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Existence of Solutions for a Class of Superlinear Anisotropic Robin Problems with Variable Exponent

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Abstract. In this work we study the following nonlinear anisotropic elliptic equations

$$(P)\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u)+|u|^{p_M(x)-2}u=f(x,u) & in \quad \Omega \\ \sum_{i=1}^{N}|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u.\nu_i+\beta(x)|u|^{p_m(x)-2}u=0 & on \quad \partial\Omega. \end{cases}$$

We set up that the problem (P) admits a sequence of weak solutions and multiplicity result under suitable conditions.

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1. Introduction

In the last few decades, one of the topics from the field of partial differential equations that has continuously attracted interest is that concerning the Sobolev space with variable exponents, $W^{1,p(.)}$ (where $p(.)$ is a function depending on x); see for example the monograph [5] and the references therein. Naturally, problems involving the $p(x)$-Laplacian operator were intensively studied.

On the other hand, it has been experimentally shown that the above-mentioned fluids may have their viscosity undergoing a significant change; see [1]. Consequently, the mathematical modelling of such fluids requires the introduction of the so-called anisotropic variable spaces. Indeed, there is by now a large number of papers and increasing interest about anisotropic problems. With no hope of being complete, let us mention some pioneering works on anisotropic Sobolev spaces [14, 17]. Therefore, in the recent years, the study of various mathematical problems modeled by quasilinear elliptic and parabolic equations with both anisotropic and variable exponent has received considerable attention.

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with smooth boundary. In this paper we study the following nonlinear anisotropic elliptic equations

$$(P)\begin{cases} -\triangle_{\overrightarrow{p}(x)}(u)+|u|^{p_M(x)-2}u=f(x,u) & in \quad \Omega \\ \sum_{i=1}^{N}|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u.\nu_i+\beta(x)|u|^{p_m(x)-2}u=0 & on \quad \partial\Omega, \end{cases}$$

where $\triangle_{\overrightarrow{p}(.)}$ represents the $\overrightarrow{p}(.)$-Laplace operator, that is,

$$\triangle_{\overrightarrow{p}(x)}(u)=\sum_{i=1}^{N}\partial_{x_i}(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u),$$

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ν_i are the components of the outer normal unit vector, $\beta \in L^\infty(\partial\Omega)$ fulfill $\beta(.) \geq \beta_0$ for some constant $\beta_0 > 0$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function,
$\overrightarrow{p}(x) = (p_1(x), p_2(x), ..., p_N(x))$,

$$p_M(x) = \max_{i\in\{1,2,...,N\}} p_i(x), \quad p_m(x) = \min_{i\in\{1,2,...,N\}} p_i(x)$$

and for $i = 1, ..., N$, we assume that p_i is a continuous function on $\overline{\Omega}$ such that $\inf_{\Omega} p_i(x) > 1$.
We set,

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}) | \min_{x\in\overline{\Omega}} h(x) > 1\}.$$

For any $h \in C_+(\overline{\Omega})$, we define

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \quad and \quad h^- = \inf_{x\in\overline{\Omega}} h(x).$$

Moreover, let's put the positive real numbers P_M^+, P_m^+, P_m^- which defined as the following

$$P_M^+ = \max\{p_1^+, ..., p_N^+\},\ P_m^+ = \max\{p_1^-, ..., p_N^-\},\ P_m^- = \min\{p_1^-, ..., p_N^-\}.$$

Throughout this paper, we assume that

$$\sum_{i=1}^{N} \frac{1}{p_i^-} > 1, \tag{1}$$

Define $P_-^*, P_{-,\infty} \in \mathbb{R}^+$ by

$$P_-^* = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i^-} - 1},\ P_{-,\infty} = \max\{P_m^+, P_-^*\}.$$

Let $F(x,t) = \int_0^t f(x,s)\,ds$, and we assume that f satisfies the following conditions:

(f_0) $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $f(x,0) = 0 \quad \forall x \in \Omega$, and

$$|f(x,t)| \leq c(1 + |t|^{q(x)-1}),$$

for all $(x,t) \in \Omega \times \mathbb{R}$ where $c > 0$ is a constant, and $q(x) \in C_+(\overline{\Omega})$ such that $q^+ = \sup_{x\in\overline{\Omega}} q(x) < P_{-,\infty}$.

(f_1) There exist $\theta > P_M^+$ and $M > 0$ such that

$$|\ t\ | \geq M \Rightarrow 0 < \theta F(x,t) \leq t f(x,t)$$

for a.e. $x \in \Omega$ and each $t \in \mathbb{R}$.

(f_2) $f(x,t) = o(|\ t\ |^{P_M^+ - 1})$ as $t \to 0$ and uniformly for $x \in \Omega$, with $q^- > P_M^+$.

(f_3) $\lim_{t\to 0} \dfrac{F(x,t)}{|t|^{P_m^-}} = \infty$, uniformly in Ω.

(f_4) There exists $\delta > 0$ be small enough such that

$$\mathcal{F}(x,t) = P_m^- F(x,t) - f(x,t)t > 0,$$

for every $x \in \Omega$, $|t| \leq \delta$ and $t \neq 0$.

(f_5) f is odd in t with $|t| \leq \delta$.

The main results of this article are as follows:

Theorem 1.1. *Suppose* (f_0), (f_1) *and* (f_2). *Then, the problem* (P) *has at least a nontrivial weak solution.*

Theorem 1.2. *Under the assumptions* (f_0) *and* $(f_3) - (f_5)$, *the problem* (P) *has a sequence of weak solutions such that* $\|u_n\|_{L^\infty} \to 0$ *as* $n \to \infty$.

Let us : $f(x,t) = f_1(x,t) + f_2(x,t)$, $\quad F_i(x,t) = \int_0^t f_i(x,s)\,ds$ and $\mathcal{F}_i(x,t) = P_m^- F_i(x,t) - f_i(x,t)t$ for $i = 1,2$.
We assume that the functions $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the conditions :

(H_1) For $i = 1,2$. $f_i \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ and $f_i(x,0) = 0 \quad \forall x \in \Omega$, and
$$|f_i(x,t)| \leq c_i(1 + |t|^{q_i(x)-1}),$$
for all $(x,t) \in \Omega \times \mathbb{R}$ where $c_i > 0$ is a constant, and $q_i(x) \in C_+(\overline{\Omega})$ such that $q_i^+ = \sup_{x\in\overline{\Omega}} q_i(x) < P_{-,\infty}$.

(H_2) $\lim_{t\to 0} \frac{F_1(x,t)}{|t|^{P_m^-}} = \infty$, uniformly in Ω.

(H_3) There exists $\delta > 0$ such that for every $x \in \Omega$,
$$\frac{f_1(x,t)}{|t|^{P_m^- -2}t} \text{ is strictly decrasing in}]0,\delta[.$$

(H_4) There exist two constants b_1 and b_2 such that
$$\liminf_{t\to 0} \frac{F_2(x,t)}{F_1(x,t)} \geq b_1 > -1,$$
and
$$\liminf_{t\to 0} \frac{\mathcal{F}_2(x,t)}{\mathcal{F}_1(x,t)} \geq b_2 > -1,$$
uniformly in Ω.

(H_5) $f_i(x,-t) = -f_i(x,t)$, $(i = 1,2)$ for all $x \in \Omega$, $|t| \leq \gamma$.

Theorem 1.3. *Under the assumptions* $(H_1) - (H_5)$, *the problem* (P) *has a sequence of weak solutions such that* $\|u_n\|_{L^\infty} \to 0$ *as* $n \to \infty$.

As an example, we consider the problem
$$(1.4)\begin{cases} -\triangle_{\overrightarrow{p}(x)}(u) + |u|^{p_M(x)-2}u = \lambda\left(m(x)|u|^{q_1(x)-2}u + n(x)|u|^{q_2(x)-2}u\right) & in \quad \Omega \\ \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u.\nu_i + \beta(x)|u|^{p_m(x)-2}u = 0, & on \quad \partial\Omega, \end{cases}$$

where $m(x)$, $n(x)$, $q_i(x) \in C(\overline{\Omega})$, $1 < q_i(x) < P_{-,\infty}$, $(i = 1,2)$ for all $x \in \overline{\Omega}$, and the parameter λ is a positive number.

Corollary 1.4. *Assume that* $1 < q_1(x) < P_m^-$, $1 < q_1(x) < q_2(x)$, $m(x) > 0$ *for all* $x \in \Omega$. *Then, For any* $\lambda > 0$ *the problem* (1.4) *has a sequence of weak solutions such that* $\|u_n\|_{L^\infty} \to 0$ *as* $n \to \infty$.

The problems studied here involve a variable exponent. The $\triangle_{\overrightarrow{p}(.)}$- laplacian operator possesses more complicated nonlinearities than the $p(.)$ -laplacian operator, mainly due to the fact that it is not homogeneous. As far as we are aware, contributions discussed a anisotropic Robin problems with variable exponents have seldom been studied. So it is necessary for us to investigate the related problems deeply. A distinguishing feature that we have assumed some conditions only at zero, however, there are no conditions imposed on f at infinity, we borrow the main ideas from Wang in [19] and also form [18].

This paper is organized as follows. In Section 2, we recall some preliminaries on variable exponent spaces. In Sections 3, we give the proof of results via a variational structure.

2. Preliminaries

In this part, we give some properties of the variable exponent Lebesgue space and anisotropic Sobolev spaces.
For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space:

$$L^{p(x)}(\Omega) = \{u : u\, is\, a\, measurable\, real{-}valued\, function\, such\, that \int_\Omega |u(x)|^{p(x)}\, dx < \infty\},$$

endowed with the *Luxemburg norm*

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\}.$$

Proposition 2.1. *(see [6, 9, 10])*

(1) *The space $(L^{p(x)}(\Omega), |u|_{p(x)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have*

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-}\right) |u|_{p(x)}|v|_{q(x)} \leq 2|u|_{p(x)}|v|_{q(x)}.$$

(2) *If $p_1(x)$, $p_2(x) \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$, $\forall x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*

Proposition 2.2. *(see [8]) Denote $\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$.
Then for $u \in L^{p(x)}(\Omega)$, $(u_n) \subset L^{p(x)}(\Omega)$ we have*

(1) $|u|_{p(x)} < 1(= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1(= 1; > 1)$,
(2) $|u|_{p(x)} > 1 \Rightarrow |u|^{p^-}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}$,
(3) $|u|_{p(x)} < 1 \Rightarrow |u|^{p^+}_{p(x)} \leq \rho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}$,
(4) $|u|_{p(x)} \to 0(\to \infty) \Leftrightarrow \rho_{p(x)}(u) \to 0(\to \infty)$,
(5) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Leftrightarrow \lim_{n\to\infty} \rho_{p(x)}(u_n - u) = 0$.

Next, we define the anisotropic space with variable exponents where we will treat the problem (P) by

$$W^{1,\overrightarrow{p}(x)}(\Omega) = \{u \in L^{p_M(x)}(\Omega) : \partial_{x_i} u \in L^{p_i(x)}(\Omega),\, \forall i \in \{1, ..., N\}\},$$

with the norm

$$\|u\| = \|u\|_{1,\overrightarrow{p}(.)} = |u|_{p_M(.)} + \sum_{i=1}^{N} |\partial_{x_i} u|_{p_i(.)},$$

which is a reflexive and separable Banach space (see [3, 7]).

Let us put

$$p^{\partial}(x) = \begin{cases} (N-1)p(x)/(N-p(x)) & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Now, we recall some results which concerning the embedding theorem .

Proposition 2.3. *(see [2, 16]) Suppose that $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary and relation (1) is fulfilled.*

(1) *For any $q \in C(\overline{\Omega})$ verifying $1 < q(x) < P_{-,\infty}$ $\forall x \in \overline{\Omega}$, the embedding*

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact.

(2) *If $\overrightarrow{p}(x) \in (C_+(\overline{\Omega}))^N$, and $q \in C(\overline{\Omega})$ verifying*

$$1 < q(x) < \min_{x \in \partial\Omega} \{p_1^{\partial}(x), ..., p_N^{\partial}(x)\} \;\; \forall x \in \partial\Omega,$$

the embedding

$$W^{1,\overrightarrow{p}(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$$

is continuous and compact.

Proposition 2.4. *(see [13, 15]) Let*

$$\mathcal{A}(u) = \int_{\Omega} \sum_{i=1}^{N} \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)}\, dx$$

- *$\mathcal{A}$ is well defined on $W^{1,\overrightarrow{p}(x)}(\Omega)$, $\mathcal{A} \in C^1(W^{1,\overrightarrow{p}(x)}(\Omega), \mathbb{R})$ and*

$$\langle \mathcal{A}'(u), \varphi \rangle = \int_{\Omega} \sum_{i=1}^{N} \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u)\varphi + |u|^{p_M(x)-2} u\varphi\, dx,$$

  for all $u, \varphi \in W^{1,\overrightarrow{p}(x)}(\Omega)$. In addition $\mathcal{A}'$ is continuous, bounded and strictly monotone.
- *$\mathcal{A}$ is weakly lower semi-continuous.*
- *$\mathcal{A}'$ is an operator of type (S_+).*

In this work, we use the proposition below which is the main tool to prove the existence of a sequence of solutions.

Proposition 2.5. *(see[12]) Let $I \in C^1(X, \mathbb{R})$ where X is a Banach space. Assume that I satisfies the (PS) condition , is even and bounded from below, and $I(0) = 0$. If for any $n \in \mathbb{N}$, there exists a k-dimensional subspace X_n and ρ_n such that*

$$\sup_{X_n \bigcap S_{\rho_n}} I < 0,$$

where $S_\rho = \{u \in X : \|u\| = \rho\}$, then I has a sequence of critical values $c_n < 0$ satisfying $c_n \to 0$ as $n \to +\infty$.

Let $X = W^{1,\overrightarrow{p}(x)}(\Omega)$. The functional I associated with the problem (P) is defined as

$$I : X \longrightarrow \mathbb{R}, \quad I = I_1 - I_2,$$

where

$$I_1(u) = \int_\Omega \left[\sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)} \right] dx + \int_{\partial\Omega} \beta(x) \frac{|u|^{p_m(x)}}{p_m(x)} \, d\sigma,$$

and

$$I_2(u) = \int_\Omega F(x,u)\, dx.$$

Under assumption (H_1), we have I is well defined on X and $I \in C^1(X, \mathbb{R})$. So we can define a weak solution as below.

Definition 2.1. A function u is a weak solution of the problem (P) if and only if

$$\int_\Omega \left[\sum_{i=1}^N \partial_{x_i}(|\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u)\varphi + |u|^{p_M(x)-2} u\varphi \right] dx + \int_{\partial\Omega} \beta(x)|u|^{p_m(x)-2} u\varphi \, d\sigma$$

$$- \int_\Omega f(x,u)\varphi \, dx = 0,$$

for all $\varphi \in X$.

3. Proof of main results

To prove Theorem 1.1, we shall use the Mountain Pass theorem. We first start with the following lemmas.

Lemma 3.1. *Under (f_0) and (f_1), the functional I satisfies the (PS) condition.*

Proof. Let $(u_n)_n$ be a (PS) sequence for the functional I: I bounded and $I'(u_n) \to 0$. Let us show that $(u_n)_n$ is bounded in X. Using the hypothesis (f_1), since $I(u_n)$ is bounded and $\beta(.) \geq \beta_0 > 0$, we have

$$\begin{aligned} C_1 \;&\geq\; \int_\Omega \left[\sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} + \frac{|u_n|^{p_M(x)}}{p_M(x)} \right] dx + \int_{\partial\Omega} \beta(x) \frac{|u_n|^{p_m(x)}}{p_m(x)} \, d\sigma - \int_\Omega F(x,u_n)\, dx \\ &\geq\; \frac{1}{P_M^+} \int_\Omega \left[\sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} + |u_n|^{p_M(x)} \right] dx + \frac{1}{P_M^+} \int_{\partial\Omega} \beta(x)|u_n|^{p_m(x)} \, d\sigma - \\ &\quad \int_\Omega \frac{u_n}{\theta} f(x,u_n)\, dx + C_2, \end{aligned}$$

where C_1 and C_2 are two constants. Note that

$$\begin{aligned} \langle I'(u_n), u_n \rangle \;&=\; \int_\Omega \left[\sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} + |u_n|^{p_M(x)} \right] dx + \int_{\partial\Omega} \beta(x)|u_n|^{p_m(x)} \, d\sigma \\ &-\; \int_\Omega f(x,u_n) u_n \, dx \end{aligned}$$

which implies

$$\begin{aligned} C_1 &\geq (\frac{1}{P_M^+}-\frac{1}{\theta})\int_\Omega \left[\sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)}+|u_n|^{p_M(x)}\right]dx+(\frac{1}{P_M^+}-\frac{1}{\theta})\beta_0\int_{\partial\Omega}|u_n|^{p_m(x)}\,d\sigma \\ &+ \frac{1}{\theta}\langle I'(u_n),u_n\rangle + C_2. \end{aligned} \tag{2}$$

Suppose, by contradiction that $(u_n)_n$ unbounded in X, so $\| u_n \|\geq 1$ for rather large values of n. For each $i \in \{1, ..., N\}$ and n we define

$$\alpha_{i,n} = \begin{cases} P_M^+ & \text{if } |\partial_{x_i}u_n|_{p_i(.)} < 1, \\ P_m^- & \text{if } |\partial_{x_i}u_n|_{p_i(.)} > 1. \end{cases}$$

Using 2) and 3) of proposition 2.2, we have

$$\begin{aligned} \int_\Omega \left[\sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)}+|u_n|^{p_M(x)}\right]dx &\geq \left[\sum_{i=1}^N |\partial_{x_i}u|_{p_i(x)}^{P_m^-}-N+|u|_{p_M(x)}^{P_m^-}-1\right] \\ &\geq \frac{\|u\|^{P_m^-}}{(N+1)^{P_m^- -1}}-(N+1). \end{aligned}$$

Furthermore, $I'(u_n)\to 0$ assure that there exists $C_3>0$ such that

$$-C_3 \| u_n \| \leq \langle I'_+(u_n),u_n\rangle \leq C_3 \| u_n \|$$

for rather large values of n. Consequently,

$$C_1 \geq (\frac{1}{P_M^+}-\frac{1}{\theta})\frac{\|u\|^{P_m^-}}{(N+1)^{P_m^- -1}}-(\frac{1}{P_M^+}-\frac{1}{\theta})(N+1)-\frac{C_3}{\theta}\| u_n \| + C_2.$$

Since $P_m^- > 1$ and $(\frac{1}{P_M^+}-\frac{1}{\theta})>0$, we have

$$(\frac{1}{P_M^+}-\frac{1}{\theta})\frac{\|u\|^{P_m^-}}{(N+1)^{P_m^- -1}}-(\frac{1}{P_M^+}-\frac{1}{\theta})(N+1)-\frac{C_3}{\theta}\| u_n \| + C_2 \to +\infty \text{ as n} \to +\infty,$$

what is a contradiction. So $(u_n)_n$ is a bounded sequence in X. The proof of lemma 3.1 is complete. □

Lemma 3.2. *There exist $r>0$ and $\alpha>0$ such that $I(u)\geq\alpha$, for all $u\in X$ with $\| u \| = r$.*

Proof. The conditions (f_0) and (f_2) assure that

$$| F(x,t) | \leq \varepsilon \, | \, t \, |^{P_M^+} + C(\varepsilon) \, | \, t \, |^{q(x)} \quad \text{for all } (x,t)\in\Omega\times\mathbb{R}.$$

For $\| u \|$ small enough, we have

$$I(u) \geq \frac{1}{P_M^+}\int_\Omega \left[\sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)}+|u|^{p_M(x)}\right]dx - \int_\Omega F(x,u)\,dx \tag{3}$$

For such an element u we have $|\partial_{x_i} u|_{p_i(.)} < 1$ and , by 3) of proposition 2.2, we obtain

$$\begin{aligned} \frac{\| u \|^{P_M^+}}{N^{P_M^+ -1}} = N(\frac{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(.)} + |u|_{p_M(.)}}{N})^{P_M^+} &\leq \sum_{i=1}^N |\partial_{x_i} u|^{P_M^+}_{p_i(.)} + |u|^{P_M^+}_{p_M(.)} \\ &\leq \sum_{i=1}^N |\partial_{x_i} u|^{p_i^+}_{p_i(.)} + |u|^{P_M^+}_{p_M(.)} \\ &\leq \int_\Omega \left[\sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} + |u|^{P_M(x)} \right] dx \end{aligned} \quad (4)$$

Relations (3)-(4) imply

$$I(u) \geq \frac{1}{P_M^+ N^{P_M^+ -1}} \| u \|^{P_M^+} - \varepsilon \int_\Omega | u |^{P_M^+} \, dx - C(\varepsilon) \int_\Omega | u |^{q(x)} \, dx \quad (5)$$

By the condition (f_0), it follows

$$P_M^+ < q^- \leq q(x) < P_{-,\infty}$$

then

$$X \subset L^{P_M^+}(\Omega) \text{ and } X \subset L^{q(x)}(\Omega),$$

with a continuous and compact embedding, what implies the existence of C_4, $C_5 > 0$ such that

$$\| u \|_{L^{P_M^+}} \leq C_4 \| u \| \quad \text{and} \quad | u |_{q(x)} \leq C_5 \| u \|$$

for all $u \in X$. Since $\| u \|$ is small enough, we deduce

$$\int_\Omega | u |^{q(x)} \leq | u |^{q^-}_{q(x)} \leq C_6 \| u \|^{q^-} .$$

Replacing in (5), it results that

$$I(u) \geq \frac{1}{P_M^+ N^{P_M^+ -1}} \| u \|^{P_M^+} - \varepsilon C_4^{P_M^+} \| u \|^{P_M^+} - C_7 \| u \|^{q^-},$$

with C_i are positives constants. Let us choose $\varepsilon > 0$ such that $\varepsilon C_4^{P_M^+} \leq \frac{1}{2 P_M^+ N^{P_M^+ -1}}$, we obtain

$$\begin{aligned} I(u) &\geq \frac{1}{2 P_M^+ N^{P_M^+ -1}} \| u \|^{P_M^+} - C_7 \| u \|^{q^-} \\ &\geq \| u \|^{P_M^+} (\frac{1}{2 P_M^+ N^{P_M^+ -1}} - C_7 \| u \|^{q^- - P_M^+}). \end{aligned}$$

Since $P_M^+ < q^-$, the function $t \to (\frac{1}{2 P_M^+ N^{P_M^+ -1}} - C_7 t^{q^- - P_M^+})$ is strictly positive in a neighborhood of zero. It follows that there exist $r > 0$ and $\alpha > 0$ such that

$$I(u) \geq \alpha \; \forall u \in X : \| u \| = r.$$

The proof is completed. □

Proof of Theorem 1.1. In order to apply the Mountain Pass Theorem, we must prove that

$$I(su) \to -\infty \text{ as s} \to +\infty,$$

for a certain $u \in X$. From the condition (f_1), we obtain

$$F(x,t) \geq c \mid t \mid^{\theta} \text{ for all (x,t)} \in \bar{\Omega} \times \mathbb{R}.$$

Let $u \in X$ and $s > 1$ we have

$$\begin{aligned} I(su) &= \int_\Omega \left[\sum_{i=1}^N \frac{s^{p_i(x)}}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} + \frac{s^{p_M(x)}}{p_M(x)} |u|^{p_M(x)} \right] dx + \int_{\partial\Omega} \beta(x) \frac{s^{p_m(x)}}{p_m(x)} |u|^{p_m(x)} \, d\sigma \\ &- \int_\Omega F(x,(su))\, dx, \\ &\leq s^{P_M^+} \int_\Omega \left[\sum_{i=1}^N \frac{1}{p_i(x)} |\partial_{x_i} u|^{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)} \right] dx + s^{P_M^+} \int_{\partial\Omega} \beta(x) \frac{|u|^{p_m(x)}}{p_m(x)} \, d\sigma \\ &- cs^{\theta} \int_\Omega \mid u \mid^{\theta} \, dx. \end{aligned}$$

The fact $\theta > P_M^+$, gives that

$$I(su) \to -\infty \text{ as s} \to +\infty.$$

It follows that there exists $e \in X$ such that $\parallel e \parallel > r$ and $I(e) < 0$.
According to the Mountain Pass Theorem, I admits a critical value $\mu \geq \alpha$ which is characterized by

$$\mu = \inf_{h \in \Lambda} \sup_{t \in [0,1]} I(h(t))$$

where

$$\Lambda = \{h \in C([0,1], X) : h(0) = 0 \text{ and h(1) = e}\}.$$

Then, the functional I has a critical point u with $I(u) \geq \alpha$. But, $I(0) = 0$, that is, $u \neq 0$. Therefore, the problem (P) has a nontrivial solution.

We split the proof of the second result into five lemmas as follows.

Lemma 3.3. *There exists $\lambda_1 > 0$ such that*

$$\lambda_1 = \inf_{u \in V} \frac{\int_\Omega \left\{ \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)} \right\} dx}{\int_\Omega |u|^{P_m^-} \, dx}.$$

where

$$V = \{u \in X : \|u\| > N+1\},$$

Proof. For a given $u \in V$, there exists $j \in \{1, 2, ..., N\}$ such that $|\partial_{x_j} u|_{p_j(x)} > 1$, or $|u|_{p_M(x)} > 1$.
If $|\partial_{x_j} u|_{p_j(x)} > 1$, then

$$\int_\Omega \frac{|\partial_{x_j} u|^{p_j(x)}}{p_j(x)} \, dx \geq \frac{1}{P_M^+} |\partial_{x_j} u|_{p_j(x)}^{P_m^-}. \tag{6}$$

Since $L^{p_j(x)}(\Omega)$ is continuously embedded in $L^{P_m^-}(\Omega)$, we infer that $|\partial_{x_j}u|_{P_m^-} \leq C_j|\partial_{x_j}u|_{p_j(x)}$, where $C_j > 0$, in other way

$$|\partial_{x_j}u|_{p_j(x)}^{P_m^-} \geq \frac{1}{C_j^{P_m^-}} \int_\Omega |\partial_{x_j}u|^{P_m^-}\, dx. \tag{7}$$

Using the relation (11) proved in [11], we obtain

$$\int_\Omega |\partial_{x_j}u|^{P_m^-}\, dx \geq M_j \int_\Omega |u|^{P_m^-}\, dx, \tag{8}$$

where $M_j > 0$.
From relations (6), (7) and (8), we deduce that there exists a constant A such that

$$\int_\Omega \frac{|\partial_{x_j}u|^{p_j(x)}}{p_j(x)}\, dx \geq A \int_\Omega |u|^{P_m^-}\, dx,$$

where $A = \min_{j\in\{1,\ldots,N\}} \frac{M_j}{P_M^+ C_j^{P_m^-}}$.
Therefore

$$\int_\Omega \left\{ \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)} \right\} dx \geq A \int_\Omega |u|^{P_m^-}\, dx.$$

If $|u|_{p_M(x)} > 1$, then

$$\int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)}\, dx \geq \frac{1}{P_M^+}|u|_{p_M(x)}^{P_m^-}.$$

As the embedding $L^{p_M(x)}(\Omega) \hookrightarrow L^{P_m^-}(\Omega)$ is continuous, so there exists $D > 0$ such that $|u|_{P_m^-} \leq D|u|_{p_M(x)}$, or $|u|_{p_M(x)}^{P_m^-} \geq \frac{1}{D^{P_m^-}}|u|_{P_m^-}^{P_m^-}$.
It follows

$$\int_\Omega \frac{|u|^{p_M(x)}}{p_M(x)}\, dx \geq \frac{1}{P_M^+ D^{P_m^-}} \int_\Omega |u|^{P_m^-}\, dx.$$

Consequently,

$$\int_\Omega \left\{ \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)} \right\} dx \geq K \int_\Omega |u|^{P_m^-}\, dx,$$

where $K = \frac{1}{P_M^+ D^{P_m^-}}$.
Hence for $u \in V$, we have

$$\int_\Omega \left\{ \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)} \right\} dx \geq K_1 \int_\Omega |u|^{P_m^-}\, dx,$$

where $K_1 = \min\{A, K\}$.
According to this inequality, we can see easily that $\lambda_1 > 0$. □

Lemma 3.4. *There exist $a > 0$, and $\tilde{f} \in C(\Omega \times \mathbb{R})$ such that*

(1) $\tilde{f}(x, -t) = -\tilde{f}(x, t)$, *for every $x \in \Omega$ and $t \in \mathbb{R}$.*
(2) $\tilde{f}(x, t) = f(x, t)$ *for all $|t| < a$.*
(3) $\tilde{\mathcal{F}}(x, t) = P_m^- \tilde{F}(x, t) - \tilde{f}(x, t)t \geq 0$, *for every $x \in \Omega$ and $t \in \mathbb{R}$, where* $\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s)\, ds$.

(4) $\tilde{\mathcal{F}}(x,t) = 0$ *for every* $|t| > 2a$ *or* $t = 0$.

Proof. From (f_3), there exists $a \in]0, \frac{\delta}{2}[$ such that

$$F(x,t) \geq \frac{\lambda_1}{4}|t|^{P_m^-}, \quad \forall |t| \leq 2a.$$

Let us define a cut-off function h as follow

$$h(t) = \begin{cases} 1 & \text{if } |t| \leq a. \\ 0 & \text{if } |t| \geq 2a, \end{cases}$$

and $h^{'}(t)t \leq 0,\ |h^{'}(t)| \leq \frac{2}{a}$.
Using this cut-off function, we can define

$$\tilde{F}(x,t) = h(t)F(x,t) + \frac{\lambda_1}{4}(1-h(t))|t|^{P_m^-}.$$

By the definition of the function h and $\tilde{F}$, we deduce there exists $B > 0$ such that

$$\frac{\lambda_1}{4}|t|^{P_m^-} \leq \tilde{F}(x,t) \leq B + \frac{\lambda_1}{2}|t|^{P_m^-}, \forall (x,t) \in \Omega \times \mathbb{R}. \tag{9}$$

On the other hand, we have

$$\tilde{f}(x,t) = \frac{\partial}{\partial t}\tilde{F}(x,t) = h^{'}(t)F(x,t) + h(t)f(x,t) + \frac{\lambda_1}{4}P_m^-(1-h(t))|t|^{P_m^- - 2}t - \frac{\lambda_1}{4}h^{'}(t)|t|^{P_m^-}.$$

and

$$\tilde{\mathcal{F}}(x,t) = h(t)\mathcal{F}(x,t) + h^{'}(t)t(\frac{\lambda_1}{4}|t|^{P_m^-} - F(x,t)). \tag{10}$$

From the definition of h, (f_4), and (f_5), obviously 1., 2., 3., and 4. of the Lemma above are fulfilled for $|t| \leq a$ and $|t| \geq 2a$. In order to finish the proof, let's take $a \leq |t| \leq 2a$, we know that $F(x,t) \geq \frac{\lambda_1}{4}|t|^{P_m^-}$ and $h^{'}(t)t \leq 0$, then from (10) we have $\tilde{\mathcal{F}}(x,t) > 0$. ☐

Now, we extend the functional associated with the problem (P) to the functional

$$\tilde{I}(u) = I_1(u) - \int_\Omega \tilde{F}(x,u)\,dx, \qquad \text{for} \quad u \in X.$$

Lemma 3.5. *Suppose that* (f_4) *is satisfied, so if* $\langle \tilde{I}^{'}(u), u\rangle = 0$ *then* $u = 0$.

Proof. We have

$$P_m^-\left[\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)}\,dx + \int_{\partial\Omega}\beta(x)\frac{|u|^{p_m(x)}}{p_m(x)}\,d\sigma\right] = P_m^-\int_\Omega \tilde{F}(x,u)\,dx,$$

and

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i}u|^{p_i(x)} + |u|^{p_M(x)}\,dx + \int_{\partial\Omega}\beta(x)|u|^{p_m(x)}\,d\sigma = \int_\Omega \tilde{f}(x,u)u\,dx.$$

Therfore, we have

$$P_m^-\left[\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)}\,dx + \int_{\partial\Omega}\beta(x)\frac{|u|^{p_m(x)}}{p_m(x)}\,d\sigma\right] \leq$$

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} + |u|^{p_M(x)}\, dx + \int_{\partial\Omega} \beta(x)|u|^{p_m(x)}\, d\sigma.$$

Then

$$P_m^- \int_\Omega \tilde{F}(x,u)\, dx \le \int_\Omega \tilde{f}(x,u)u\, dx,$$

finally

$$\int_\Omega P_m^- \tilde{F}(x,u) - \int_\Omega \tilde{f}(x,u)u\, dx \le 0,$$

by (f_4), we obtain that $u = 0$. □

Lemma 3.6. *The functional $\tilde{I}$ is even and satisfies the (PS) condition.*

Proof. Using Lemma 3.4 we can see easily that $\tilde{I}$ is even and $\tilde{I} \in C^1(X, \mathbb{R})$. Since $\beta(.) \ge 0$ and for $\|u\| > N+1$, we have

$$\begin{aligned}
\tilde{I}(u) &= \int_\Omega \left[\sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)}\right] dx + \int_{\partial\Omega} \beta(x)\frac{|u|^{p_m(x)}}{p_m(x)}\, d\sigma - \int_\Omega \tilde{F}(x,u)\, dx \\
&\ge \int_\Omega \left\{\sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)}\right\} dx - \int_\Omega \tilde{F}(x,u)\, dx \\
&\ge \int_\Omega \left\{\sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)}\right\} dx - \frac{\lambda_1}{2}\int_\Omega |u|^{P_m^-} - B|\Omega|. \\
&\ge \frac{1}{2}\int_\Omega \left\{\sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} + \frac{|u|^{p_M(x)}}{p_M(x)}\right\} dx - B|\Omega| \\
&\ge \frac{1}{2P_M^+}\left\{\sum_{i=1}^N |\partial_{x_i} u|_{p_i(x)}^{P_m^-} - N + |u|_{p_M(x)}^{P_m^-} - 1\right\} - B|\Omega| \\
&\ge \frac{\|u\|^{P_m^-}}{2P_M^+(N+1)^{P_m^- - 1}} - \frac{N+1}{2P_M^+} - B|\Omega|,
\end{aligned}$$

where $|\Omega|$ is the measure of Ω, therefore $\tilde{I}(u)$ is coercive. Then any $(PS)_c$ sequence of $\tilde{I}$ is bounded. Using a standard argument, we show that $\tilde{I}$ verifies $(PS)_c$ condition on X for all c. □

Lemma 3.7. *For every $k \in \mathbb{N}$, there exists ρ_k such that*

$$\sup_{X_k \bigcap S_{\rho_k}} \tilde{I} < 0.$$

Proof. For every $k \in \mathbb{N}$ we have k independent smooth functions φ_i, for $i = 1, 2, ..., k$, let's define the subspace $X_k = span\{\varphi_1, ..., \varphi_k\}$. For the moment, using again (f_0) we can find a constant $d > \frac{\lambda_1}{4}$ satisfying $\tilde{F}(x,t) > d|t|^{P_m^-}$ for $|t| < \varepsilon < \frac{\delta}{2}$. For any $v \in X_k \setminus \{0\}$ such that $\|v\| = 1$, and $|\rho_k v(x)| < \varepsilon$ with $0 < \rho_k < 1$, and $x \in \Omega$, and from Lemma 3.4, we have

$$\begin{aligned}\tilde{I}(\rho_k v) &= I_1(\rho_k v) - \int_\Omega \tilde{F}(x, \rho_k v)\, dx \\ &\leq I_1(\rho_k v) - d\rho_k^{P_m^-} \int_\Omega |v|^{P_m^-}\, dx.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}I_1(\rho_k v) &\leq \frac{\rho_k^{P_m^-}}{P_m^-} \sum_{i=1}^N \int_\Omega |\partial_{x_i} v|^{p_i(x)}\, dx + \frac{\rho_k^{P_m^-}}{P_m^-} \int_\Omega |v|^{p_M(x)}\, dx + \frac{\rho_k^{P_m^-}}{P_m^-} \int_{\partial\Omega} \beta(x)|v|^{p_m(x)}\, d\sigma \\ &\leq \frac{\rho_k^{P_m^-}}{P_m^-} \left(\sum_{i=1}^N \int_\Omega |\partial_{x_i} v|^{p_i(x)}\, dx + \int_\Omega |v|^{p_M(x)}\, dx + \int_{\partial\Omega} \beta(x)|v|^{p_m(x)}\, d\sigma \right).\end{aligned}$$

Therefore, from 2) of proposition 2.3 and $\beta \in L^\infty$, there exists $C > 0$ such that

$$\tilde{I}(\rho_k v) \leq C \frac{\rho_k^{P_m^-}}{P_m^-} \left(\sum_{i=1}^N \int_\Omega |\partial_{x_i} v|^{p_i(x)}\, dx + \int_\Omega |v|^{p_M(x)}\, dx \right) - d\rho_k^{P_m^-} \int_\Omega |v|^{P_m^-}\, dx.$$

Since all norms on the finite dimensional vector space X_k are equivalent, put $u = \rho_k v$, so for ρ_k small enough and for d large enough, we obtain

$$\sup_{X_k \bigcap S_{\rho_k}} \tilde{I} < 0.$$

□

Proof of Theorem 1.2. From Lemma 3.5, Lemma 3.6 and Lemma 3.7, we deduce that the conditions of Proposition 2.5 are fulfilled, and then there exists a sequence of negative critical values c_k for the functional $\tilde{I}$ which verifies $c_k \to 0$ when k is large enough.

Thereby, for any u_k satisfying $\tilde{I}(u_k) = c_k$ and $\tilde{I}'(u_k) = 0$, we have $(u_k)_k$ is $(PS)_0$ sequence of $\tilde{I}(u)$, then the subsequence still denoted (u_k) has a limit.

Using again Lemma 3.5 and Lemma 3.4, we infer that 0 is the only critical point when the energy is zero, then the subsequence (u_k) has to converge to zero.

In order to achieve the proof we apply the results of regularity in [4]. Hence $(u_k) \in C(\overline{\Omega})$ and $|u_k|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$.

Finally from Lemma 3.4, we get $|u_n|_{C(\Omega)} \leq a$, so the sequence $(u_k)_k$ are solutions of the problem (P). □

Proof of Theorem 1.3. Firstly, we begin by proving of the third result into two claims as follows.

Claim: There exists $\delta > 0$ be small enough such that

$$\mathcal{F}_1(x,t) = P_m^- F_1(x,t) - f(x,t)t > 0,$$

for every $x \in \Omega$, $|t| \leq \delta$ and $t \neq 0$,

Proof. In view of the condition (H_3), for the case $t > 0$ and $s \in]0, t[$ we have

$$f_1(x,s) > \frac{f_1(x,t)}{|t|^{P_m^- - 2} t} |s|^{P_m^- - 2} s.$$

Integrating this inequality over $]0,t[$, we deduce

$$F_1(x,t) = \int_0^t f_1(x,s)\,ds > \frac{f_1(x,t)}{|t|^{P_m^- -2}t}\int_0^t |s|^{P_m^- -2}s\,ds = \frac{1}{P_m^-} f_1(x,t)t.$$

Thus, $P_m^- F_1(x,t) - f_1(x,t)t > 0$, in $t \in]0,\delta[$.

Since $f(x,.)$ is odd and so $F(x,.)$ is even. It is easy to see that $P_m^- F_1(x,t) - f_1(x,t)t > 0$, in $t \in]-\delta, 0[$.

Therfore,the claim follows. □

Claim: Assume that $(H_1) - (H_5)$ are satisfied, then $f = f_1 + f_2$ satisfies the conditions (f_0) and $(f_3) - (f_5)$.

Proof. It is easy to see that

$$\frac{F(x,t)}{|t|^{P_m^-}} = \frac{F_1(x,t) + F_2(x,t)}{|t|^{P_m^-}} = \left(1 + \frac{F_2(x,t)}{F_1(x,t)}\right)\frac{F_1(x,t)}{|t|^{P_m^-}}$$

and

$$\mathcal{F}(x,t) = \mathcal{F}_1(x,t) + \mathcal{F}_2(x,t) = \left(1 + \frac{\mathcal{F}_2(x,t)}{\mathcal{F}_1(x,t)}\right)\mathcal{F}_1(x,t).$$

Now, we choose $\varepsilon \in (0, \min\left(\frac{b_1+1}{2}, \frac{b_2+1}{2}\right)$, then by (H_2) , (H_3) and (H_4) there exist $\delta > 0$ such that

$$1 + \frac{F_2(x,t)}{F_1(x,t)} \geq 1 + b_1 - \varepsilon \geq \frac{b_1+1}{2} > 0$$

and

$$1 + \frac{\mathcal{F}_2(x,t)}{\mathcal{F}_1(x,t)} \geq 1 + b_2 - \varepsilon \geq \frac{b_2+1}{2} > 0,$$

for $|t| \leq \delta$ and a.e. $x \in \Omega$.

Then, we obtain easily (f_3) and (f_4) . So, (f_5) with $\delta \leq \gamma$, follows from (H_5). □

Finally, by using Theorem 1.2, we infer that the problem (P) has a sequence of weak solutions such that $\|u_n\|_{L^\infty} \to 0$ as $n \to \infty$. □

Proof of Corollary 1.1. Take

$$f_1(x,t) = \lambda m(x)|t|^{q_1(x)-2}t \text{ and } f_2(x,t) = \lambda n(x)|t|^{q_2(x)-2}t,$$

we can see that f_1 and f_2 still satisfies the above conditions used in the Theorem 1.3. Thus, the result is a consequence of Theorem 1.3. □

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