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On Lacunary $\mathcal{I}$-Convergence of Sequence in Credibility Space

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Abstract. The aim of this research article is to introduce the concept of strongly lacunary ideal convergence of sequences of fuzzy variables in a credibility space in different directions. We define strongly lacunary ideal convergence via credibility measure, credibility distribution function and expected value of the fuzzy variables which are indeed elements of the sequences. Furthermore, we initiate two more notions of convergences by reducing the size of the domain of fuzzy variables. We show existence of such types of sequences and establish interrelationships between different notions.

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1. Introduction

The notion of statistical convergence for sequences of real numbers was put forward by Fast [9]. But the research on this concept got momentum once the works of Šalát [32] and Fridy [10] came into literature. Statistical convergence of sequences has been studied by various authors from different aspects. In recent days, the works on statistical convergence are being mainly investigated in two environment: one using theory of uncertainty and another one is using the theory of credibility. The notion of statistical convergence in an uncertainty space is introduced by Nath and Tripathy [36] and different properties are studied by Das et al. [2, 3] and Nath et al. [30]. The same notion has been extended to double and triple sequences of complex uncertain variable by Das et al. [4, 5]. They [6, 7] also characterized statistical convergence in triple sequence by considering complex uncertain variables. On the other hand, Savaş et al. [34] initiated the study of statistical convergence in a credibility space.
The notion of $\mathcal{I}$-convergence was studied at initial stage by Kostyrko et al. [17]. Kostyrko et al. [18] examined some of basic properties of $\mathcal{I}$-convergence and dealt with extremal $\mathcal{I}$-limit points. Later on it was studied by Kişi et al. [14], Khan et al. [15, 16], Šalát et al. [33], Tripathy and Hazarika [35], Tripathy and Tripathy [37] and many others. Fridy and Orhan [12] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in [11, 39]. The notion of lacunary ideal convergence of real sequences was introduced in [38].

The theory of fuzzy sets was originally proposed by Zadeh through membership degree function in the year 1965 [41]. For measuring a fuzzy event, he further initiated the notions of possibility measure and necessary measure. It is observed that both

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the measures are normal, non-negative and monotone in nature [8, 31]. But they do not satisfy the law of truth conservation and both of them are inconsistent with laws of exclude middle and contradiction. The reason for such behavior is that they do not satisfy the self-duality property which is intuitive and important both theory and practice purposes. To address this issue, Liu and Liu presented the concept of credibility measure [26], which is a self-dual measure. Credibility theory, founded by Liu in 2006 [24] and refined by Liu in 2007 [25], is a new branch of mathematics for studying the behavior of fuzzy phenomena. Lately, the merging of fuzzy variables (FVs) has gained prominence within credibility theory, finding practical applications in engineering and mathematical finance problems.

Credibility theory and uncertainty theory are two distinct concepts used to analyze properties of sequence spaces. In credibility theory, sequences of FVs are examined, while in uncertainty theory, uncertain variables in sequence spaces are considered. Despite their differences, both theories explore the convergence of sequences from similar perspectives. It's important to note that a FV maps from a possibility space to the set of real numbers [31], whereas an uncertain variable maps from the uncertainty space to the set of real numbers [25]. Since then, the credibility theory has been developed rapidly and applied widely, one may refer to [1, 13, 19, 20, 21, 22, 24, 25, 27, 29, 40]. The generalization of lacunary ideal convergence for sequences in credibility theory remains an open question. Thus, the primary objective of this article is to introduce a novel form of convergence in credibility theory tailored for sequences of FVs.

This paper is devoted to present a new kind of convergence for FVs sequences. In Section 2, some preliminary definitions and theorems related to FVs sequences, credibility space are presented. In Section 3, we introduce the notion of lacunary ideal convergence of FVs in a credibility spaces in different aspects. We establish some inclusive interrelationships and characterize the notions through producing fundamental properties of the lacunary ideal convergence of sequences using credibility theory.

2. Preliminaries

In this initial section, we present the definitions of several key concepts and relevant results that will be instrumental in establishing the outcomes of our research. Firstly, we introduce the notion of a credibility measure, along with the concept of expected value for FVs based on such measures. Additionally, we discuss the bounded convergence theorem proposed by Liu and Wang [28], as well as explore the interrelationships among various convergence concepts for sequences of FVs in a credibility space.

Definition 2.1. (Liu and Liu [26]) A set function Cr is defined as a credibility measure if it fulfills the following axioms: Let Θ be a nonempty set, and $\mathcal{P}(\Theta)$ be the power set of Θ, which represents all possible subsets of Θ (in other words, the largest algebra over Θ). Each subset in $\mathcal{P}$ is considered an event. For any subset $V \in \mathcal{P}(\Theta)$, $\mathrm{Cr}\{V\}$ denotes the degree of possibility that the fuzzy event V occurs. The set function $\mathrm{Cr}\{.\}$ is classified as a credibility measure iff it satisfies the following conditions:

Axiom i. $\mathrm{Cr}\{\Theta\} = 1$;

Axiom ii. $\mathrm{Cr}\{V\} \leq \mathrm{Cr}\{U\}$ whenever $V \subset U$;

Axiom iii. Cr is self-dual, i.e., $\mathrm{Cr}\{V\} + \mathrm{Cr}\{V^c\} = 1$, for any $V \in \mathcal{P}(\Theta)$;
Axiom iv. $\mathrm{Cr}\{\cup_i V_i\} = \sup_i \mathrm{Cr}\{V_i\}$ for any collection $\{V_i\}$ in $\mathcal{P}(\Theta)$ with $\sup_i \mathrm{Cr}\{V_i\} < 0.5$.
The triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is called a credibility space.

Liu and Liu [26] investigated the usage of FVs within credibility theory, treating them as functions from the credibility space to the set of real numbers.

Definition 2.2. (Li and Liu [20]) The expected value of FV ξ is given by

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\, dr - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \leq r\}\, dr$$

provided that at least one of the two integrals is finite.

Lemma 2.1. *(Wang and Liu [40]) If the sequence $\{\zeta_i\}$ convergence in credibility to ζ, then $\{\zeta_i\}$ converges almost surely to ζ.*

Lemma 2.2. *(Liu, [23]) If the sequence $\{\zeta_i\}$ convergence in mean to ζ, then $\{\zeta_i\}$ converges credibility to ζ.*

Lemma 2.3. *(Wang and Liu [40]) If the sequence $\{\zeta_i\}$ convergence in credibility to ζ, then $\{\zeta_i\}$ converges almost surely to ζ.*

Definition 2.3. (Freedman, Sember and Raphael [42]) A lacunary sequence is an increasing integer sequence $\{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$, with $q_r = \frac{k_r}{k_{r-1}}$.

3. New Concepts

In this section, we introduce special types of FV sequences in a given credibility space. The notion of statistical convergence in a credibility space is investigated by Savaş et al. [34]. Kişi et al. [14] examined the extended form of this notion, called lacunary statistical convergence notion within the same setting. This research article introduce more generalized version of both the above-mentioned notions named as lacunary ideal convergence in a credibility space from five aspects. Three of these are using credibility measure function (named as strongly lacunary ideal convergence in credibility), using expected value operator (named as strongly lacunary ideal convergence in mean), using credibility distribution operator (classified as strongly lacunary ideal convergence in distribution). The remaining two are defined by reducing the domain of the sequences (referred as strongly lacunary ideal convergence in almost surely and uniformly almost surely).

Definition 3.1. Strongly lacunary $\mathcal{I}$-convergence in almost surely:

Let $\{\zeta_k\}$ be a sequence of FV in a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$. The sequence $\{\zeta_i\}$ is said to be strongly lacunary $\mathcal{I}$-convergent in almost surely to a FV ζ if there exists a set $A \in \mathcal{P}(\Theta)$ with unit credibility measure such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \varepsilon \right\} \in \mathcal{I},$$

for any $\phi \in A$ and $\varepsilon > 0$.

Definition 3.2. Strongly lacunary $\mathcal{I}$-convergence in credibility:

The sequence $\{\zeta_k\}$ of FV in $(\Theta, \mathcal{P}(\Theta), Cr)$ is said to be strongly lacunary $\mathcal{I}$-convergent in credibility to ζ if for any preassigned $\varepsilon > 0$ and $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon \right\} \in \mathcal{I}.$$

Definition 3.3. Strongly lacunary $\mathcal{I}$-convergence in mean:

The sequence $\{\zeta_k\}$ of FV in a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ is called strongly lacunary $\mathcal{I}$-convergent in mean to a FV ζ if all the FVs ζ, ζ_k $(k = 1, 2, 3, ...)$ posses finite expected values and for any arbitrarily chosen $\varepsilon > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} E[||\zeta_k - \zeta||] \geq \varepsilon \right\} \in \mathcal{I}.$$

Definition 3.4. Strongly lacunary $\mathcal{I}$-convergence in distribution:

Let $\{\zeta_k\}$ be a FV sequence defined in a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ and Φ_k be the credibility distribution functions for the FV ζ_k. Then, we say that the sequence $\{\zeta_k\}$ is called strongly lacunary $\mathcal{I}$-convergent in distribution to a FV ζ whose credibility distribution function is Φ if for any given $\varepsilon > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |\Phi_k(x) - \Phi(x)| \geq \varepsilon \right\} \in \mathcal{I},$$

where x is any real number where Φ is continuous.

Definition 3.5. Strongly lacunary $\mathcal{I}$-convergence in uniformly almost surely:

The sequence $\{\zeta_k\}$ of FVs in the space space $(\Theta, \mathcal{P}(\Theta), Cr)$ is called strongly lacunary $\mathcal{I}$-convergent with respect to uniformly almost surely to a FV ζ if there exists some events A_k $(k \in \mathbb{N})$ each of whose credibility measure approaches zero such that the sequence is uniformly lacunary converges to the same limit in the sense of ideal. In this case

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \varepsilon \right\} \in \mathcal{I},$$

for all $\phi \in \Theta - A_k$, and $\varepsilon > 0$.

4. Main Results

In this section, we produce several results which establishes interconnection between several types of strongly lacunary ideal convergent sequences of FVs in a credibility space.

Strongly lacunary $\mathcal{I}$-convergence in mean VS Strongly lacunary $\mathcal{I}$-convergence in credibility

Theorem 4.1. *If a FV sequence is strongly lacunary ideal convergent in mean to some limit then it is also strongly lacunary ideal convergent to the same limit in credibility.*

Proof. Let $\{\zeta_k\}$ be sequence of FV in credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$, which strongly lacunary $\mathcal{I}$-convergence in mean to the FV ζ.
Then, for any positive real number δ, we have by Markov inequality
$\mathrm{Cr}\{||\zeta_k - \zeta|| \geq \delta\} \leq \frac{E[||\zeta_k - \zeta||]}{\delta}$.

So, for any arbitrary $\varepsilon > 0$ and an increasing sequence of natural numbers $\{h_r\}$, we have

$$\left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon\right\}$$
$$= \left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{\phi : |\zeta_k(\phi) - \zeta(\phi)| \geq \delta\} \geq \varepsilon\right\}$$
$$\subseteq \left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} E\left[\phi : \tfrac{|\zeta_k(\phi) - \zeta(\phi)|}{\delta}\right] \geq \varepsilon\right\}$$
$$= \left\{r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} E\left[\tfrac{||\zeta_k - \zeta||}{\delta}\right] \geq \varepsilon\right\} \in \mathcal{I}.$$

Consequently, the FV sequence is strongly lacunary $\mathcal{I}$-convergent to ζ within the structure. □

Remark 4.1. The converse part of the above relation is not true in general. This claim is demonstrated in the following example.

Example 4.1. Strongly lacunary ideal convergence in credibility does not imply strongly lacunary ideal convergence in mean.
To establish this, we consider the credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ where the space $\Theta = \{\phi_1, \phi_2, ...\}$, and the credibility measure for the events are defined by $\mathrm{Cr}\,\{\phi_t\} = 1/t$ for $t = 1, 2, ...$
Consider the FV sequence $\{\zeta_k\}$, where each of the FVs are identified as

$$\zeta_k\,(\phi_t) = \begin{cases} k, & \text{if } t = k \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

for $k = 1, 2, ...$ and let another FV $\zeta = 0$.
Consider the FVs defined by (1) which does not strongly lacunary ideal converge in mean to ζ. But, for any small number $\varepsilon > 0$ and $\delta \in \left[\frac{1}{2}, 1\right)$, we get

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon\right\} \in \mathcal{I}$$

That is, the sequence $\{\zeta_k\}$ strongly lacunary ideal converges in credibility to ζ.

Strongly lacunary $\mathcal{I}$-convergence in almost surely VS Strongly lacunary $\mathcal{I}$-convergence in credibility

Remark 4.2. The concept of strongly lacunary ideal convergence in credibility and in almost surely are two independent to each other. The claim is illustrated in the following two consecutive examples.

Example 4.2. Strongly lacunary ideal convergence almost surely does not imply strongly lacunary ideal convergence in credibility.

Consider, $\Theta = \{\phi_1, \phi_2, ...\}$, $\mathrm{Cr}\{\phi_1\} = 1$ and $\mathrm{Cr}\{\phi_t\} = (t-1)/t$ for $t = 2, 3, ...$ and the FVs are defined by

$$\zeta_k(\phi_t) = \begin{cases} k, & \text{if } t = k \\ 0, & \text{otherwise} \end{cases}$$

for $k = 1, 2, ...$ and $\zeta = 0$. Then, the sequence $\{\zeta_k\}$ strongly lacunary ideal converges almost surely to ζ. However, for any small number $\delta > 0$ and $\varepsilon \in \left(0, \frac{1}{2}\right)$, the sequence $\{\zeta_k\}$ is not strongly lacunary ideal convergence in credibility. Also,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon \right\} \notin \mathcal{I}.$$

That is to say, the sequence $\{\zeta_k\}$ does not converge in credibility to ζ.

Example 4.3. Strongly lacunary ideal convergence in credibility does not imply strongly lacunary ideal convergence almost surely, too.
Take $\Theta = \{\phi_1, \phi_2, ...\}$, $\mathrm{Cr}\{\phi_t\} = 1/t$ for $t = 1, 2, ...$ and the FVs are defined by

$$\zeta_k(\phi_t) = \begin{cases} (t+1)/t, & \text{if } t = k, k+1, k+2, ... \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

for $k = 1, 2, ...$ and $\zeta = 0$. For any small number $\delta > 0$ and $\varepsilon \in \left[\frac{1}{2}, 1\right)$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon \right\} \in \mathcal{I}$$

which gives that $\{\zeta_k\}$ strongly lacunary ideal converges in credibility to ζ. But, it is obvious that $\{\zeta_k\}$ is not strongly lacunary $\mathcal{I}$-convergent in almost surely to ζ.

Strongly lacunary $\mathcal{I}$-convergence in mean VS Strongly lacunary $\mathcal{I}$-convergence in almost surely

Remark 4.3. The notions of strongly lacunary ideal convergence in mean and in almost surely do not imply each other. The following two consecutive examples justifies the claim.

Example 4.4. Strongly lacunary ideal convergence almost surely does not imply strongly lacunary ideal convergence in mean, too.
Consider $\Theta = \{\phi_1, \phi_2, ...\}$, $\mathrm{Cr}\{\phi_t\} = 1/t$ for $t = 1, 2, ...$ and the FVs are defines by

$$\zeta_k(\phi_t) = \begin{cases} k, & \text{if } t = k \\ 0, & \text{if not} \end{cases}$$

for $k = 1, 2, ...$ and $\zeta = 0$. Then, the sequence $\{\zeta_k\}$ strongly lacunary ideal converges almost surely to ζ. However, for any $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} E[||\zeta_k - \zeta||] \geq \varepsilon \right\} \notin \mathcal{I}.$$

Namely, the sequence $\{\zeta_k\}$ does not strongly lacunary ideal converge in mean to ζ.

Example 4.5. Strongly lacunary ideal convergence in mean does not imply strongly lacunary ideal convergence almost surely.

Consider the FVs defined by (2) which does not strongly lacunary ideal convergence almost surely to ζ. But

$$E\left[|\zeta_k - \zeta|\right] = \frac{k+1}{2k^2} \to 0.$$

So, for every $\varepsilon > 0$, we obtain

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} E[||\zeta_k - \zeta||] \geq \varepsilon \right\} \in \mathcal{I}.$$

which gives that $\{\zeta_k\}$ strongly lacunary ideal converges in mean to ζ.

To examine the relation between strongly lacunary ideal convergence with respect to uniformly almost surely and strongly lacunary ideal convergence in almost surely, we need to establish the following two propositions.

Theorem 4.2. *A sequence $\{\zeta_k\}$ of FVs in a credibility spaces strongly lacunary ideal converges almost surely to another FV ζ iff for any preassigned positive real number ε and δ*

$$\left\{ r \in \mathbb{N} : \mathrm{Cr} \left\{ \bigcap_{r \in I_{r_k}} \bigcup_{k \in I_r} \left\{ \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \right\} \geq \varepsilon \right\} \in \mathcal{I}.$$

Proof. According to the definition of strongly lacunary ideal converges almost surely, we have that there is $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \in \mathcal{I}$$

for each $\delta > 0$ and every $\phi \in A$.
For any $\varepsilon > 0$, there is a $m \in \mathbb{N}$ such that
$\frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| < \varepsilon$, where $k > m$
for any $A \in \mathcal{P}(\Theta)$, that is equivalent to

$$\left\{ r \in \mathbb{N} : \mathrm{Cr} \left\{ \bigcup_{r \in I_{r_k}} \bigcap_{k \in I_r} \left\{ \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \right\} \geq \varepsilon \right\} \in \mathcal{F}(\mathcal{I}).$$

From the duality axiom of credibility measure we obtain

$$\left\{ r \in \mathbb{N} : \mathrm{Cr} \left\{ \bigcap_{r \in I_{r_k}} \bigcup_{k \in I_r} \left\{ \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \right\} \geq \varepsilon \right\} \in \mathcal{I}.$$

□

Theorem 4.3. *Let $\zeta_1, \zeta_2, \ldots$ be FVs. Then, $\{\zeta_k\}$ is strongly lacunary $\mathcal{I}$-convergent uniformly almost surely to the FV ζ iff for any $\varepsilon, \delta > 0$, we have*

$$\left\{ r \in \mathbb{N} : \mathrm{Cr} \left\{ \bigcup_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \eta \right\} \right\} \geq \delta \right\} \in \mathcal{I}$$

Proof. Let the FV sequence $\{\zeta_k\}$ be strongly lacunary ideal convergent in uniformly almost surely to the FV ζ.
Then, for any $\delta > 0$, we can find subsets $A_j \subset \Theta$, for $j \in \mathbb{N}$ in the credibility space with $Cr\{A_j\} < \delta$, for each j (i.e. $Cr\{A_j\} \to 0$) and the sequence is strongly lacunary ideal convergent uniformly over the domain $P(\Theta) - A_j$, for each j.
This means, for any preassigned $\eta > 0$, there exists a positive real $m > 0$ such that $\frac{1}{h_r} \sum\limits_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| < \eta, \ \forall k > m$ and $\phi \in P(\Theta) - A_j$.

From the above, we can write

$$\bigcup_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \eta \right\} \subset A.$$

Taking credibility measure of both the sets in the above, we get

$$\mathrm{Cr} \left\{ \bigcup_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \eta \right\} \right\} < \mathrm{Cr}\{A\} < \delta.$$

Consequently, from the strongly lacunary ideal convergence view, we can write

$$\left\{ r \in \mathbb{N} : \mathrm{Cr} \left\{ \bigcup_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \eta \right\} \right\} \geq \delta \right\} \in \mathcal{I}$$

For the converse part, let us assume that the given conditions hold.
Then, for any given $\delta > 0$ with $a \geq 1$ there exists $N_0 \in \mathbb{N}$ such that

$$\mathrm{Cr} \left\{ \bigcup_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \frac{1}{a} \right\} \right\} \leq \frac{\delta}{2^a}, \quad \text{whenever } k \geq N_0.$$

Taking consideration of the above, let us now consider the set

$$A = \bigcup_{a=1}^{\infty} \bigcup_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \frac{1}{a} \right\}$$

Then, it is obvious that, $\mathrm{Cr}\{A\} \leq \sum\limits_{a=1}^{\infty} \frac{\delta}{2^a} = \delta$, which can be arbitrarily small (means tends to zero).

Consequently, we can say that the FV sequence is strongly lacunary ideal convergent with respect to uniformly almost convergent within the credibility space. □

Strongly lacunary $\mathcal{I}$-convergence in uniformly almost surely VS Strongly lacunary $\mathcal{I}$-convergence in almost surely

Theorem 4.4. *Strongly lacunary $\mathcal{I}$-convergent with respect to uniformly almost surely of a sequence of FV implies its strongly lacunary $\mathcal{I}$-convergence in almost surely with the preservation of limit in a given credibility space.*

Proof. Strongly lacunary ideal convergence of a FV sequence $\{\zeta_k\}$ with respect to uniformly almost surely to ζ gives the following from the Theorem 4.3,

$$\left\{ r \in \mathbb{N} : \mathrm{Cr} \left\{ \bigcup\nolimits_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \eta \right\} \right\} \geq \delta \right\} \in \mathcal{I}$$

Since

$$\begin{aligned} &\mathrm{Cr} \left\{ \bigcap\nolimits_{r \in I_{r_k}} \bigcup\nolimits_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \right\} \\ &\quad \leq \mathrm{Cr} \left\{ \bigcup\nolimits_{k \in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \right\}, \end{aligned}$$

so,

$$\left\{ r \in \mathbb{N} : \text{Cr}\left\{ \bigcap_{r\in I_{r_k}} \bigcup_{k\in I_r} \left\{ \phi : \frac{1}{h_r} \sum_{k\in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \right\} \geq \varepsilon \right\} \in \mathcal{I}.$$

Thus by the Theorem 4.2, we can say the sequence $\{\zeta_k\}$ of FVs strongly lacunary ideal converges to ζ in almost surely. □

Open Question: Is the converse part of the above theorem true? We contemplate that it is not. We are keeping the query about the example in this regard as an open question hereby.

Strongly lacunary $\mathcal{I}$-convergence in distribution VS Strongly lacunary $\mathcal{I}$-convergence in credibility

Theorem 4.5. *Suppose that $\phi, \phi_1, \phi_2, ...$ are the credibility distributions of the FVs ζ, ζ_1, ζ_2, ..., respectively. If the sequence $\{\zeta_k\}$ strongly lacunary ideal convergence in credibility to ζ, then $\{\zeta_k\}$ strongly lacunary ideal convergence in distribution to ζ.*

Proof. Let x be any given continuity point of the credibility distribution ϕ . On the one hand, for any $y > x$, we get

$$\begin{aligned} \{\phi : \zeta_k(\phi) \leq x\} &= \{\phi : \zeta_k(\phi) \leq x, \zeta(\phi) \leq y\} \cup \{\phi : \zeta_k(\phi) \leq x, \zeta(\phi) > y\} \\ &\subset \{\phi : \zeta(\phi) \leq y\} \cup \{\phi : |\zeta_k(\phi) - \zeta(\phi)| \geq y - x\} \end{aligned}$$

which implies that

$$\Phi_k(x) \leq \Phi(y) + \text{Cr}\{\phi : |\zeta_k(\phi) - \zeta(\phi)| \geq y - x\}.$$

Since $\{\zeta_k\}$ strongly lacunary ideal convergence in credibility to ζ, we get

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \text{Cr}\{\phi : |\zeta_k(\phi) - \zeta(\phi)| \geq y - x\} \geq \delta \right\} \in \mathcal{I}.$$

for any $\delta > 0$.
Thus, we obtain

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \sup_k \{|\Phi_k(x) - \Phi(x)| \geq \delta\} \right\} \in \mathcal{I} \tag{3}$$

for all δ as $y \to x$.
On the other hand, for any $z < x$, we have

$$\begin{aligned} \{\phi : \zeta(\phi) \leq z\} &= \{\phi : \zeta(\phi) \leq z, \zeta_k(\phi) \leq x\} \cup \{\phi : \zeta(\phi) \leq z, \zeta_k(\phi) > x\} \\ &\subset \{\phi : \zeta_k(\phi) \leq x\} \cup \{\phi : |\zeta_k(\phi) - \zeta(\phi)| \geq x - z\} \end{aligned}$$

which implies that

$$\Phi(z) \leq \Phi_k(x) + Cr\{\phi : |\zeta_k(\phi) - \zeta(\phi)| \geq x - z\}.$$

Since

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \text{Cr}\{\phi : |\zeta_k(\phi) - \zeta(\phi)| \geq x - z\} \geq \delta \right\} \in \mathcal{I}.$$

Thus, we obtain

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k\in I_r} \inf_k \{|\Phi_k(x) - \Phi(x)| \geq \delta\} \right\} \in \mathcal{I} \tag{4}$$

for all δ as $x \to z$. It follows from (3) and (4) that $\{\zeta_k\}$ is strongly lacunary ideal convergent in distribution to ζ. The theorem is proved. □

The converse of the above theorem is not true, in general. This means, strongly lacunary ideal convergence in distribution of a FV sequence doesn't imply strongly lacunary ideal convergence in credibility to the same limit. We present an example which proves our claim.

Example 4.6. Consider the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ to be $\{\Phi_1, \Phi_2, ...\}$ with $\mathrm{Cr}\{\Phi_1\} = \mathrm{Cr}\{\Phi_2\} = \frac{1}{2}$. Define the FV as

$$\zeta(\Phi) = \begin{cases} 1, & \text{if } \Phi = \Phi_1; \\ -1, & \text{if } \Phi = \Phi_2. \end{cases}$$

We also take $\{\zeta_k\} = -\zeta$ for $k \in I_r$. Then, $\{\zeta_i\}$ and ζ have the same credibility distribution functions and so $\{\zeta_k\}$ strongly lacunary $\mathcal{I}$-converges in distribution to ζ. But, for any given $\varepsilon > 0$ and $\delta > 0$, we have

$$\left\{ r \in \mathbb{N} : \mathrm{Cr}\left\{ \tfrac{1}{h_r} \sum_{k \in I_r} ||\zeta_k - \zeta|| \geq \delta \right\} \geq \varepsilon \right\}$$
$$\subseteq \left\{ r \in \mathbb{N} : \mathrm{Cr}\left\{ \tfrac{1}{h_r} \sum_{k \in I_r} ||2\zeta_k|| \geq \delta \right\} \geq \varepsilon \right\} \notin \mathcal{I}.$$

Therefore, the FV sequence $\{\zeta_k\}$ does not strongly lacunary $\mathcal{I}$-converge in credibility to ζ.

Strongly lacunary $\mathcal{I}$-convergence in credibility VS Strongly lacunary $\mathcal{I}$-convergence in uniformly almost surely

Theorem 4.6. *If $\{\zeta_k\}$ is strongly lacunary $\mathcal{I}$-convergent uniformly almost surely to a FV ζ, then $\{\zeta_k\}$ is strongly lacunary $\mathcal{I}$-convergent in credibility to ζ.*

Proof. If $\{\zeta_k\}$ is strongly lacunary $\mathcal{I}$-convergent uniformly almost surely to a FV ζ, then we have from Theorem 4.3,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\left\{ \bigcap_{r \in I_{r_k}} \bigcup_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \geq \varepsilon \right\} \in \mathcal{I}.$$

But

$$\left\{ r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{|\zeta_k(\phi) - \zeta(\phi)| \geq \delta\} \geq \varepsilon \right\}$$
$$\subseteq \left\{ r \in \mathbb{N} : \tfrac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\left\{ \bigcap_{r \in I_{r_k}} \bigcup_{k \in I_r} |\zeta_k(\phi) - \zeta(\phi)| \geq \delta \right\} \geq \varepsilon \right\}$$

As the right hand side set belongs to $\mathcal{I}$, hence

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \mathrm{Cr}\{||\zeta_k - \zeta|| \geq \delta\} \geq \varepsilon \right\} \in \mathcal{I}.$$

This implies $\{\zeta_k\}$ is strongly lacunary $\mathcal{I}$ -convergent in credibility to ζ. □

Definition 4.1. The FV sequence $\{\zeta_i\}$ is said to be strongly $\mathcal{I}$-Cesàro summable to ζ iff there exists $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{i=1}^{n} \|\zeta_i(\phi) - \zeta(\phi)\| \geq \varepsilon \right\} \in \mathcal{I},$$

for each $\varepsilon > 0$ and every $\phi \in A$. In this case, we write $\zeta_i \overset{\sigma_1[\mathcal{I}]}{\to} \zeta$.

Definition 4.2. The FV sequence $\{\zeta_i\}$ is said to be strongly $\mathcal{I}$-lacunary summable to ζ iff there exists $A \in \mathcal{P}(\Theta)$ with $\mathrm{Cr}\{A\} = 1$ such that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} \|\zeta_i(\phi) - \zeta(\phi)\| \geq \varepsilon \right\} \in \mathcal{I},$$

for each $\varepsilon > 0$ and every $\phi \in A$. In this case, we write $\zeta_i \overset{N_\theta[\mathcal{I}]}{\to} \zeta$.

Theorem 4.7. *Let $\zeta, \zeta_1, \zeta_2, \ldots$ be FVs. In order for $\sigma_1[\mathcal{I}] \subseteq N_\theta[\mathcal{I}]$ it is necessary and sufficient that* $\lim_r \inf q_r > 1$.

Proof. For the sufficiency we assume $\lim_r \inf q_r > 1$, then there exist $\zeta(\theta) \in (\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ and $\mathrm{Cr}(\zeta(\theta)) > 0$ such that $1 + \mathrm{Cr}(\zeta(\phi)) \leq q_r$ for each $r \geq 1$. Since $h_r = k_r - k_{r-1}$, we get $\frac{k_r}{h_r} \leq \frac{1+\mathrm{Cr}(\zeta(\phi))}{\mathrm{Cr}(\zeta(\phi))}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\mathrm{Cr}(\zeta(\phi))}$; as $\mathrm{Cr}(\zeta(\phi)) > 0$ and $q_r = \frac{k_r}{k_{r-1}}$. Assume $\varepsilon > 0$ and we define the set

$$S = \left\{ k_r \in \mathbb{N} : \frac{1}{k_r} \sum_{i=1}^{k_r} \|\zeta_i(\phi) - \zeta(\phi)\| < \varepsilon \right\},$$

for $\phi \in A$. We can easily say that $S \in \mathcal{F}(\mathcal{I})$, which is a filter of the ideal $\mathcal{I}$, so we have

$$\begin{aligned} \frac{1}{h_r} \sum_{i \in I_r} \|\zeta_i(\phi) - \zeta(\phi)\| &= \frac{1}{h_r} \sum_{i=1}^{k_r} \|\zeta_i(\phi) - \zeta(\phi)\| - \frac{1}{h_r} \sum_{i=1}^{k_{r-1}} \|\zeta_i(\phi) - \zeta(\phi)\| \\ &= \frac{k_r}{h_r} \cdot \frac{1}{k_r} \sum_{i=1}^{k_r} \|\zeta_i(\phi) - \zeta(\phi)\| - \frac{k_{r-1}}{h_r} \cdot \frac{1}{k_{r-1}} \sum_{i=1}^{k_{r-1}} \|\zeta_i(\phi) - \zeta(\phi)\| \\ &\leq \left(\frac{1+\mathrm{Cr}(\zeta(\phi))}{\mathrm{Cr}(\zeta(\phi))} \right) \varepsilon - \frac{1}{\mathrm{Cr}(\zeta(\phi))} \varepsilon' \end{aligned}$$

for $\phi \in A$. Choose $\sigma = \left(\frac{1+\mathrm{Cr}(\zeta(\phi))}{\mathrm{Cr}(\zeta(\phi))} \right) \varepsilon - \frac{1}{\mathrm{Cr}(\zeta(\phi))} \varepsilon'$. So, for all $\phi \in A$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{i \in I_r} \|\zeta_i(\phi) - \zeta(\phi)\| < \sigma \right\} \in \mathcal{F}(\mathcal{I}).$$

Hence, we obtain $\sigma_1[\mathcal{I}] \subseteq N_\theta[\mathcal{I}]$.

Conversely, suppose that $\sigma_1[\mathcal{I}] \subseteq N_\theta[\mathcal{I}]$ and $\lim_r \inf q_r = 1$. Since θ is lacunary sequence, we can select a subsequence k_{r_j} of θ providing,

$$\frac{k_{r_j}}{k_{r_j - 1}} < 1 + \frac{1}{j} \text{ and } \frac{k_{r_j - 1}}{k_{r_{j-1}}} > j, \text{ where } r_j \geq r_{j-1} + 2.$$

Identify $\zeta = (\zeta_i(\phi))$ by

$$\zeta_i(\phi) = \begin{cases} 1, & \text{if } i \in I_{r_j} \text{ for some } j = 1, 2, \ldots; \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any $\zeta(\phi)$,

$$\left\{r\in\mathbb{N}:\frac{1}{h_{r_j}}\sum_{i\in I_{r_j}}\|\zeta_i(\phi)-\zeta(\phi)\|<\varepsilon\right\}\in\mathcal{F}(\mathcal{I})\,;\ j=1,2,...$$

and

$$\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{i\in I_r}\|\zeta_i(\phi)-\zeta(\phi)\|<\varepsilon\right\}\in\mathcal{F}(\mathcal{I})\ \text{ for } r\neq r_j.$$

It gives that $(\zeta_i(\phi))\notin N_\theta[\mathcal{I}]$.

But, $(\zeta_i(\phi))$ is strongly $\mathcal{I}$ -Cesàro summable, since if we consider w is sufficiently large, there is a unique j for which $k_{r_j-1}<w\le k_{r_{j+1}-1}$ and for any $\varepsilon>0$, we write

$$\frac{1}{w}\sum_{i=1}^{w}\|\zeta_i(\phi)-\zeta(\phi)\|\le\frac{k_{r_{j-1}}+h_{r_j}}{k_{r_j}-1}\le\frac{1}{j}+\frac{1}{j}=\frac{2}{j}<\varepsilon,$$

so we get

$$\left\{w\in\mathbb{N}:\frac{1}{w}\sum_{i=1}^{w}\|\zeta_i(\phi)-\zeta(\phi)\|\ge\varepsilon\right\}\in\mathcal{I}.$$

Hence, we get $(\zeta_i(\phi))\in\sigma_1[\mathcal{I}]$. This contradicts to our assumption. Therefore, $\lim_r\inf q_r>1$. □

Theorem 4.8. *Let θ be a lacunary sequence. If $\lim_r\sup q_r<\infty$, then*

$$\zeta_i\overset{N_\theta[\mathcal{I}]}{\to}\zeta\Rightarrow\zeta_i\overset{\sigma_1[\mathcal{I}]}{\to}\zeta.$$

Proof. Let $\lim_r\sup q_r<\infty$. So, there exist $H(\phi)\in(\Theta,\mathcal{P}(\Theta),\mathrm{Cr})$ and $\mathrm{Cr}(H(\phi))>0$ such that $q_r<\mathrm{Cr}(H(\phi))$ for all $r\ge1$. Let $\zeta_i\overset{N_\theta[\mathcal{I}]}{\to}\zeta$ and we define the sets

$$T=\left\{r\in\mathbb{N}:\frac{1}{h_r}\sum_{i\in I_r}\|\zeta_i(\phi)-\zeta(\phi)\|<\varepsilon_1\right\}$$

and

$$S=\left\{n\in\mathbb{N}:\frac{1}{n}\sum_{i\in1}^{n}\|\zeta_i(\phi)-\zeta(\phi)\|<\varepsilon_2\right\},$$

for every ε_1,$\varepsilon_2>0$ and $\phi\in A$. Let

$$\tau_j=\frac{1}{h_j}\sum_{i\in I_j}\|\zeta_i(\phi)-\zeta(\phi)\|<\varepsilon_1$$

for all $\phi\in A$ and $\tau_j\in T$. It is obvious that $T\in\mathcal{F}(\mathcal{I})$.

Choose n is any integer with $k_{r-1}<n\le k_r$, where $r>T$. Then, we have

$$
\begin{aligned}
&\frac{1}{n}\sum_{i\in 1}^{n}\left\|\zeta_i(\phi)-\zeta(\phi)\right\| \leq \frac{1}{k_{r-1}}\sum_{i\in 1}^{k_r}\left\|\zeta_i(\phi)-\zeta(\phi)\right\| \\
&=\frac{1}{k_{r-1}}\left(\sum_{i\in I_1}\left\|\zeta_i(\phi)-\zeta(\phi)\right\|+\sum_{i\in I_2}\left\|\zeta_i(\phi)-\zeta(\phi)\right\|+\ldots+\sum_{i\in I_r}\left\|\zeta_i(\phi)-\zeta(\phi)\right\|\right) \\
&=\frac{k_1}{k_{r-1}}\left(\frac{1}{h_1}\sum_{i\in I_1}\left\|\zeta_i(\phi)-\zeta(\phi)\right\|+\frac{k_2-k_1}{k_{r-1}}\frac{1}{h_2}\sum_{i\in I_2}\left\|\zeta_i(\phi)-\zeta(\phi)\right\|\right. \\
&\quad\left.+\ldots+\frac{k_r-k_{r-1}}{k_{r-1}}\frac{1}{h_r}\sum_{i\in I_r}\left\|\zeta_i(\phi)-\zeta(\phi)\right\|\right) \\
&=\frac{k_1}{k_{r-1}}\tau_1+\frac{k_2-k_1}{k_{r-1}}\tau_2+\ldots+\frac{k_r-k_{r-1}}{k_{r-1}}\tau_r \\
&\leq\left(\sup_{j\in T}\tau_j\right).\frac{k_1}{k_{r-1}}<\varepsilon_1.\mathrm{Cr}\left(H(\phi)\right)
\end{aligned}
$$

for $\phi \in A$. Choose $\varepsilon_2 = \frac{\varepsilon_1}{\mathrm{Cr}(H(\phi))}$ and in view of the fact that

$$\bigcup\{n \in \mathbb{N} : k_{r-1} < n \leq k_r,\ r \in T\} \subset S$$

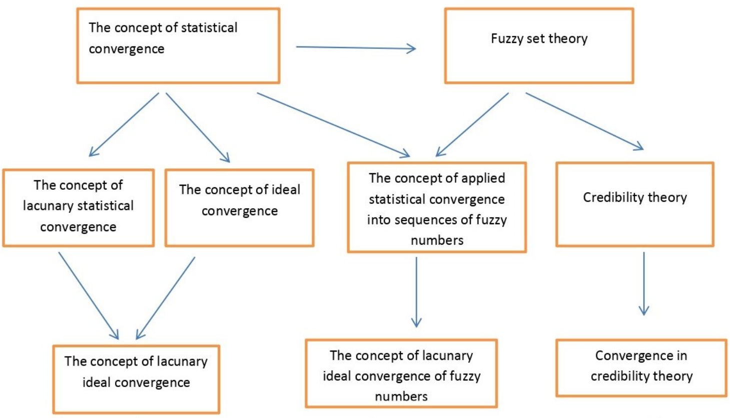
where $T \in \mathcal{F}(\mathcal{I})$. It follows from our assumption on θ that the set S also belongs to $\mathcal{F}(\mathcal{I})$. Hence, $N_\theta[\mathcal{I}] \subseteq \sigma_1[\mathcal{I}]$. □

Corollary 4.9. *Let $\zeta_1, \zeta_2, \ldots$ be FVs. If $\theta = \{k_r\}$ be a lacunary sequence with $1 < \liminf q_r \leq \limsup q_r < \infty$, then*

$$\zeta_i \overset{N_\theta[\mathcal{I}]}{\to} \zeta \Leftrightarrow \zeta_i \overset{\sigma_1[\mathcal{I}]}{\to} \zeta.$$

Proof. This is an immediate consequence of Theorem 4.7 and Theorem 4.8. □

In this article, we can present a diagram illustrating the connections between the fundamental concepts as follows:



Conclusion

Credibility theory, initially established by Liu to investigate fuzzy phenomena, has seen further development. Liu and Liu expanded on his work by providing an axiomatic foundation and delving deeper into credibility theory. More recently, mathematicians have begun exploring the connection between the concepts of convergence in credibility and lacunary ideal convergence. We presented many sorts of convergence in this study, such as strongly lacunary ideal convergence in credibility, strongly lacunary ideal convergence in mean, strongly lacunary ideal convergence in distribution and strongly lacunary ideal convergence uniformly almost surely of the FV, and achieved some interesting findings. These findings can be applied to investigate the convergence challenges in sequences of FVs exhibiting chaotic patterns within the realm of credibility theory. To explore optimal pathways in turnpike theory within a fuzzy environment, one can employ the concepts and outcomes presented here as valuable theoretical tools. The conclusions of this study are more general and a natural extension of the conventional convergence of FV sequences.

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