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Direct Approximations of Szász-Beta-Schurer Operators using Hermite Polynomial

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Abstract. The aim of present article is to introduce the Szász-Beta-Schurer operators in terms of Hermite Polynomial. We calculate some estimates and then discuss convergence theorems and order of approximation in terms of Korovkin theorem and first order modulus of smoothness respectively. Next, we study pointwise approximation results in terms of Peetre's K-functional, second order modulus of smoothness, Lipschitz type space and r^{th} order Lipschitz type maximal function. Lastly, weighted approximation results and statistical approximation theorems are proved.

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1. Introduction and Preliminaries

The approximation in operator theory is a significant area of mathematical analysis, emerged in the 19^{th} century and continues to be studied by mathematicians worldwide. Its relevance extends beyond mathematics to various fields, including the basic sciences and engineering. The primary objective of approximation theory is to represent a complex function using simpler functions with more elementary properties, such as differentiability and integrability. It has applications in computational aspects like describing the shapes of geometric objects as well as in applied and pure mathematics, including fixed point theory and numerical analysis. Control nets and control points are used to study parametric surfaces and curves, respectively. The theory has widespread applications in other scientific branches, such as data structures, computer graphics, computer algebra and numerical analysis. In 1885, Weierstrass [27] gave an elegant result in approximation theory named as Weierstrass approximation theorem. Several renowned mathematicians have worked on providing simpler and more understandable proofs for this theorem.

In order to provide a succinct proof of the Weierstrass approximation theorem using binomial distribution, Bernstein [4] invented a sequence of polynomials known as Bernstein polynomials in 1912 as follows:

$$B_s(g;y) = \sum_{l=0}^{s} g\left(\frac{l}{s}\right)\binom{s}{l} y^l(1-y)^{s-l}, \quad y \in [0,1], \tag{1}$$

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where g is a bounded function defined on $[0,1]$. The approximation with the sequences of operators given in (1) are restricted for bounded function on $[0,1]$. To approximate on $[0,\infty)$, Szász [23] gave modification of the sequences given in (1) which play an important role in the development of operator theory as below:

$$S_s(g;y) = e^{-sy}\sum_{l=0}^{\infty}\frac{(sy)^l}{l!}g\left(\frac{l}{s}\right),\ s\in\mathbb{N}, \tag{2}$$

where the real valued function $g\in C[0,\infty)$. The linear positive operators introduced in (2) are restricted for the space of continuous functions only. To approximate in longer class of functions, i.e., space of functions which are measurable in Lebesgue sense, several integral versions of these sequences of operators are introduced, e.g., Szász-Kantorovich type and Szász-Durremeyer type operators etc. (see [20], [21]). Many mathematicians, e.g., Acu et al. ([1], [2]), Mohiuddine et al. ([12], [13]) Mursaleen et al. ([14], [15]), Raiz et al.[18, 19], Khan et al. [10], Nasiruzzaman [16] and Wafi et al. ([24], [25]) gave various generalizations for such type of sequences. Grażyna [9] presented a class of sequence of operators $G_s^{\alpha}(.;.)$, $s\in\mathbb{N}$, $\alpha\geq 0$, given by the formula

$$G_s^{\alpha}(g;y) = e^{-(sy+\alpha y^2)}\sum_{l=0}^{\infty}\frac{y^l}{l!}H_l(s,\alpha)g\left(\frac{l}{s}\right),\qquad y\in\mathbb{R}_0^+, \tag{3}$$

where H_l is the two variable Hermite polynomial (see [3]) given by

$$H_l(n,\alpha) = l!\sum_{m=0}^{[\frac{l}{2}]}\frac{n^{l-2m}\alpha^m}{(l-2m)!m!}. \tag{4}$$

The operators (3) are linear and positive. Basic facts on positive linear operators, their generalizations and applications can be found in [5].

The sequences of operators presented in (3) are restricted for continuous function only. Motivated with the above development, we introduce a sequences of positive linear operators to give approximations in bigger class of function, i.e., the space of Lebesgue measurable functions which is named as Szász-Beta-Schurer operators in view of Hermite Polynomial as:

$$H_{s+p}^{\alpha}(g;y) \;=\; \sum_{l=0}^{\infty}P_{s+p,l}^{\alpha}(y)\int_0^{\infty}Q_{s+p}(v)g(v)dv,\qquad for\quad y\in\mathbb{R}_0^+, \tag{5}$$

where

$$P_{s+p,l}^{\alpha}(y) = e^{-((s+p)y+\alpha y^2)}\frac{y^l}{l!}H_l((s+p),\alpha)\ and$$
$$Q_{(s+p)}(v) = \frac{1}{\beta(l+1,(s+p))}\left[\frac{v^l}{(1+v)^{l+1+(s+p)}}\right],$$

with β (Beta) function, $\beta(l+1,(s+p)) = \int_0^{\infty}\frac{v^l}{(1+v)^{l+1+(s+p)}}dv$.

Lemma 1.1. *Let $H^{\alpha}_{(s+p)}(.;.)$ be the sequence of operators given by (5) and $e_i(t) = t^i$, $i \in \{0, 1, 2\}$. Then, one get*

$$\begin{aligned} H^{\alpha}_{(s+p)}(1;y) &= 1, \\ H^{\alpha}_{(s+p)}(e_1;y) &= \frac{1}{(s+p)-1}((s+p)y + 2\alpha y^2 + 1); \quad (s+p) > 1, \\ H^{\alpha}_{(s+p)}(e_2;y) &= \frac{1}{((s+p)-2)((s+p)-1)}\Big[(s+p)^2y^2 + 4(s+p)(\alpha y^3 + y) \\ &+ 4\alpha^2y^4 + 10\alpha y^2 + 2\Big]; (s+p) > 2. \end{aligned}$$

for each $y \in \mathbb{R}^+_0$.

Proof. From the Eq. (5), we have

$$H^{\alpha}_{(s+p)}(e_i;y) = \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \int_0^{\infty} Q_{(s+p)}(v) t^i dv.$$

Now, for $i = 0$,

$$\begin{aligned} H^{\alpha}_{(s+p)}(e_0;y) = &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{1}{\beta(l+1,(s+p))} \int_0^{\infty} \frac{v^l}{(1+v)^{l+1+(s+p)}} dv \\ &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{1}{\beta(l+1,(s+p))} \beta(l+1,(s+p)) \\ &= 1. \end{aligned}$$

For $i = 1$,

$$\begin{aligned} H^{\alpha}_{(s+p)}(e_1;y) &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{1}{\beta(l+1,(s+p))} \int_0^{\infty} \frac{v^{l+1}}{(1+v)^{l+1+(s+p)}} dv \\ &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{1}{\beta(l+1,(s+p))} \beta(l+2,(s+p)-1) \\ &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{(l+1)}{((s+p)-1)} \\ &= \frac{(s+p)}{(s+p)-1}\left[G^{\alpha}_{(s+p)}(e_1;y) + \frac{1}{(s+p)}\right] \\ &= \frac{1}{(s+p)-1}((s+p)y + 2\alpha y^2 + 1). \end{aligned}$$

For $i = 2$,

$$\begin{aligned} H^{\alpha}_{(s+p)}(e_2;y) &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{1}{\beta(l+1,(s+p))} \int_0^{\infty} \frac{v^{l+2}}{(1+v)^{l+1+(s+p)}} dv \\ &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{1}{\beta(l+1,(s+p))} \beta(l+3,(s+p)-2) \\ &= \sum_{l=0}^{\infty} P^{\alpha}_{(s+p),l}(y) \frac{(l+2)}{((s+p)-2)} \frac{(l+1)}{((s+p)-1)} \end{aligned}$$

$$= \frac{1}{((s+p)-2)((s+p)-1)}\Big[(s+p)^2 G^{\alpha}_{(s+p)}(e_2;y) + 3(s+p)G^{\alpha}_{(s+p)}(e_1;y) + 2G^{\alpha}_{(s+p)}(e_0;y)\Big]$$

$$= \frac{1}{((s+p)-2)((s+p)-1)}\Big((s+p)^2y^2 + 4(s+p)(\alpha y^3 + y) + 4\alpha^2 y^4 + 10\alpha y^2 + 2\Big).$$

□

Lemma 1.2. *Let $H^{\alpha}_{(s+p)}(.;.)$ be the operators given by (5) and central moments $\eta_i(t;y) = (t-y)^i$, $i \in \{0,1,2\}$. Then, one get*

$$\begin{aligned}
H^{\alpha}_{(s+p)}(\eta_0;y) &= 1,\\
H^{\alpha}_{(s+p)}(\eta_1;y) &= -\frac{1}{(s+p)-1}(2\alpha y^2 + y + 1), \quad (s+p) > 1,\\
H^{\alpha}_{(s+p)}(\eta_2;y) &= \frac{1}{((s+p)-2)((s+p)-1)}\Big[(s+p)(y^2+2y) + 4\alpha^2 y^4 \\
&\quad + 2\alpha y^2(4y+5) + 2(y+1)^2\Big], \quad (s+p) > 2,\\
H^{\alpha}_{(s+p)}(\eta_4;y) &= O\Big(\frac{1}{(s+p)^2}\Big), \quad (s+p) > 4,
\end{aligned}$$

for each $y \in \mathbb{R}^+_0$.

Proof. Using the definition of $H^{\alpha}_{(s+p)}(.;.)$, we get for $i = 0$, it is obvious that

$$H^{\alpha}_{(s+p)}(\eta_0;y) = 1.$$

Now, we consider for $i = 1$, that is $H^{\alpha}_{(s+p)}(\eta_1;y)$ as follows:

$$\begin{aligned}
H^{\alpha}_{(s+p)}(\eta_1;y) &= \sum_{l=0}^{\infty} P^{\alpha}_{s+p,l}(y)\frac{1}{\beta(l+1,s+p)}\int_0^{\infty}\frac{v^l}{(1+v)^{l+1+s+p}}(y-v)dv\\
&= y\sum_{l=0}^{\infty} P^{\alpha}_{s+p,l}(y)\frac{1}{\beta(l+1,s+p)}\int_0^{\infty}\frac{v^l}{(1+v)^{l+1+s+p}}dv\\
&\quad -\sum_{l=0}^{\infty} P^{\alpha}_{s+p,l}(y)\frac{1}{\beta(l+1,s+p)}\int_0^{\infty}\frac{v^{l+1}}{(1+v)^{l+1+s}}dv\\
&= yH^{\alpha}_{s+p}(e_0;y) - H^{\alpha}_{s+p}(e_1;y)\\
&= -\frac{1}{s+p-1}(2\alpha y^2 + y + 1).
\end{aligned}$$

Further, for $i = 2$, that is $H^{\alpha}_{s+p}(\eta_2;y)$ as follows:

$$\begin{aligned}
H^{\alpha}_{s+p}(\eta_2;y) &= \sum_{l=0}^{\infty} P^{\alpha}_{s+p,l}(y)\frac{1}{\beta(l+1,s+p)}\int_0^{\infty}\frac{v^l}{(1+v)^{l+1+s+p}}(y-v)^2dv\\
&= y^2 H^{\alpha}_{s+p}(e_0;y) - 2yH^{\alpha}_{s+p}(e_1;y) + H^{\alpha}_{s+p}(e_2;y)
\end{aligned}$$

$$
\begin{aligned}
&= y^2 - 2y\frac{1}{s+p-1}((s+p)y + 2\alpha y^2 + 1) + \frac{1}{(s+p-2)(s+p-1)} \\
&\quad \times\Big[(s+p)^2 y^2 + 4(s+p)(\alpha y^2 + y) + 4\alpha^2 y^4 + 10\alpha y^2 + 2\Big] \\
&= \frac{1}{(s+p-2)(s+p-1)}\Big[(s+p-2)(s+p-1)y^2 - 2y(s+p-2)(s+py+2\alpha y^2+1) \\
&\quad +(s+p)^2 y^2 + 4(s+p)(\alpha y^2 + y) + 4\alpha^2 y^4 + 10\alpha y^2 + 2\Big] \\
&= \frac{1}{(s+p-2)(s+p-1)}\Big[(s+p)\{-4\alpha y^3 + y^2(4\alpha+1) + 2y\} + 4\alpha^2 y^4 \\
&\quad +8\alpha y^3 + y^2(10\alpha+2) + 4y + 2\Big].
\end{aligned}
$$

Similarly, we can prove the rest part of this Lemma. □

In subsequent sections, we deal with convergence rate of operators and order of approximation. Fuhrer, direct results are discussed as locally and globally in different spaces. In the last section, A-Statistical approximation results are investigated in several functional spaces.

2. Convergence Rate and Approximation Order

Definition 2.1. Let g be a continuous function defined on positive semi-axes. Then the modulus of smoothness is given by

$$\omega(g;\delta) = \sup_{|y_1-y_2|\le\delta} |g(y_1)-g(y_2)|, \qquad y_1, y_2 \in [0,\infty).$$

Theorem 2.1. *Let $H^{\alpha}_{s+p}(.;.)$ be a sequence of operator introduced in Eq. (5). Then, for all $g \in C_B[0,\infty)$, $H^{\alpha}_{s+p}(g;y) \rightrightarrows g$ on each closed and bounded subset of $[0,\infty)$ where $\rightrightarrows$ represents uniform convergent.*

Proof. In view of Korovkin type theorem which regard the uniform convergence of the sequence of linear positive operators, it is enough to see that

$$\lim_{s\to\infty} H^{\alpha}_{s+p}(t^i;y) = y^i,\ \ i = 0,1,2,$$

uniformly on every closed and bounded subset of $[0,\infty)$. In the light of Lemma 1.1, this result can easily be proved. □

In view of result given by Shisha et al. [22], we can prove the order of convergence in terms of Ditzian-Totik the modulus of continuity.

Theorem 2.2. *For $g \in C_B[0,\infty)$ and the operators $H^{\alpha}_{s+p}(.;.)$ introduced in Eq. (5), we have*

$$|H^{\alpha}_{s+p}(g;y) - g(y)| \le 2\omega(g;\delta),$$

where $\delta = \sqrt{H^{\alpha}_{s+p}((t-y)^2;y)}$.

3. Locally Approximation Results

In this section, we think back to some functional spaces and functional relation as: $C_B[0,\infty)$: Represent a space of bounded and continuous real valued functions. Now, Peetre's K-functional is given by

$$K_2(g,\delta)=\inf_{h\in C_B^2[0,\infty)}\left\{\|g-h\|_{C_B[0,\infty)}+\delta\|h''\|_{C_B^2[0,\infty)}\right\},$$

where $C_B^2[0,\infty)=\{h\in C_B[0,\infty):h',h''\in C_B[0,\infty)\}$ provided with the norm $\|g\|=\sup_{0\le y<\infty}|g(y)|$ and Ditzian-Totik modulus of smoothness of second order is given by

$$\omega_2(g;\sqrt{\delta})=\sup_{0<k\le\sqrt{\delta}}\ \sup_{y\in[0,\infty)}|f(y+2k)-2f(y+k)+f(y)|.$$

We recall a relation from DeVore and Lorentz ([5] page no. 177, Theorem 2.4), as:

$$K_2(g;\delta)\le C\omega_2(g;\sqrt{\delta}), \tag{6}$$

where C is a constant absolute. Now in view to prove the further result, we take the auxiliary operator as:

$$\widehat{H}^{\alpha}_{s+p}(g;y)=H^{\alpha}_{s+p}(g;y)+g(y)-g\left(\frac{(s+p)y+2\alpha y^2+1}{s+p-1}\right) \tag{7}$$

where $g\in C_B[0,\infty)$, $y\ge0$ and $n>2$. From Eq. (7), one can yield

$$\widehat{H}^{\alpha}_{s+p}(1;y)=1,\ \widehat{H}^{\alpha}_{s+p}(\eta_1;x)=0 \text{ and } |\widehat{H}^{\alpha}_{s+p}(g;y)|\le3\|g\|. \tag{8}$$

Lemma 3.1. *For $s+p>2$ and $y\ge0$, one yield*

$$|\widehat{H}^{\alpha}_{s+p}(g;y)-g(y)|\le\theta(y)\|g''\|,$$

where $g\in C_B^2[0,\infty)$ and $\theta(y)=\widehat{H}^{\alpha}_{s+p}(\eta_2;y)+(\widehat{H}^{\alpha}_{s+p}(\eta_1;y))^2$.

Proof. For $g\in C_B^2[0,\infty)$ and in view of relation Taylor expansion , we get

$$g(t)=g(y)+(t-y)g'(y)+\int_y^t(t-v)g''(v)dv. \tag{9}$$

Now, applying the auxiliary operators $\widehat{H}^{\alpha}_{s+p}(.;.)$ given in Eq.(7) on both the sides in above Eq. (9), we get

$$\widehat{H}^{\alpha}_{(s+p)}(g;y)-g(y)\ =\ g'(y)\widehat{H}^{\alpha}_{(s+p)}(\eta_1;y)+\widehat{H}^{\alpha}_{(s+p)}\Big(\int_y^t(t-v)g''(v)dv;y\Big).$$

Using the Eqs. (8) and (9), we get

$$\widehat{H}^{\alpha}_{(s+p)}(g;y)-g(y)=\widehat{H}^{\alpha}_{(s+p)}\Big(\int_y^t (t-v)g''(v)dv;y\Big)$$

$$=H^{\alpha}_{(s+p)}\Big(\int_y^t (t-v)g''(v)dv;y\Big)-\int_y^{\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}}\left(\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}-v\right)g''(v)dv,$$

$$|\widehat{H}^{\alpha}_{(s+p)}(g;y)-g(y)|\le\left|H^{\alpha}_{(s+p)}\Big(\int_y^t (t-v)g''(v)dv;y\Big)\right|$$
$$+\left|\int_y^{\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}}\left(\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}-v\right)g''(v)dv\right|. \quad (10)$$

Since,

$$\left|\int_y^t (t-v)g''(v)dv\right|\le(t-y)^2\parallel g''\parallel, \quad (11)$$

then

$$\left|\int_y^{\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}}\left(\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}-v\right)g''(v)dv\right|\le\left(\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}-y\right)^2\parallel g''\parallel. \quad (12)$$

In view of (10), (11) and (12), we find

$$|\widehat{H}^{\alpha}_{(s+p)}(g;y)-g(y)|\le\left\{\widehat{H}^{\alpha}_{(s+p)}(\eta_2;y)+\left(\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}-y\right)^2\right\}\|g''\|$$
$$=\theta(y)\|g''\|.$$

Which proves the required result. □

Theorem 3.2. *Let $g\in C_B^2[0,\infty)$. Then, there corresponds a non-negative constant $\tilde{C}>0$ such that*

$$|\ H^{\alpha}_{(s+p)}(g;y)-g(y)\ |\le\tilde{C}\omega_2\big(g;\sqrt{\theta(y)}\big)+\omega(g;H^{\alpha}_{(s+p)}(\eta_1;y)),$$

where $\theta(y)$ is given by in Lemma 3.1.

Proof. For $g\in C_B^2[0,\infty)$ and $g\in C_B[0,\infty)$ and with the definition of $\widehat{H}^{\alpha}_{(s+p)}(.;.)$, we get

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\le|\widehat{H}^{\alpha}_{(s+p)}(g-h;y)|+|(g-h)(y)|+|\widehat{H}^{\alpha}_{(s+p)}(h;y)-h(y)|$$
$$+\left|g\Big(\frac{(s+p)y+2\alpha y^2+1}{((s+p)-1)}\Big)-g(y)\right|.$$

In the light of Lemma 3.1 and inequalities in Eq. (8), one get

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq 4\|g-h\|+|\widehat{H}^{\alpha}_{(s+p)}(h;y)-h(y)|+\left|g\left(\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1}\right)-g(y)\right|$$
$$\leq 4\|g-h\|+\theta(y)\|h''\|+\omega\Big(g;H^{\alpha}_{(s+p)}((t-y);y)\Big).$$

Using Eq. (6), we yield the desired result. □

Now, we discuss the next result in Lipschitz type space [17], which is given as:

$$Lip^{\zeta_1,\zeta_2}_{\tilde{M}}(\gamma):=\left\{g\in C_B[0,\infty):|g(t)-g(y)|\leq \tilde{M}\frac{|t-y|^{\gamma}}{(t+\zeta_1 y+\zeta_2 y^2)^{\frac{\gamma}{2}}}:y,t\in(0,\infty)\right\},$$

where $\tilde{M}>0$, $0<\gamma\leq 1$ and $\zeta_1,\zeta_2>0$.

Theorem 3.3. *For the sequence of positive linear operators (5) and $g\in Lip^{\zeta_1,\zeta_2}_{M}(\gamma)$, one has*

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y+\zeta_2 y^2}\right)^{\frac{\gamma}{2}}, \tag{13}$$

where $0<\gamma\leq 1$, $\zeta_1,\zeta_2\in(0,\infty)$ and $\lambda(y)=H^{\alpha}_{(s+p)}(\eta_2;y)$.

Proof. For $\gamma=1$ and $y\geq 0$, we get

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq H^{\alpha}_{(s+p)}(|g(t)-g(y)|;y)$$
$$\leq \tilde{M}H^{\alpha}_{(s+p)}\left(\frac{|t-y|}{(t+\zeta_1 y+\zeta_2 y^2)^{\frac{1}{2}}};y\right).$$

Since $\dfrac{1}{t+\zeta_1 y+\zeta_2 y^2}<\dfrac{1}{\zeta_1 y+\zeta_2 y^2}$, for all $y\in(0,\infty)$, we yield

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq \frac{\tilde{M}}{(\zeta_1 y+\zeta_2 y^2)^{\frac{1}{2}}}(H^{\alpha}_{(s+p)}(\eta_2;y))^{\frac{1}{2}}$$
$$\leq \tilde{M}\left(\frac{\lambda(y)}{\zeta_1 y+\zeta_2 y^2}\right)^{\frac{1}{2}},$$

which implies that Theorem 3.3 works for $\gamma=1$. Next, we consider for $\gamma\in(0,1)$ and in view of Hölder's inequality using $p=\frac{2}{\gamma}$ and $q=\frac{2}{2-\gamma}$, we have

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq \left(H^{\alpha}_{(s+p)}(|g(t)-g(y)|^{\frac{2}{\gamma}};y)\right)^{\frac{\gamma}{2}}$$
$$\leq \tilde{M}\left(H^{\alpha}_{(s+p)}\left(\frac{|t-y|^2}{(t+\zeta_1 y+\zeta_2 y^2)};y\right)\right)^{\frac{\gamma}{2}}.$$

Since $\dfrac{1}{t+\zeta_1 y+\zeta_2 y^2}<\dfrac{1}{\zeta_1 y+\zeta_2 y^2}$, for all $y\in(0,\infty)$, one get

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq \tilde{M}\left(\frac{H^{\alpha}_{(s+p)}(|t-y|^2;y)}{\zeta_1 y+\zeta_2 y^2}\right)^{\frac{\gamma}{2}}\leq \tilde{M}\Big(\frac{\lambda(y)}{\zeta_1 y+\zeta_2 y^2}\Big)^{\frac{\gamma}{2}}.$$

Hence, we yield the required result. □

Next, we deal the approximation locally in view of r^{th} order modulus of smoothness then, Lipschitz-type maximal function which is introduced by Lenze [11] as:

$$\widetilde{\omega}_r(g;y)=\sup_{t\neq y,t\in(0,\infty)}\frac{|g(t)-g(y)|}{|t-y|^r},\ y\in[0,\infty)\text{ and }r\in(0,1]. \tag{14}$$

Theorem 3.4. *Let $g\in C_B[0,\infty)$ and $r\in(0,1]$. Then, for all $y\in[0,\infty)$, we have*

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq\widetilde{\omega}_r(g;y)\Big(\lambda(y)\Big)^{\frac{r}{2}}.$$

Proof. It is noted that

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq H^{\alpha}_{(s+p)}(|g(t)-g(y)|;y).$$

Using Eq. (14), one get

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq\widetilde{\omega}_(s+p)(g;y)H^{\alpha}_{(s+p)}(|t-y|^r;y).$$

Using Hölder's inequality using $p=\frac{2}{r}$ and $q=\frac{2}{2-r}$, we have

$$|H^{\alpha}_{(s+p)}(g;y)-g(y)|\leq\widetilde{\omega}_r(g;y)\big(H^{\alpha}_{(s+p)}(|t-y|^2;y)\big)^{\frac{r}{2}}.$$

Hence, we completes the proof. □

4. Approximation Properties Globally

Suppose that $\nu(y)=1+y^4, 0\leq y<\infty$ be the weight function. Then, $B_\nu[0,\infty)=\{g(y):|g(y)|\leq\tilde{M}_g(1+y^4)$, here $\tilde{M}_g$ is a constant based on g and $C_\nu[0,\infty)$ denotes space of continuous function in $B_\nu[0,\infty)$ equipped with the norm $\|g(y)\|_\nu=\sup\limits_{y\in[0,\infty)}\frac{|g(y)|}{\nu(y)}$ and $C^{\tilde{k}}_\nu[0,\infty)=\{g\in C_\nu[0,\infty):\lim\limits_{y\to\infty}\frac{g(y)}{\nu(y)}=\tilde{k}$, where $\tilde{k}$ is a constant depending on $g\}$.

Ditzian-Totik modulus of continuity for the function g defined on the closed interval $[0,b]$ with $b>0$ is defined by

$$\omega_b(g,\delta)=\sup_{|t-y|\leq\delta}\ \sup_{y,t\in[0,b]}|g(t)-g(y)|. \tag{15}$$

One can easily note that for any $g\in C_\nu[0,\infty)$, the modulus of smoothness given by in the Eq. (15) approaches to zero.

Theorem 4.1. *([7], [8]) Suppose that the sequence of positive linear operators $(L_s)_{s\geq1}$ acting from $C_\nu[0,\infty)$ to $B_\nu[0,\infty)$ satisfies the conditions*

$$\lim_{s\to\infty}||L_s(e_i;.)-e_i||_\nu=0,\ \textit{where}\ i=0,1,2,$$

then, for $g\in C^{\tilde{k}}_\nu[0,\infty)$, we have

$$\lim_{s\to\infty}||L_sg-g||_\nu=0.$$

Theorem 4.2. *Let $g\in C^{\tilde{k}}_\nu[0,\infty)$. Then, we have*

$$\lim_{s\to\infty}\|H^{\alpha}_{(s+p)}(g;.)-g\|_\nu=0.$$

Proof. In order to prove Theorem 4.2, it is sufficient to check that

$$\lim_{s\to\infty} \|H^{\alpha}_{(s+p)}(e_i;.) - e_i\|_{\nu} = 0, \text{ for } \; i = 0, 1, 2.$$

In the light of Lemma 1.1, it is obvious $\|H^{\alpha}_{(s+p)}(e_0;.) - 1\|_{\nu} = 0$, and

$$\begin{aligned}\|H^{\alpha}_{(s+p)}(e_1;.) - e_1\|_{\nu(y)} &= \sup_{y\in[0,\infty)} \frac{1}{\nu(y)} \left| \frac{(s+p)y + 2\alpha y^2 + 1}{(s+p)-1} - y \right| \\ &= \frac{1}{(s+p)-1} \sup_{y\in[0,\infty)} \frac{2\alpha y2}{1+y^4} + \frac{1}{(s+p)-1} \sup_{y\in[0,\infty)} \frac{2}{1+y^4}.\end{aligned}$$

For a large value of $(s+p)$, we get $\|H^{\alpha}_{(s+p)}(e_1;.) - e_1\|_{\nu} \to 0$.
Also,

$$\begin{aligned}\|H^{\alpha}_{(s+p)}(e_2;.) - e_2\|_{\nu} \leq & \left(\frac{4\alpha^2}{((s+p)-2)((s+p)-1)} \right) \sup_{y\in[0,\infty)} \frac{y^4}{1+y^4} \\ & + \left(\frac{4(s+p)\alpha}{((s+p)-2)((s+p)-1)} \right) \sup_{y\in[0,\infty)} \frac{y^3}{1+y^4} \\ & + \left(\frac{10\alpha + 3(s+p) - 2}{((s+p)-2)((s+p)-1)} \right) \sup_{y\in[0,\infty)} \frac{y^2}{1+y^4} \\ & + \left(\frac{4(s+p)y}{((s+p)-2)((s+p)-1)} \right) \sup_{y\in[0,\infty)} \frac{y}{1+y^4} \\ & + \left(\frac{2}{((s+p)-2)((s+p)-1)} \right) \sup_{y\in[0,\infty)} \frac{1}{1+y^4}.\end{aligned}$$

Which implies that $\|H^{\alpha}_{(s+p)}(e_2;.) - e_2\|_{\nu} \to 0$ as $s \to \infty$. Hence, we completes the proof of Theorem 4.2 □

Theorem 4.3. *Let $g \in C^{\tilde{k}}_{\nu}[0,\infty)$ and $\zeta > 0$. Then, we have*

$$\lim_{m\to\infty} \sup_{y\in[0,\infty)} \frac{|H^{\alpha}_{(s+p)}(g;y) - g(y)|}{(1+y^4)^{1+\zeta}} = 0.$$

Proof. Since $|g(y)| \leq \|g\|_{\nu}(1+y^4)$, for any real fixed number $y_0 > 0$, we get

$$\begin{aligned}\sup_{y\in[0,\infty)} \frac{|H^{\alpha}_{(s+p)}(g;y) - g(y)|}{(1+y^4)^{1+\zeta}} &\leq \sup_{y\leq y_0} \frac{|H^{\alpha}_{(s+p)}(g;y) - g(y)|}{(1+y^4)^{1+\zeta}} + \sup_{y\geq y_0} \frac{|H^{\alpha}_{(s+p)}(g;y) - g(y)|}{(1+y^4)^{1+\zeta}} \\ &\leq \|H^{\alpha}_{(s+p)}(g;.) - g\|_{C[0,y_0]} \\ &\quad + \|g\|_{\nu} \sup_{y\geq y_0} \frac{|H^{\alpha}_{(s+p)}(1+t^2;y)|}{(1+y^4)^{1+\zeta}} + \sup_{y\geq y_0} \frac{|g(y)|}{(1+y^4)^{1+\zeta}} \\ &= \tilde{T}_1 + \tilde{T}_2 + \tilde{T}_3, \quad \textit{say}. \end{aligned} \tag{16}$$

Now,

$$\tilde{T}_3 = \sup_{y\geq y_0} \frac{|g(y)|}{(1+y^4)^{1+\zeta}} \leq \sup_{y\geq y_0} \frac{\|g\|_{\nu}(1+y^4)}{(1+y^4)^{1+\zeta}} \leq \frac{\|g\|_{\nu}}{(1+y_0^4)^{\zeta}}.$$

In view of Lemma 1.1, it gives

$$\lim_{(s+p)\to\infty} \sup_{y\in[y_0,\infty)} \frac{H^{\alpha}_{(s+p)}(1+t^2;y)}{1+y^4} = 1.$$

Therefore, for any arbitrary $\epsilon > 0$, there corresponds $(s+p)_1 \in \mathbb{N}$ with

$$\sup_{y\in[y_0,\infty)} \frac{H^{\alpha}_{(s+p)}(1+t^2;y)}{1+y^4} \leq \frac{(1+y_0^4)^{\zeta}}{\|g\|_{\nu}} \frac{\epsilon}{3} + 1, \text{ for all } (s+p) \geq (s+p)_1.$$

Therefore

$$\tilde{T}_2 = ||g||_{\nu} \sup_{y\in[y_0,\infty)} \frac{H^{\alpha}_{(s+p)}(1+t^2;y)}{(1+y^4)^{1+\zeta}} \leq \frac{||g||_{\nu}}{(1+y_0^4)^{\zeta}} + \frac{\epsilon}{3}, \text{ for all } (s+p) \geq (s+p)_1. \quad (17)$$

Hence, we get

$$\tilde{T}_2 + \tilde{T}_3 < 2\frac{\|g\|_{\nu}}{(1+y^4)^{\zeta}} + \frac{\epsilon}{3}.$$

If we take y_0 to be so large that $\frac{\|g\|_{\nu}}{(1+y^4)^{\zeta}} < \frac{\epsilon}{6}$, tnen, we have

$$\tilde{T}_2 + \tilde{T}_3 < \frac{2\epsilon}{3} \text{ for all } (s+p) \geq (s+p)_1. \quad (18)$$

Now, from Theorem 2.1, there corresponds $(s+p)_2 > (s+p)$ with

$$\tilde{T}_1 = \|H^{\alpha}_{(s+p)}(g;\cdot) - g\|_{C[0,y_0]} < \frac{\epsilon}{3} \text{ for all } (s+p)_2 \geq (s+p). \quad (19)$$

Let $(s+p)_3 = \max((s+p)_1, (s+p)_2)$. Then, using the Eqs. (16), (18) and (19), we get

$$\sup_{y\in[0,\infty)} \frac{|H^{\alpha}_{(s+p)}(g;y) - g(y)|}{(1+y^4)^{1+\zeta}} < \epsilon,$$

which, completes the proof. □

5. A-Statistical Approximation

In this section, we recall some notations from [6]. Suppose that $A = (a_{s\mu})$ represents non-negative infinite suitability matrix. Then, a sequence $y := (y_{\mu})$ is called to be A-statistically convergent to L, that is $(s+p)t_A - \lim y = L$, if for every $\epsilon > 0$

$$\lim_{(s+p)} \sum_{\mu:|y_k - L|\geq\epsilon} a_{(s+p)\mu} = 0.$$

Let $q = (q_(s+p))$ be a sequence with following assertions holds

$$st_A - \lim_{(s+p)} q_(s+p) = 1 \text{ and } st_A - \lim_{(s+p)} q_(s+p)^(s+p) = a, \; 0 \leq a < 1. \quad (20)$$

Theorem 5.1. *Consider $A = (a_{(s+p)\mu})$ be a non-negative regular suitability matrix and the sequence $q = (q_(s+p))$ with condition (20) with $q_(s+p) \in (0,1)$, $(s+p) \in \mathbb{N}$. Then, for each $g \in C^0_{\nu}[0,\infty)$, $st_A - \lim_{(s+p)} \|H^{\alpha}_{(s+p)}(g;y) - g\|_{\nu} = 0$.*

Proof. By using Lemma 1.1, we have

$$st_A - \lim_{(s+p)} \|H^{\alpha}_{(s+p)}(e_0; y) - e_0\|_{\nu} = 0$$

, and

$$\begin{aligned}\|H^{\alpha}_{(s+p)}(e_i;.) - e_i\|_{\nu} &= \sup_{y\in[0,\infty)} \frac{1}{1+y^4}\left|\frac{(s+p)y+2\alpha y^2+1}{(s+p)-1} - y\right| \\ &= \frac{1}{1+y^4}\sup_{y\in[0,\infty)} \frac{2\alpha y2}{(s+p)-1} + \frac{1}{1+y^4}\sup_{y\in[0,\infty)} \frac{2}{(s+p)-1}.\end{aligned}$$

Now

$$\tilde{I_1} := \left\{(s+p) : \|H^{\alpha}_{(s+p)}(e_1; y) - y\| \geq \epsilon\right\},$$

$$\tilde{I_2} := \left\{(s+p) : \frac{2\alpha}{(s+p)-1} \geq \frac{\epsilon}{2}\right\},$$

$$\tilde{I_3} := \left\{(s+p) : \frac{2}{(s+p)-1} \geq \frac{\epsilon}{2}\right\}.$$

Which implies that $\tilde{I_1} \subseteq \tilde{I_2} \cup \tilde{I_3}$, this shows that $\sum_{\mu\in\tilde{I_1}} a_{(s+p)\mu} \leq \sum_{\mu\in\tilde{I_2}} a_{(s+p)\mu} + \sum_{\mu\in\tilde{I_3}} a_{(s+p)\mu}$. Therefore, we get

$$st_A - \lim_{(s+p)} \|H^{\alpha}_{(s+p)}(e_1; y) - y\|_{\nu} = 0. \tag{21}$$

Now by using Lemma 1.1, we have

$$\begin{aligned}\|H^{\alpha}_{(s+p)}(e_2;.) - e_2\|_{1+y^4} \leq \sup_{y\in[0,\infty)} \frac{1}{\nu(y)}\Bigg|\frac{1}{((s+p)-2)((s+p)-1)}\Big\{(s+p)^2y^2 \\ + 4(s+p)(\alpha y^3 + y) + 4\alpha^2y^4 + 10\alpha y^2 + 2\Big\} - y^2\Bigg|.\end{aligned}$$

For a given $\varepsilon > 0$, we have the following sets

$$\tilde{G_1} := \left\{(s+p) : \left\|H^{\alpha}_{(s+p)}(e_2; y) - y^2\right\|_{\nu} \geq \epsilon\right\}$$

$$\tilde{G_2} := \left\{(s+p) : \frac{4\alpha^2}{((s+p)-2)((s+p)-1)} \geq \frac{\epsilon}{5}\right\}$$

$$\tilde{G_3} := \left\{(s+p) : \frac{4(s+p)\alpha}{((s+p)-2)((s+p)-1)} \geq \frac{\epsilon}{5}\right\}$$

$$\tilde{G_4} := \left\{(s+p) : \frac{10\alpha + 3(s+p) - 2}{((s+p)-2)((s+p)-1)} \geq \frac{\epsilon}{5}\right\}$$

$$\tilde{G_5} := \left\{(s+p) : \frac{4(s+p)}{((s+p)-2)((s+p)-1)} \geq \frac{\epsilon}{5}\right\}$$

$$\tilde{G_6} := \left\{(s+p) : \frac{2}{((s+p)-2)((s+p)-1)} \geq \frac{\epsilon}{5}\right\}.$$

One can note that $\tilde{G_1} \subseteq \tilde{G_2} \bigcup \tilde{G_3} \bigcup \tilde{G_4} \bigcup \tilde{G_5} \bigcup \tilde{G_6}$. Thus, we have

$$\sum_{\mu\in\tilde{G_1}} a_{m\mu} \leq \sum_{\mu\in\tilde{G_2}} a_{m\mu} + \sum_{\mu\in\tilde{G_3}} a_{(s+p)\mu} + \sum_{\mu\in\tilde{G_4}} a_{(s+p)\mu} + \sum_{\mu\in\tilde{G_5}} a_{(s+p)\mu} + \sum_{\mu\in\tilde{G_6}} a_{(s+p)\mu}.$$

As $(s+p) \to \infty$, we have

$$st_A - \lim_n \|H^{\alpha}_{(s+p)}(e_2;.) - e_2\|_{\nu} = 0. \tag{22}$$

Hence, we completes the proof of Theorem 5.1. □

Now, we discuss the rate of A-Statistical approximation convergence in view of the Peetre's K-functional for operators $H^{\alpha}_{(s+p)}(.;.)$.

Theorem 5.2. *Let $g \in C^2_B[0,\infty)$. Then,*

$$st_A - \lim_{(s+p)} \|H^{\alpha}_{(s+p)}(g;\cdot) - f\|_{C_B[0,\infty)} = 0.$$

Proof. In view of Taylor's result, we get

$$g(t) = g(y) + g'(y)(t-y) + \frac{1}{2}g''(\eta)(t-y)^2,$$

where $t \leq \eta \leq y$. Operating $H^{\alpha}_{(s+p)}(.;.)$, both the sides in the above equation, we get

$$H^{\alpha}_{(s+p)}(g;y) - g(y) = g'(y)H^{\alpha}_{(s+p)}(\eta_1;y) + \frac{1}{2}g''(\eta)H^{\alpha}_{(s+p)}(\eta_2;y),$$

which yields that

$$\begin{aligned}\|H^{\alpha}_{(s+p)}(g;\cdot) - g\|_{C_B[0,\infty)} \leq& \|g'\|_{C_B[0,\infty)}\|H^{\alpha}_{(s+p)}(e_1-,.)\|_{C_B[0,\infty)} \\ &+ \|g''\|_{C_B[0,\infty)}\|H^{\alpha}_{(s+p)}(e_1-,.)^2\|_{C_B[0,\infty)} \\ =&\tilde{W}_1 + \tilde{W}_2, \quad say.\end{aligned} \tag{23}$$

From the Eqs. (21) and (22), one has

$$\begin{aligned}\lim_{(s+p)} \sum_{\mu\in\mathbb{N}:\tilde{W}_1\geq\frac{\epsilon}{2}} a_{(s+p)\mu} &= 0,\\ \lim_{(s+p)} \sum_{\mu\in\mathbb{N}:\tilde{W}_2\geq\frac{\epsilon}{2}} a_{(s+p)\mu} &= 0.\end{aligned}$$

From Eq. (23), we have

$$\lim_{(s+p)} \sum_{\mu\in\mathbb{N}:\|H^{\alpha}_{(s+p)}(g;\cdot)-g\|_{C_B[0,\infty)}\geq\epsilon} a_{(s+p)\mu} \leq \lim_{(s+p)} \sum_{\mu\in\mathbb{N}:\tilde{W}_1\geq\frac{\epsilon}{2}} a_{(s+p)\mu} + \lim_{(s+p)} \sum_{\mu\in N:\tilde{W}_2\geq\frac{\epsilon}{2}} a_{(s+p)\mu}.$$

Thus $st_A - \lim_{(s+p)} \|H^{\alpha}_{(s+p)}(g;\cdot) - g\|_{C_B[0,\infty)} \to 0.$ as $(s+p) \to \infty$.

Hence, we arrive the proof. □

Theorem 5.3. *Let $g \in C_B^2[0,\infty)$. Then,*

$$\|H_{(s+p)}^{\alpha}(g;\cdot) - g\|_{C_B[0,\infty)} \leq M\omega_2(g;\sqrt{\delta}),$$

where $\delta = \|H_{(s+p)}^{\alpha}(e_1 - \cdot;\cdot)\|_{C_B[0,\infty)} + \|H_{(s+p)}^{\alpha}((e_1 - \cdot)^2;\cdot)\|_{C_B[0,\infty)}$, *and* $\|g\|_{C_B^2[0,\infty)} = \|g\|_{C_B[0,\infty)} + \|g'\|_{C_B[0,\infty)} + \|g''\|_{C_B[0,\infty)}$.

Proof. Let $h \in C_B^2[0,\infty)$. Using Eq. (23), one get

$$\begin{aligned}\|H_{(s+p)}^{\alpha}(h) - h\|_{C_B[0,\infty)} \leq & \|h'\|_{C_B[0,\infty)}\|H_{(s+p)}^{\alpha}((e_1 - \cdot);\cdot)\|_{C_B[0,\infty)} \\ & + \frac{1}{2}\|h''\|_{C_B[0,\infty)}H_{(s+p)}^{\alpha}((e_1 - \cdot)^2;\cdot)\|_{C_B[0,\infty)} \\ \leq & \delta\|h\|_{C_B^2[0,\infty)}. \end{aligned} \tag{24}$$

For every $g \in C_B[0,\infty)$ and $h \in C_B^2$, from Eq. (24), we obtain

$$\begin{aligned}\|H_{(s+p)}^{\alpha}(g;\cdot) - g\|_{C_B[0,\infty)} \leq & \|H_{(s+p)}^{\alpha}(g;\cdot) - H_s^{\alpha}(h;\cdot)\|_{C_B[0,\infty)} \\ & + \|H_{(s+p)}^{\alpha}(h;\cdot) - h\|_{C_B[0,\infty)} + \|h - g\|_{C_B[0,\infty)} \\ \leq & 2\|h - g\|_{C_B[0,\infty)} + \|H_{(s+p)}^{\alpha}(h;\cdot) - h\|_{C_B[0,\infty)} \\ \leq & 2\|h - g\|_{C_B[0,\infty)} + \delta\|h\|_{C_B^2}.\end{aligned}$$

In view of Peetre's K-functional, one get

$$\|H_{(s+p)}^{\alpha}(g;\cdot) - g\|_{C_B[0,\infty)} \leq 2K_2(g;\delta)$$

and

$$\|H_{(s+p)}^{\alpha}(g;\cdot) - g\|_{C_B[0,\infty)} \leq \tilde{M}\big\{\omega_2(g;\sqrt{\delta}) + \min(1,\delta)\|g\|_{C_B[0,\infty)}\big\}.$$

Using Eq. (22), we obtain that

$$st_A - \lim_{(s+p)} \delta = 0, \text{ thus } st_A - \lim_{(s+p)} \omega(g;\sqrt{\delta}) = 0,$$

which completes the proof of required result. □

6. Conclusion

In this paper, we introduce a sequence of linear positive operators in integral form via Hermite Polynomial to approximate the functions which belongs to Lebesgue measurable space named as Szász-Beta type operators defined by (5). Further, we calculate the some estimates which are used to prove convergence rate and approximation order. Moreover, the various approximation results, e.g., locally and globally approximation results and A-Statistical approximation are investigated using these sequences of operators to achieve better approximations in several functional spaces.

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