1) *has an unbounded sequence of weak solutions for every* $\mu \in [0, \mu_{G,\lambda})$*.*

Proof. Our aim is to apply Theorem 2.1(b) to problem (1). Take $X = E$ and consider the functionals $\Phi, \Psi : E \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi(u) : &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{Q(t_j)} \int_0^{u_i(t_j)} I_{ij}(s)ds - \int_0^T e^{Q(t)} H(u(t))dt \\ &= \frac{1}{2}\int_0^T e^{Q(t)} \Big[|u^{'}|^2 + (A(t)u(t), u(t))\Big]dt + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{Q(t_j)} \int_0^{u_i(t_j)} I_{ij}(s)ds \\ &\quad - \int_0^T e^{Q(t)} H(u(t))dt\end{aligned}$$

and

$$\Psi(u) := \int_0^T e^{Q(t)} \Big[F(t,u) + \frac{\mu}{\lambda} G(t,u)\Big]dt$$

and put

$$J_{\overline{\lambda}}(u) := \Phi(u) - \overline{\lambda}\Psi(u),$$

for each $u \in E$.

Note that the weak solutions of (1) are exactly the critical points of $J_{\overline{\lambda}}$. Then the functionals Φ, Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in E^*$, given by

$$\begin{aligned}\Phi^{'}(u) &= \int_0^T e^{Q(t)} \Big[(u^{'}(t), v^{'}(t)) + (A(t)u(t), v(t)) - (\nabla H(u(t)), v(t))\Big]dt \\ &\quad + \sum_{j=1}^{p}\sum_{i=1}^{N} e^{Q(t_j)} I_{ij}(u_i(t_j))v_i(t_j)\end{aligned} \tag{10}$$

for any $v \in E$. Furthermore, $\Phi' : E \to E^*$ admits a continuous inverse. On the other hand, the fact that E is embedded into $C([0,T])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_0^T e^{Q(t)} [(\nabla F(t,u(t)), v(t)) + \frac{\overline{\mu}}{\overline{\lambda}} (\nabla G(t,u(t)), v(t))]dt. \tag{11}$$

Now, Φ is sequentially weakly lower semicontinuous.. To see this, let $u_n \in E$ with $u_n \to u$ weakly in E, and using the sequential weakly lower semicontinuity of the norm, we have $\liminf_{n\to+\infty}\|u_n\| \geq \|u\|$ and $u_n \to u$ uniformly on $[0,T]$. hence, since H is continuous,

$$\lim\inf_{n\to+\infty}\Big(\frac{1}{2}||u_n||^2+\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_{n_i}(t_j)} e^{Q(t_j)}I_{ij}(s)ds-\int_0^T e^{Q(t)}H(u_n(t))dt\Big)$$
$$\geq \frac{1}{2}||u||^2+\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i(t_j)} e^{Q(t_j)}I_{ij}(s)ds-\int_0^T e^{Q(t)}H(u(t))dt,$$

i.e. $\liminf_{n\to+\infty}\Phi(u_n)\geq\Phi(u)$. This implies Φ is sequentially weakly lower semicontinuous.

From the definition of Φ, since $(E,\|.\|)$ is compactly embedded in $C([0,T],\mathbb{R}^N)$, we observe that Φ is strongly continuous. Since $-L|x|^2\leq H(x)\leq L|x|^2$ for every $x\in\mathbb{R}^N$ and $-L_{ij}|s|^2\leq I_{ij}(s)\leq L_{ij}|s|^2$ for every $s\in\mathbb{R}$ for all $i=1,2,...,N$ and $j=1,2,...,p$, in view of (4), we see that

$$\frac{(1-k)e^{Q_1}}{2}\|u\|^2\leq\Phi(u)\leq\frac{(1+k)e^{Q_2}}{2}\|u\|^2 \tag{12}$$

Furthermore, $\lim_{||u||\to+\infty}\Phi(u)=+\infty$ for all $u\in X$ and so Φ is coercive.

First of all, we will show that $\overline{\lambda}<\frac{1}{\gamma}$. Hence, let $\{\theta_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty}\theta_n=+\infty$. Put $r_n:=\frac{(1-k)e^{Q_1}}{2C^2}\theta_n^2$ for all $n\in\mathbb{N}$. Then, for all $u\in E$ with $\Phi(u)\leq r_n$ one has

$$\begin{aligned}\Phi^{-1}(-\infty,r_n) &= \{u\in X;\Phi(u)<r_n\}\\ &= \{u\in X;\Phi(u)<\frac{(1-k)e^{Q_1}}{2C^2}\theta_n^2\}\\ &= \{u\in X;|u|<\theta_n\}.\end{aligned}$$

Then, for all $n\in\mathbb{N}$,

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}((-\infty,r_n))}\frac{\Big(\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)\Big)-\Psi(u)}{r_n-\Phi(u)}\\ &\leq \frac{\sup_{v\in\Phi^{-1}((-\infty,r_n))}\Psi(v)}{r_n}\\ &\leq \frac{2C^2}{(1-k)e^{Q_1}}\Big(\frac{\int_0^T e^{Q(t)}\max_{|v|<\theta_n}F(t,v(t))dt}{\theta_n^2}+\frac{\overline{\mu}}{\overline{\lambda}}\frac{\int_0^T e^{Q(t)}\max_{|v|<\theta_n}G(t,v(t))dt}{\theta_n^2}\Big).\end{aligned}$$

Moreover, from the assumption (A_2) and (9)

$$\begin{aligned}\gamma &= \liminf_{r\to+\infty}\varphi(r)\leq\liminf_{n\to+\infty}\varphi(r_n)\\ &\leq \frac{2C^2}{(1-k)e^{Q_1}}\Big(f_\infty+\frac{\overline{\mu}}{\overline{\lambda}}G_\infty\Big)<+\infty.\end{aligned}$$

The assumption $\overline{\mu}\in(0,\mu_{G,\overline{\lambda}})$ immediately yields $\gamma<\frac{1}{\overline{\lambda}}$. Let $\overline{\lambda}$ be fixed. We claim that the functional $J_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}}<\frac{2B_\infty}{(1+k)MDe^{Q_2}},$$

there exist a sequence $\{d_n\}$ and a positive constant τ such that $\lim_{n\to+\infty} d_n = +\infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{2\int_{t_1}^{t_p} e^{Q(t)}F(t, d_n\varepsilon)dt}{DM(1+k)e^{Q_2}d_n^2}. \tag{13}$$

Let $\{w_n\}$ be a sequence in E defined by

$$w_n(t) := \begin{cases} (T + \frac{t_p - T}{t_1}t)\frac{d_n\varepsilon}{t_p} & \text{if } t \in [0, t_1), \\ d_n\varepsilon & \text{if } t \in [t_1, t_p], \\ \frac{d_n\varepsilon}{t_p}t & \text{if } t \in (t_p, T]. \end{cases} \tag{14}$$

It is clear that $w_n \in E$ for all $n \in \mathbb{N}$, and $mDd_n^2 \leq \|w_n\|^2 \leq MDd_n^2$. Therefore, from (12),

$$\Phi(w_n) \leq \frac{e^{Q_2}}{2}(1+k)MDd_n^2 \tag{15}$$

From (A_1) and since G is nonnegative, due to definition of Ψ, we infer

$$\Psi(w_n) \geq \int_{t_1}^{t_p} e^{Q(t)}F(t, d_n\varepsilon)dt, \tag{16}$$

so from (13), (15) and (16), we have

$$\begin{aligned} J_{\overline{\lambda}}(w_n) &= \Phi(w_n) - \overline{\lambda}\Psi(w_n) \\ &\leq \frac{e^{Q_2}}{2}(1+k)MDd_n^2 - \overline{\lambda}\int_{t_1}^{t_p} e^{Q(t)}F(t, d_n\varepsilon)dt \\ &= \frac{e^{Q_2}}{2}(1+k)MDd_n^2(1-\overline{\lambda}\tau) \end{aligned} \tag{17}$$

for every $n \in \mathbb{N}$ large enough. Since $\overline{\lambda}\tau > 1$ and $\lim_{n\to+\infty} d_n = +\infty$, we have

$$\lim_{n\to+\infty} J_{\overline{\lambda}}(w_n) = -\infty.$$

Then, the functional $J_{\overline{\lambda}}$ is unbounded from below, and it follows that $J_{\overline{\lambda}}$ has no global minimum. Therefore, by Theorem 2.1(b), there exists a sequence $\{u_n\}$ of critical points of $J_{\overline{\lambda}}$ such that $\lim_{n\to+\infty} ||u_n|| = +\infty$ and the conclusion is achieved. □

For a given non-negative constant θ and a given positive constant d, with $(1-k)\theta^2 \neq (1+k)MDC^2d^2e^{Q_2-Q_1}$, put

$$a_d(\theta) := \frac{\int_0^T e^{Q(t)} \max_{|v|<\theta}\left[F(t, v(t)) + \frac{\mu}{\lambda}G(t, v(t))\right]dt - \int_{t_1}^{t_p} e^{Q(t)}F(t, d\varepsilon)dt}{\theta^2 - \frac{1+k}{1-k}MDC^2d^2e^{Q_2-Q_1}},$$

$\mu_1 :=$

$$\frac{(1-k)e^{Q_1}\theta_1^2 - (1+k)MDC^2d^2e^{Q_2} + 2C^2\lambda\int_0^T e^{Q(t)}\max_{|v|<\theta_1} F(t,v)dt + 2C^2\lambda\int_{t_1}^{t_p} e^{Q(t)}F(t, d\varepsilon)dt}{2C^2\int_0^T e^{Q(t)}\max_{|v|<\theta_1} G(t,v)dt},$$

$\mu_2 :=$

$$\frac{(1-k)e^{Q_1}\theta_2^2 - (1+k)MDC^2d^2e^{Q_2} + 2C^2\lambda\int_0^T e^{Q(t)}\max_{|v|<\theta_2} F(t,v)dt + 2C^2\lambda\int_{t_1}^{t_p} e^{Q(t)}F(t, d\varepsilon)dt}{2C^2\int_0^T e^{Q(t)}\max_{|v|<\theta_2} G(t,v)dt}.$$

Now, we present an application of Theorem 2.2 which we will use to obtain one nontrivial weak solution.

Theorem 3.2. *Assume that there exist three nonnegative constant θ_1, θ_2 and d with hold (A_1) such that*

$$\theta_1^2 < (\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1} < \theta_2^2 \tag{18}$$

(A$_3$) $a_d(\theta_2) < a_d(\theta_1)$.

Moreover, $\lambda \in \frac{(1-k)e^{Q_1}}{2C^2}\Big]\frac{1}{a_d(\theta_1)}, \frac{1}{a_d(\theta_2)}\Big[$ *and whose potential $G(t,x)$ for all $(t,x) \in [0,T]\times\mathbb{R}^N$, is non-negative. Then for every $\mu \in (\mu_1,\mu_2)$, the problem* (1) *admits at least one nontrivial weak solution $u_1 \in E$.*

Proof. Fix λ, G and μ as in the conclusion and take Φ and Ψ as in the proof of Theorem (3.1). We observe that the regularity assumptions of Theorem (2.2) on Φ and Ψ are satisfied. To this end, put

$$w(t) := \begin{cases} (T + \dfrac{t_p - T}{t_1}t)\dfrac{d\varepsilon}{t_p} & \text{if } t \in [0,t_1), \\ d\varepsilon & \text{if } t \in [t_1,t_p], \\ \frac{d\varepsilon}{t_p}t & \text{if } t \in (t_p,T], \end{cases} \tag{19}$$

it is easy to verify that $w \in E$. Now, put $r_1 := \frac{(1-k)e^{Q_1}}{2C^2}\theta_1^2$, $r_2 := \frac{(1-k)e^{Q_1}}{2C^2}\theta_2^2$ and $\frac{(1-k)e^{Q_1}}{2}mDd^2 \le \Phi(w) \le \frac{(1+k)e^{Q_2}}{2}MDd^2$. In particular, from (18), we conclude

$$r_1 < \Phi(w) < r_2.$$

On the other hand, for all $u \in E$, we have

$$\begin{aligned} \Phi^{-1}(-\infty,r_2) &= \{u \in X; \Phi(u) < r_2\} \\ &= \{u \in X; |u| < \theta_2\}. \end{aligned}$$

From which it follows

$$\begin{aligned} \sup_{u\in\Phi^{-1}(-\infty,r_2)} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r_2)} \int_0^T e^{Q(t)}\Big[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big]dt \\ &\le \int_0^T e^{Q(t)} \max_{|v|<\theta_2}\Big[F(t,v(t)) + \frac{\mu}{\lambda}G(t,v(t))\Big]dt. \end{aligned}$$

Arguing as before, we obtain

$$\begin{aligned} \sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(u) &= \sup_{u\in\Phi^{-1}(-\infty,r_1)} \int_0^T e^{Q(t)}\Big[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))\Big]dt \\ &\le \int_0^T e^{Q(t)} \max_{|v|<\theta_1}\Big[F(t,v(t)) + \frac{\mu}{\lambda}G(t,v(t))\Big]dt, \end{aligned}$$

assumption (A_1) ensures that

$$\psi(w) \ge \int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt.$$

Then, due to $G \ge 0$, we get

$$\int_0^T e^{Q(t)} \sup_{|v|<\theta_2}\Big[F(t,v) + \frac{\mu}{\lambda}G(t,v)\Big]dt \ge \int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt,$$

and thus $a_d(\theta_2) \geq 0$. At this point, one has

$$\begin{aligned}
\beta(r_1,r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)-\Psi(w)}{r_2-\Phi(w)} \\
&\leq \frac{\int_0^T e^{Q(t)}\max_{|v|<\theta_2}\left[F(t,v(t))+\frac{\mu}{\lambda}G(t,v(t))\right]dt-\int_{t_1}^{t_p}e^{Q(t)}F(t,d\varepsilon)dt}{\theta_2^2-(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}} \\
&= \frac{2C^2}{(1-k)e^{Q_1}}\frac{\int_0^T e^{Q(t)}\max_{|v|<\theta_2}\left[F(t,v(t))+\frac{\mu}{\lambda}G(t,v(t))\right]dt-\int_{t_1}^{t_p}e^{Q(t)}F(t,d\varepsilon)dt}{\theta_2^2-(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}} \\
&= \frac{2C^2}{(1-k)e^{Q_1}}a_d(\theta_2).
\end{aligned}$$

Since $a_d(\theta_2) \geq 0$, hypothesis (A_3) implies that

$$\int_0^T e^{Q(t)}\max_{|v|<\theta_1}\left[F(t,v(t))+\frac{\mu}{\lambda}G(t,v(t))\right]dt<\int_{t_1}^{t_p}e^{Q(t)}F(t,d\varepsilon)dt.$$

So, one has

$$\begin{aligned}
\rho(r_1,r_2) &\geq \frac{\Psi(w)-\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{\Phi(w)-r_1} \\
&\geq \frac{\int_{t_1}^{t_p}e^{Q(t)}F(t,d\varepsilon)dt-\int_0^T e^{Q(t)}\max_{|v|<\theta_1}\left[F(t,v(t))+\frac{\mu}{\lambda}G(t,v(t))\right]dt}{(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}-\theta_1^2} \\
&= \frac{2C^2}{(1-k)e^{Q_1}}\frac{\int_{t_1}^{t_p}e^{Q(t)}F(t,d\varepsilon)dt-\int_0^T e^{Q(t)}\max_{|v|<\theta_1}\left[F(t,v(t))+\frac{\mu}{\lambda}G(t,v(t))\right]dt}{(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}-\theta_1^2} \\
&= \frac{2C^2}{(1-k)e^{Q_1}}a_d(\theta_1).
\end{aligned}$$

Hence, from assumption (A_3), one has $\beta(r_1,r_2)<\rho(r_1,r_2)$. Therefore, from Theorem (2.2), for each $\lambda \in \frac{(1-k)e^{Q_1}}{2C^2}\left]\frac{1}{a_d(\theta_1)},\frac{1}{a_d(\theta_2)}\right[$ the functional J_λ admits at least one critical point u_1 such that

$$r_1<\Phi(u_1)<r_2.$$

□

Set $G^\theta := \int_0^T \max_{|\xi|\leq\theta} G(t,\xi)dt$ for every $\theta>0$ and $G_d := \inf_{[0,T]\times[0,d]} G$ for every $d>0$, then $G^\theta \geq 0$ and $G_d \leq 0$. Put

$$\lambda_3 := \frac{(1+k)MDe^{Q_2}d^2}{2\int_{t_1}^{t_p}e^{Q(t)}F(t,d\varepsilon)dt},$$

$$\lambda_4 := \frac{(1-k)e^{Q_1}\theta^2}{2C^2\int_0^T e^{Q(t)}\max_{|v|\leq\theta}F(t,v)dt},$$

$$\mu_3 := \frac{(1-k)e^{Q_1}\theta^2-2C^2\lambda\int_0^T e^{Q(t)}\max_{|v|\leq\theta}F(t,v)dt}{2TC^2e^{Q_2}G^\theta}$$

and

$$\mu_4 := \frac{(1+k)MDd^2e^{Q_2} - 2\lambda \int_{t_1}^{t_2} e^{Q(t)}F(t,d\varepsilon)dt}{2Te^{Q_1}G_d}$$

Theorem 3.3. *Assume that there exist two positive constants θ and d with*

$$\frac{\theta}{C}\sqrt{\frac{(1-k)e^{Q_1-Q_2}}{MD(1+k)}} < d$$

and hold (A_1) such that

(A4) $\displaystyle \frac{C^2}{e^{Q_1}(1-k)}\frac{\int_0^T e^{Q(t)}\max_{|v|\le\theta}F(t,v(t))dt}{\theta^2} < \frac{1}{e^{Q_2}(1+k)MD}\frac{\int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt}{d^2}$

(A5) $\displaystyle \limsup_{|\xi|\to+\infty}\frac{\max_{t\in[0,T]}F(t,\xi)}{\xi^2} \le 0.$

Then, for each $\lambda \in (\lambda_3,\lambda_4)$ and for every L^1-Carathéodory function $G(t,x) : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ satisfying the condition

$$\limsup_{|\xi|\to+\infty}\frac{\max_{t\in[0,T]}G(t,\xi)}{\xi^2} < +\infty,$$

for each $\mu \in (\mu_3,\mu_4)$, the problem (1) admits at least three distinct weak solutions in E.

Proof. In order to apply Theorem 2.3 to our problem. Then, our aim is to verify (d) and (e). To this end, put w as given (19), as well as $r := \frac{(1-k)e^{Q_1}}{2C^2}\theta^2$. Clearly, the weak solutions of the problem (1) are exactly the solutions of the equation $\Phi'(u)-\lambda\Psi'(u) = 0$. We observe $0 < r < \Phi(w)$. Since $\frac{(1-k)e^{Q_1}}{2}\|u\|^2 \le \Phi(u)$ for each $u \in E$ and bearing (4) in mind, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty,r]) &= \{u\in E;\ \Phi(u)\le r\}\\ &\subseteq \{u\in E;\ |u(t)|\le\theta \text{ for each } t\in[0,T]\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\int_0^T e^{Q(t)}\Big[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))\Big]dt\\ &\le \int_0^T e^{Q(t)}\max_{|v|\le\theta}F(t,v(t))dt+\frac{\mu}{\lambda}e^{Q_2}G^\theta.\end{aligned}$$

On the other hand, by using condition (A1), we deduce

$$\begin{aligned}\Psi(w) &\ge \int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt+\frac{\mu}{\lambda}\int_0^T e^{Q(t)}G(t,d\varepsilon)dt\\ &\ge \int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt+\frac{\mu}{\lambda}Te^{Q_1}\inf_{[0,T]\times[0,d]}G\\ &= \int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt+\frac{\mu}{\lambda}Te^{Q_1}G_d.\end{aligned}$$

Therefore, we have

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} = \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\int_0^T e^{Q(t)}\Big[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))\Big]dt}{r}$$
$$\leq \frac{\int_0^T e^{Q(t)}\max\limits_{|v|\leq\theta}F(t,v(t))dt+\frac{\mu}{\lambda}e^{Q_2}G^\theta}{\frac{(1-k)e^{Q_1}}{2C^2}\theta^2}, \tag{20}$$

and

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt+\frac{\mu}{\lambda}\int_0^T e^{Q(t)}G(t,d\varepsilon)dt}{\frac{e^{Q_2}}{2}(1+k)MDd^2}$$
$$\geq \frac{\int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt+\frac{\mu}{\lambda}Te^{Q_1}G_d}{\frac{e^{Q_2}}{2}(1+k)MDd^2}. \tag{21}$$

we observe that the condition (d) of Theorem 2.1 is fulfilled.
Finally, we can fix $b>0$ such that $\limsup\limits_{|\xi|\to\infty}\frac{\max_{t\in[0,T]}G(t,\xi)}{\xi^2}<b$. Therefore, there exists a function $s\in L^1([0,T])$ such that

$$G(t,\xi)\leq b\xi^2+s(t), \tag{22}$$

for every $t\in[0,T]$ and $\xi\in\mathbb{R}^N$.
Now, fix $0<\epsilon<\frac{(1-k)e^{Q_1-Q_2}}{2C^2T\lambda}-\frac{\mu b}{\lambda}$. From (A5) there is a function $h_\epsilon\in L^1([0,T])$ such that

$$F(t,\xi)\leq\epsilon\xi^2+h_\epsilon(t), \tag{23}$$

for every $t\in[0,T]$ and $\xi\in\mathbb{R}^N$.
Taking (4) into account, it follows that, for each $u\in E$,

$$J_\lambda(u) = \Phi(u)-\lambda\Psi(u)\geq\frac{e^{Q_1}}{2}(1-k)\|u\|^2-\lambda\int_0^T e^{Q(t)}\Big[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))\Big]dt$$
$$\geq\frac{e^{Q_1}}{2}(1-k)\|u\|^2-\lambda\epsilon e^{Q_2}\int_0^T u^2(t)dt-\lambda e^{Q_2}\|h_\epsilon\|_1-\mu b e^{Q_2}\int_0^T u^2(t)dt-\mu e^{Q_2}\|s\|_1$$
$$\geq\Big(\frac{e^{Q_1}}{2}(1-k)-\lambda\epsilon TC^2e^{Q_2}-\mu TbC^2e^{Q_2}\Big)\|u\|^2-\lambda e^{Q_2}\|h_\epsilon\|_1-\mu e^{Q_2}\|s\|_1,$$

and thus

$$\lim_{\|u\|\to+\infty}(\Phi(u)-\lambda\Psi(u))=+\infty,$$

which means the functional J_λ is coercive, and the condition (e) of Theorem 2.3 is verified.
Since from relations (20)-(21),

$$\lambda\in\Bigg]\frac{\Phi(w)}{\Psi(w)},\frac{r}{\sup_{\Phi(u)\leq r}\Psi(u)}\Bigg[,$$

Theorem 2.1 (with $\overline{v} = w$) assures the desired conclusion. □

4. Applications

Remark 4.1. *The conditions $f_\infty = 0$ and $B_\infty = +\infty$ where $\varepsilon = (1, 0, ..., 0) \in \mathbb{R}^N$, Theorem (3.1) ensures that for every $\lambda > 0$ and for each $\mu \in [0, \frac{(1-k)e^{Q_1}}{2C^2 G_\infty})$ the problem (1) admits infinitely many classical periodic solutions.Moreover, if $G_\infty = 0$, then the result holds for every $\lambda > 0$ and $\mu \geq 0$.*

Remark 4.2. *Assumption (A_2) in Theorem (3.1) can be replaced by the more general condition*
$(A_2^{'})$ there exist two sequence $\{\theta_n\}$ and $\{\eta_n\}$ with

$$\frac{(1-k)e^{Q_1}}{2C^2}\eta_n^2 > \frac{(1+k)DMe^{Q_2}}{2}\theta_n^2$$

for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \eta_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{\int_0^T e^{Q(t)} \max_{|v|<\eta_n} F(t, v(t)) - \int_{t_1}^{t_p} e^{Q(t)} F(t, \theta_n \varepsilon) dt}{\frac{(1-k)e^{Q_1}}{2C^2}\eta_n^2 - \frac{(1+k)DMe^{Q_2}}{2}\theta_n^2} < \limsup_{|\xi|\to+\infty} \frac{\int_{t_1}^{t_p} e^{Q(t)} F(t, \xi\varepsilon) dt}{\frac{(1+k)DMe^{Q_2}}{2}\xi^2}$$

where $\varepsilon = (1, 0, ..., 0)$ By choosing $\theta_n = 0$ for all $n \in \mathbb{N}$, (A_2) follows from $(A_2^{'})$. Moreover, if we assume $(A_2^{'})$ instead of (A_2) and set $r_n := \frac{(1-k)e^{Q_1}}{2C^2}\eta_n^2$ for all $n \in \mathbb{N}$, by the same reasoning as in the proof of Theorem (3.1), we obtain

$$\begin{aligned} \varphi(r_n) &\leq \frac{\left(\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)\right) - \Psi(z_n)}{r_n - \Phi(z_n)} \\ &\leq \frac{\int_0^T e^{Q(t)} \max_{|v|<\eta_n} [F(t, v(t)) + \frac{\mu}{\lambda} G(t, v(t))] dt - \int_{t_1}^{t_p} e^{Q(t)} F(t, \theta_n \varepsilon) dt}{\frac{(1-k)e^{Q_1}}{2C^2}\eta_n^2 - \frac{(1+k)DMe^{Q_2}}{2}\theta_n^2} \end{aligned}$$

where

$$z_n(t) := \begin{cases} (T + \frac{t_p - T}{t_1} t)\frac{\theta_n \varepsilon}{t_p} & \text{if } t \in [0, t_1), \\ \theta_n \varepsilon & \text{if } t \in [t_1, t_p], \\ \frac{\theta_n \varepsilon}{t_p} t & \text{if } t \in (t_p, T]. \end{cases} \tag{24}$$

We have the same conclusion as Theorem (3.1) with the interval (λ_1, λ_2) replaced by $(\lambda_1^{'}, \lambda_2^{'})$, where

$$\lambda_1^{'} := \frac{e^{Q_2}(1+k)DM}{2\limsup_{\xi\to+\infty} \frac{\int_{t_1}^{t_p} e^{Q(t)} F(t, \xi\varepsilon) dt}{\xi^2}}$$

$$\lambda_2^{'} := [\lim_{n\to+\infty} \frac{\int_0^T e^{Q(t)} \max_{|v|<\eta_n} F(t, v(t)) dt - \int_{t_1}^{t_p} e^{Q(t)} F(t, \theta_n \varepsilon) dt}{\frac{(1-k)e^{Q_1}}{2C^2}\eta_n^2 - \frac{(1+k)DMe^{Q_2}}{2}\theta_n^2}]^{-1}.$$

Corollary 4.1. *Assume that (B_1) , (B_2) and (A_1) hold,*
(A_6) $f_\infty < \frac{e^{Q_1}(1-k)}{2C^2}$
(A_7) $\frac{MDe^{Q_2}(1+k)}{2} < B_\infty$
where $\varepsilon = (1, 0, ..., 0)$. Then, for every arbitrary non-negative function $G(t, x)$:

$[0,T]\times\mathbb{R}^N\to\mathbb{R}$ that is measurable with respect to t for all $x\in\mathbb{R}^N$, continuously differentiable in x for almost every $t\in[0,T]$, and satisfies (9), and for every $\mu\in[0,\mu_{G,1})$ where

$$\mu_{G,1}:=\frac{(1-k)e^{Q_1}-2C^2f_\infty}{2C^2G_\infty},$$

and λ_2 is given in the statement of Theorem (3.1), the problem

$$\begin{cases} -u^{''}(t)-q(t)u^{'}(t)+A(t)u(t)=\nabla F(t,u(t))+\mu\nabla G(t,u(t))+\nabla H(u(t)), \\ \hfill a.e.\ t\in(0,T), \\ \triangle(u_i'(t_j))=I_{ij}(u_i(t_j)),\ i=1,2,...,N, j=1,2,...,p, \\ u(0)-u(T)=u'(0)-u'(T)=0. \end{cases} \tag{25}$$

has an unbounded sequence of classical periodic solutions.

Theorem 4.2. *Assume that (B_1), (B_2) hold and let $F:\mathbb{R}^N\to\mathbb{R}$ be a continuously differentiable function such that $\liminf_{\xi\to+\infty}\frac{\max_{|x|<\xi}F(x)}{\xi^2}=0$ and $\limsup_{\xi\to+\infty}\frac{F(\xi\varepsilon)}{\xi^2}=+\infty$ where $\varepsilon=(1,0,...,0)$. Then, the problem*

$$\begin{cases} -u^{''}(t)-q(t)u^{'}(t)+A(t)u(t)=\nabla F(u(t))+\nabla H(u(t)),\ a.e.\ t\in(0,T), \\ \triangle(u_i'(t_j))=I_{ij}(u_i(t_j)),\ i=1,2,...,N, j=1,2,...,p, \\ u(0)-u(T)=u'(0)-u'(T)=0. \end{cases} \tag{26}$$

has an unbounded sequence of classical periodic solutions.

Proof. Theorem proves with Corollary (4.1) and Remark (4.1) whit $\mu=0$. □

We use the following notations

$$f_0:=\liminf_{\xi\to0^+}\frac{\int_0^T e^{Q(t)}\max_{|v|<\xi}F(t,v(t))dt}{\xi^2},$$

$$G_0:=\lim_{\xi\to0^+}\frac{\int_0^T e^{Q(t)}\max_{|v|<\xi}G(t,v(t))dt}{\xi^2},$$

$$B_0:=\limsup_{\xi\to0^+}\frac{\int_{t_1}^{t_p} e^{Q(t)}F(t,\xi\varepsilon)dt}{\xi^2},$$

$$\lambda_5:=\frac{(1+k)MDe^{Q_2}}{2B_0},$$

and

$$\lambda_6:=\frac{(1-k)e^{Q_1}}{2C^2f_0}.$$

Using Theorem 2.1(c) and arguing as in the proof of Theorem 3.1, we can obtain the following result.

Theorem 4.3. *Assume that*

(A_8) $\int_{t_1}^{t_p} e^{Q(t)}F(t,\xi\varepsilon)dt\ge0$ *for every* $t\in[0,T]$;

(A_9) $(\frac{1+k}{1-k})MDC^2e^{Q_2-Q_1}f_0<B_0$.

Then, for each $\lambda \in (\lambda_5, \lambda_6)$ and for every arbitrary non-negative function $G(t,x)$: $[0,T] \times \mathbb{R}^N \to \mathbb{R}$ that is measurable with respect to t for all $x \in \mathbb{R}^N$, continuously differentiable in x for almost every $t \in [0,T]$, and satisfies

$$G_0 < +\infty, \tag{27}$$

if we put $\mu'_{G,\lambda} := \frac{(1-k)e^{Q_1}-2\lambda C^2 f_0}{2C^2G_0}$, where $\mu'_{G,\lambda} = +\infty$ when $G_\infty = 0$, the problem (1) has an unbounded sequence of weak solutions for every $\mu \in [0, \mu'_{G,\lambda})$.

Proof. Fix $\overline{\lambda} \in (\lambda_5, \lambda_6)$ and let G be a function that satisfies the condition (27). Since $\overline{\lambda} < \lambda_6$, we obtain

$$\mu'_{G,\overline{\lambda}} := \frac{(1-k)e^{Q_1} - 2\lambda C^2 f_0}{2C^2G_0} > 0.$$

Now fix $\overline{\mu} \in (0, \mu'_{g,\overline{\lambda}})$ and set

$$J_{\overline{\lambda}}(u) := \Phi(u) - \overline{\lambda}\Psi(u),$$

We take Φ, Ψ and $J_{\overline{\lambda}}$ as in the proof of Theorem 3.1. Now, as it has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Theorem 2.1. As first step, we will prove that $\overline{\lambda} < \frac{1}{\delta}$. Then, let $\{\theta_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \theta_n = 0$ and $inf_X \Phi = 0$ and the definition of δ, we have $\delta = \liminf_{r\to 0^+} \varphi(r)$.

Putting $r_n := \frac{(1-k)e^{Q_1}}{2C^2}\theta_n^2$. we can prove that $\delta < +\infty$. From $\overline{\mu} \in (0, \mu'_{G,\overline{\lambda}})$, the following inequalities hold

$$\delta \leq \frac{2C^2}{(1-k)e^{Q_1}}\Big(f_0 + \frac{\overline{\mu}}{\overline{\lambda}}G_0\Big).$$

Therefore, $\overline{\lambda} < \frac{1}{\delta}$. Let $\overline{\lambda}$ be fixed. We claim that the functional $J_{\overline{\lambda}}$ has not a local minimum at zero. Since

$$\frac{1}{\overline{\lambda}} < \frac{2B}{(1+k)MDe^{Q_2}},$$

there exist a sequence $\{d_n\}$ of positive number and $\eta > 0$ such that $\lim_{n\to+\infty} d_n = 0^+$ and

$$\frac{1}{\overline{\lambda}} < \eta < \frac{2\int_{t_1}^{t_p} e^{Q(t)}F(t, d_n\varepsilon)dt}{(1+k)DMe^{Q_2}d_n^2},$$

for each $n \in \mathbb{N}$ large enough. Let w_n from (14) be the sequence in E. From (A_1) one has (17) holds. Note that $\overline{\lambda}\eta > 1$. we can obtain

$$J_{\overline{\lambda}}(w_n) < \frac{e^{Q_2}}{2}(1+k)MDd_n^2(1-\overline{\lambda}\eta) \leq 0 = \Phi(0) - \overline{\lambda}\Psi(0),$$

for each $n \in \mathbb{N}$ large enough. Then, we see that zero is not a local minimum of $J_{\overline{\lambda}}$. Thus, together with the fact zero is the only global minimum of Φ, we deduce that the energy functional $J_{\overline{\lambda}}$ has not a local minimum at the unique global minimum of Φ. Therefore, by Theorem 2.1, there exists a sequence $\{u_n\}$ of critical points of $J_{\overline{\lambda}}$ which converges weakly to zero. In view of the fact that the embedding $E \hookrightarrow C([0,T])$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Theorem 4.4. *Assume that there exist two nonnegative constant* θ *and* d *with hold* (A_1) *such that* $(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1} < \theta^2$ *Furthermore, suppose that*

(A_{10}) $\frac{\int_0^T e^{Q(t)} \max_{|v|<\theta} F(t,v)dt}{\theta^2} < \frac{\int_{t_1}^{t_p} e^{Q(t)} F(t,d\varepsilon)dt}{(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}}$. *Then, for each*

$$\lambda \in \frac{(1-k)e^{Q_1}}{2C^2}\left]\frac{(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}}{\int_{t_1}^{t_p} e^{Q(t)}F(t,d\varepsilon)dt}, \frac{\theta^2}{\int_0^T e^{Q(t)} \max_{|v|<\theta} F(t,v(t))dt}\right[$$

problem

$$\begin{cases} -u^{''}(t) - q(t)u^{'}(t) + A(t)u(t) = \lambda\nabla F(t,u(t)) + \nabla H(u(t)), \ a.e. \ t \in (0,T), \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)), \ i = 1,2,...,N, j = 1,2,...,p, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{28}$$

admits at least one nontrivial weak solution $\overline{u}$ *such that* $|\overline{u}(x)| < \theta$ *for all* $t \in [0,T]$.

Proof. The conclusion follows from Theorem 3.2, by taking $\theta_1 = 0$, $\theta_2 = \theta$ and $\mu = 0$ Indeed, owing to assumption (A_{10}), one has

$$\begin{aligned} a_d(\theta) &= \frac{\int_0^T e^{Q(t)} \max_{|v|<\theta} F(t,v(t))dt - \int_{t_1}^{t_2} e^{Q(t)}F(t,d\varepsilon)dt}{\theta^2 - (\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}}, \\ &< \frac{\left(1 - \frac{(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}}{\theta^2}\right)\int_0^T e^{Q(t)} \max_{|v|<\theta} F(t,v(t))dt}{\theta^2 - (\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}} \\ &= \frac{1}{\theta^2}\int_0^T e^{Q(t)} \max_{|v|<\theta} F(t,v(t))dt. \end{aligned}$$

On the other hand, one has

$$a_d(0) = \frac{\int_{t_1}^{t_2} e^{Q(t)}F(t,d\varepsilon)dt}{(\frac{1+k}{1-k})MDC^2d^2e^{Q_2-Q_1}}.$$

Hence, taking assumption (A_{10}) into account, Theorem 3.2 ensures the conclusion. □

A special case of Theorem 3.3 is the following theorem.

Theorem 4.5. *Let* $\nabla F : \mathbb{R} \to \mathbb{R}$ *be a continuous function,* $F(d\varepsilon) > 0$ *for some* $d > 0$ *in* $[t_1, t_p]$, $F(\xi) \geq 0$ *and* $\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^2} = \limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0$.
Then there is $\lambda^* > 0$ *such that for each* $\lambda > \lambda^*$ *the problem*

$$\begin{cases} -u^{''}(t) - u^{'}(t) + u(t) = \lambda\nabla F(u(t)) + \nabla H(u(t)), \ a.e. \ t \in (0,T), \\ \triangle(u'(t_j)) = I_j(u(t_j)), \ j = 1,2, \\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{29}$$

Proof. Fix $\lambda > \lambda^* := \frac{(1+k')MDd^2e^T}{2F(d\varepsilon)e^{t_p-t_1}}$ where $k' = C^2(2LT + \sum_{j=1}^p L_j)$ and $\varepsilon = (1,0,...,0)$ for some $d > 0$. Since

$$\liminf_{\xi\to 0} \frac{F(\xi)}{\xi^2} = 0,$$

there is a sequence $\{\theta_n\} \subset (0,+\infty)$ such that $\lim_{n\to+\infty} \theta_n = 0$ and

$$\lim_{n\to+\infty} \frac{\max_{|\xi|\leqslant\theta_n} F(\xi)}{\theta_n^2} = 0.$$

Indeed, one has

$$\lim_{n\to+\infty} \frac{\max_{|\xi|\leqslant\theta_n} F(\xi)}{\theta_n^2} = \lim_{n\to+\infty} \frac{\max_{|\xi|\leqslant\theta_n} F(\xi)}{\xi^2}\frac{\xi^2}{\theta_n^2} = 0,$$

Here, there exists $\theta > 0$ such that $\frac{\theta}{C}\sqrt{\frac{(1-k')e^{-T}}{MD(1+k')}} < d$. From Theorem 3.3 the conclusion follows. □

Example 4.1. Consider the system

$$\begin{cases} -u''(t) - u'(t) + A(t)u(t) = \lambda\nabla F(t,u(t)) + \nabla H(u(t)), \; a.e.\; t \in (0,3), \\ \triangle(u_i'(t_j)) = I_{ij}(u_i(t_j)), \; i = 1,2, j = 1,2, \\ u(0) - u(3) = u'(0) - u'(3) = 0. \end{cases} \tag{30}$$

where $A(t) = I$ where I an identity matrix of order 2×2, $N = p = 2$, $q(t) = 1$, $t = 1$ and $t = 2$ therefore $Q(t) = t$ for all $t \in [0,3]$. Also let $F(t,x) = F(x) = sinh(|x|^2)$, $H(x) = \frac{|x|^2}{108(1+|x|^4)}$ and $I_{ij}(s) = \frac{1}{288}s(1+sins)$ for all $(t,x) \in [0,3]\times\mathbb{R}^2$ and $s \in \mathbb{R}$. We consider $C = \sqrt{6}$, $m = 1$, $M = 2$, $L = \frac{1}{108}$, $L_{ij} = \frac{1}{288}$, $D = \frac{7}{2}$ and $k = \frac{5}{12}$,

$$\liminf_{\xi\to 0^+} \frac{\max F(\xi)}{\xi^2} = 1,$$

$$\limsup_{\xi\to 0^+} \frac{F(\xi,0)}{\xi^2} = +\infty,$$

We see that all the conditions of Theorem (4.3) are satisfied. Hence, for every $\lambda \in (0.048, +\infty)$ system (30) has an unbounded sequence of classical periodic solutions that converges uniformly to 0 in E.

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