Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 32, 2005, Pages 112–125
ISSN: 1223-6934

Results on multi-layer age-structured diffusion

Caterina Cusulin

Abstract. In this paper we present a mathematical model describing the dynamics of an age-structured population spreading in a one dimensional environment. Our model takes into account two important features of the population: its spatial diffusion and its age structure.

We consider a population living in a one dimensional stratified environment composed of layers. The vital rates are supposed to depend on the layer and on a significant variable which represents a way of weighting the age distribution. The diffusion coefficients depend on both the age and the layer. We suppose that there is no flux through the boundary of the environment. Thus we have to deal with a system of nonlinear partial differential equations with zero-flux condition on the boundary.

Under suitable assumptions, which are meaningful from a biological point of view, we prove an existence theorem for the solution of this system, using a semigroup approach.

Moreover we give a brief overview on some other results on this kind of problems, in particular we develop a method for finding the analytical solution of the problem in a particular linear case.

2000 Mathematics Subject Classification. 35K57, 47B44, 92D25.
Key words and phrases. evolution equations, reaction diffusion, multi-layer, m-accretive operators, age structure, population dynamics.

1. Introduction

In this paper we consider the modeling of an age-structured population living in a one dimensional stratified environment composed of n layers. We suppose that the vital rates depend on the population as a whole, through a significant variable which represents a way of weighting the age distribution. Actually in this model fertility and mortality at a point of the spatial domain depend upon all age classes (as in the Gurtin MacCamy model) and upon the population in a space neighbourhood. The diffusion coefficients depend on both the age and the layer, thus taking into account possible interaction between age and space structure. The purpose of such a framework is the description of natural population spreading in a stratified region such as fish or plankton populations that live at different water levels or insect populations that spread in a region that is fragmented and characterized by patches with very different life conditions and diffusion coefficients. The mathematical formulation of the problem falls within the theory of non-linear accretive operators and we provide a general existence and uniqueness theorem for the model as a basis for further analysis. Obviously it is important, from a biological point of view, to prove non-negativeness of the solution.

We are also interested in finding an analytical solution for the problem and in considering what happens when the spatial structure changes, namely when the number of layers increases, with an appropriate decrease of the depths of the layers.

Received: December 12, 2004.

In section 2 we present the formulation of the model and in Section 3 to the setting of the functional framework. For its analysis, in Section 4, we establish the accretiveness of the operator involved in the problem and in Section 5 we obtain the main result of existence. In Section 6 we briefly present some other results obtained for the model. In particular we consider tha case of two layers and, under certain assumptions on the vital rates, whose make the problem linear, we present the main ideas of a method for finding the analytical solution of the problem. An extended treatment of the problems and the results presented here will appear on [4] and on the paper [5] in preparation.

2. Model formulation

We consider an age structured population spreading in a one dimensional domain (y_0, y_n) partitioned into n layers (y_{j-1}, y_j) $(j = 1, \dots, n)$. In each layer we describe the population by a function $p_j(t, a, y)$ $(a \in (0, a_\dagger),\ y \in (y_{j-1}, y_j),\ t \in (0, T))$ that denotes the age-space density of the population in the layer j. We denote by $\mu_0(a)$ the intrinsic mortality of the population (which is supposed to be the same in all the layers), while $\mu_j(a, S_j(t, y))$ will denote the extra-mortality in layer j depending upon the "size" $S_j(t, y)$ defined as $(j = 1, \dots, n)$

$$S_j(t,y) = \int_0^{a_\dagger} \int_{y_{j-1}}^{y_j} \gamma_j(a,y,z) p_j(t,a,z)\,dz\,da, \quad y \in (y_{j-1}, y_j). \tag{1}$$

This variable is a weighted average of the population density operated with respect to age and to the space variable over the whole age interval and the whole layer respectively. Moreover we denote by $\beta_j(a, S_j(t, y))$ the age specific fertility and we supposed that it depends on the layer and on $S_j(t, y)$. Finally, in order to describe the diffusion process, we introduce the diffusion coefficients $K_j(a)$ that depend on the layer and on age and also consider a source $f_j(t, a, y)$ i.e. a population supply which is supposed to be explicitly given.

We use the following notations for the domains and the boundaries:

$$\Omega = (0, a_\dagger) \times (y_0, y_n),\ \Omega_j = (0, a_\dagger) \times (y_{j-1}, y_j),$$
$$\Gamma_0 = \{(0, y); y \in (y_0, y_n)\},\ \Gamma_{a_\dagger} = \{(a_\dagger, y); y \in (y_0, y_n)\},$$
$$\Gamma_{y_j} = \{(a, y_j); a \in (0, a_\dagger)\},\ j = 0, \dots, n.$$

With these premises, we have that the behaviour of the population in each layer is described by the following problem of Gurtin-MacCamy type $(j = 1, \dots, n)$

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} + \mu_0(a) p_j + \mu_j(a, S_j(t,y)) p_j - K_j(a) \frac{\partial^2 p_j}{\partial y^2} = f_j \quad in \quad (0,T) \times \Omega_j, \tag{2}$$

$$p_j(t,0,y) = \int_0^{a_\dagger} \beta_j(a, S_j(t,y)) p_j(t,a,y)\,da \qquad in \quad (0,T) \times (y_{j-1}, y_j), \tag{3}$$

$$p_j(0,a,y) = p_{j0}(a,y) \qquad in \quad \Omega_j, \tag{4}$$

where the boundary condition (3) at age $a = 0$ is the renewal condition giving the newborns rate and p_{j0} is a given initial datum. Moreover, we have to impose the continuity of the density and of the flux at the interface between two layers, namely we have the following conditions for $j = 1, \dots, n-1$,

$$p_j = p_{j+1} \text{ on } (0,T) \times \Gamma_{y_j}, \tag{5}$$

$$K_j(a)\frac{\partial p_j}{\partial y} = K_{j+1}(a)\frac{\partial p_{j+1}}{\partial y} \text{ on } (0,T)\times\Gamma_{y_j}. \tag{6}$$

Finally we assume that the population does not go through the boundary and then we impose the following boundary conditions

$$K_1(a)\frac{\partial p_1}{\partial y} = 0 \text{ on } (0,T)\times\Gamma_{y_0}, \tag{7}$$

$$K_n(a)\frac{\partial p_n}{\partial y} = 0 \text{ on } (0,T)\times\Gamma_{y_n}\ . \tag{8}$$

Concerning the mathematical conditions on the problem we shall consider some meaningful assumptions in the context of population dynamics with spatial diffusion. First we assume that the intrinsic mortality μ_0 satisfies the following assumptions

$$\begin{aligned}&\mu_0(\cdot)\in L^1_{loc}([0,a_\dagger)),\\ &\mu_0(a)\geq 0 \quad a.\ e.\ in \quad [0,a_\dagger], \quad \int_0^\infty \mu_0(a)\ da = +\infty.\end{aligned} \tag{9}$$

which are standard in the theory of age structured population dynamics. In particular these conditions guarantee that the survival probability

$$\Pi_0(a) = e^{-\int_0^a \mu_0(\sigma)d\sigma} \tag{10}$$

vanishes at the maximum age $a_\dagger$.

Concerning fertility and mortality, we assume that the functions $\beta_j(a,x)$ and $\mu_j(a,x)$ are measurable and locally Lipschitz functions on $\mathbf{R}$ in the variable x, uniformly with respect to age , i.e., there exist $L_\mu(R)>0$ and $L_\beta(R)>0$, such that as $|x|\leq R$ and $|\bar{x}|\leq R$ we have

$$|\beta_j(a,x)-\beta_j(a,\bar{x})| \leq L_\beta(R)\,|x-\bar{x}|\,, \tag{11}$$

$$|\mu_j(a,x)-\mu_j(a,\bar{x})| \leq L_\mu(R)\,|x-\bar{x}|\,. \tag{12}$$

We also consider, as reliable, that fertility and mortality and the weight function are non-negative, and the fertility and the weight function γ_j are essentially bounded

$$0\leq \beta_j(a,x)\leq \beta_+, \tag{13}$$

$$0\leq \mu_j(a,x) \quad with \quad \mu_j(a,0)=0 \tag{14}$$

and

$$\gamma_j \in L^\infty(\Omega\times(y_0,y_n)), \quad \gamma_j(a,y,z)\geq 0. \tag{15}$$

Finally we impose the following conditions on the diffusion coefficients

$$K_j(\cdot)\in L^\infty(0,a_\dagger) \ \ and \quad K_j(a)\geq K_0>0. \tag{16}$$

Before starting with the treatment of the problem we perform a change of variables that introduces a slight simplification of the problem. Namely we introduce the function replacements (see (10))

$$p_j(t,a,y) = \widetilde{p}_j(t,a,y)\Pi_0(a), \tag{17}$$

$$\widetilde{S}_j(t,y) = \int_0^{a+}\int_{y_{j-1}}^{y_j} \widetilde{\gamma}_j(a,y,z)\widetilde{p}_j(t,a,z)dzda, \tag{18}$$

$$\widetilde{\beta}_j(a,\widetilde{S}_j(t,y)) = \beta_j(a,\widetilde{S}_j(t,y))\Pi_0(a), \tag{19}$$

$$\widetilde{\gamma}_j(a,y,z) = \gamma_j(a,y,z)\Pi_0(a), \tag{20}$$

$$\widetilde{p}_{j0}(a,y) = \frac{p_{j0}(a,y)}{\Pi_0(a)}, \tag{21}$$

$$\widetilde{f}_j(t,a,y) = \frac{f_j(t,a,y)}{\Pi_0(a)}, \tag{22}$$

$$\widetilde{\mu}_j(a, \widetilde{S}_j(t,y)) = \mu_j(a, \widetilde{S}_j(t,y)), \tag{23}$$

which transform the system into the next one for the unknowns $\widetilde{p}_j(t,a,y)$. However, for the writing simplicity we shall no longer indicate the $\sim$ symbol, but we keep in mind that in the next form we refer to the functions defined by (17)-(23).

Hence, the new system we shall deal with is composed of the equations

$$\frac{\partial p_j}{\partial t} + \frac{\partial p_j}{\partial a} + \mu_j(a, S_j(t,y))p_j - K_j(a)\frac{\partial^2 p_j}{\partial y^2} = f_j \qquad in \quad (0,T)\times\Omega_j \tag{24}$$

that replace (2) while the initial and the boundary conditions read exactly like (3)-(8).

3. Functional setting of the problem

Here we want to set up a functional framework for problem (24), (3)-(8), in order to treat the existence results that will be proved in the next section. To this purpose we define the following functions on $(0,T)\times\Omega$

$$p(t,a,y) = \begin{cases} p_1(t,a,y), \ y \in (y_0, y_1), \\ \dots \\ p_n(t,a,y), \ y \in (y_{n-1}, y_n), \end{cases} \tag{25}$$

$$S(t,y) = \begin{cases} S_1(t,y), \ y \in (y_0, y_1), \\ \dots \\ S_n(t,y), \ y \in (y_{n-1}, y_n), \end{cases} \tag{26}$$

$$\beta(a,y,x) = \begin{cases} \beta_1(a,x), \quad y \in (y_0, y_1), \\ \dots \\ \beta_n(a,x), \quad y \in (y_{n-1}, y_n), \end{cases} \tag{27}$$

$$\mu(a,y,x) = \begin{cases} \mu_1(a,x), \quad y \in (y_0, y_1), \\ \dots \\ \mu_n(a,x), \quad y \in (y_{n-1}, y_n), \end{cases} \tag{28}$$

$$\gamma(a,y,z) = \begin{cases} \gamma_1(a,y,z), \ y,z \in (y_0, y_1), \\ \dots \\ \gamma_n(a,y,z), \ y,z \in (y_{n-1}, y_n), \\ \\ 0 \ \ elsewhere \end{cases} \tag{29}$$

$$K(a,y) = \begin{cases} K_1(a), \ y \in (y_0, y_1), \\ \dots \\ K_n(a), \ y \in (y_{n-1}, y_n), \end{cases} \tag{30}$$

and

$$p_0(a,y); f(t,a,y) = \begin{cases} p_{10}(a,y); f_1(t,a,y), \ y \in (y_0, y_1), \\ \dots \\ p_{n0}(a,y); f_n(t,a,y), \ y \in (y_{n-1}, y_n). \end{cases} \tag{31}$$

It should be specified that $S(t,y)$ can be rewritten as

$$S(t,y) = \int_\Omega \gamma(a,y,z)p(t,a,z)dzda. \tag{32}$$

We now consider the spaces $V = H^1(y_0, y_n)$, V' its dual, $H = L^2(y_0, y_n)$ and $H_\Omega = L^2(\Omega)$ and we define the operator $A : D(A) \subset H_\Omega \to H_\Omega$ by

$$(Au, \psi)_{H_\Omega} = \int_\Omega (u_a \psi + \mu(a, y, S(y))u\psi + K(a, y)u_y \psi_y) da dy, \quad \forall \psi \in L^2(0, a_\dagger; V), \tag{33}$$

where

$$S(y) = \int_\Omega \gamma(a, y, z)u(a, z)dz da, \tag{34}$$

and

$$\begin{aligned} D(A) = \Big\{ & u \in L^2(0, a_\dagger; V), \quad u_a \in L^2(0, a_\dagger; V'), \\ & u(0, y) = \int_0^{a_\dagger} \beta(a, y, S(y))u(a, y)da, \quad Au \in H_\Omega \Big\}. \end{aligned} \tag{35}$$

Everywhere in the following we shall use the standard notation for the Sobolev spaces on Ω or on (y_0, y_n). Moreover $W^{1,p}(0, T; H_\Omega) = \Big\{ u \in L^p(0, T; H_\Omega);\ \dfrac{du}{dt} \in L^p(0, T; H_\Omega) \Big\}$, where $\dfrac{du}{dt}$ is in the sense of distributions and $1 \le p \le \infty$. Recall (see e. g. [1]) that any $u \in W^{1,p}(0, T; H_\Omega)$ is absolutely continuous on $[0, T]$ and $\dfrac{du}{dt}$ exists a. e. on $(0, T)$. We may notice that u at $a = 0$ makes sense, since $u \in L^2(0, a_\dagger; V)$ and $u_a \in L^2(0, a_\dagger; V')$ implies $u \in C([0, a_\dagger]; H)$. We also specify that $(\cdot, \cdot)_{H_\Omega}$ means the scalar product in H_Ω.

So, we are led to the Cauchy problem

$$\frac{dp}{dt} + Ap = f \quad a.e. \quad t \in (0, T), \tag{36}$$

$$p(0) = p_0. \tag{37}$$

Henceforth, for the writing simplicity, we shall not indicate the function arguments in the integrands.

It is not difficult to prove that if p is a strong solution (see definition in [1]) to (36)-(37) then it satisfies (24), (3)-(8) in the sense of distributions.

4. The m-accretiveness of the operator A

In order to prove existence of a solution to problem (36)-(37), we are going to show the quasi m-accretiveness of the operator A. We first note that the assumptions (11) -(16) imply the same properties for the newly defined functions (27)-(30). Namely for $|x| \le R$ and $|\bar{x}| \le R$, we have that,

$$|\beta(a, y, x) - \beta(a, y, \bar{x})| \le L_\beta(R) |x - \bar{x}|, \tag{38}$$

$$|\mu(a, y, x) - \mu(a, y, \bar{x})| \le L_\mu(R) |x - \bar{x}|, \tag{39}$$

and, for any $x \in \mathbf{R}$

$$0 \le \beta(a, y, x) \le \beta_+, \tag{40}$$

$$0 \le \mu(a, y, x) \quad with \quad \mu(a, y, 0) = 0, \tag{41}$$

$$0 \le \gamma(a, y, z) \le \gamma_\infty, \tag{42}$$

$$K(a, y) \ge K_0, \tag{43}$$

where $\gamma_\infty \overset{def.}{=} \max_{j=1,\dots,n} \{ \|\gamma_j\|_{L^\infty(\Omega_j)} \}$ and $L_\mu(R)$, $L_\beta(R)$, β_+, K_0 are the same as before.

Let us consider the following functions

$$u \mapsto E(u) \equiv \mu(a,y,S(y))u(a,y) \tag{44}$$

and

$$u \mapsto F(u) \equiv \beta(a,y,S(y))u(a,y) \tag{45}$$

which, under the conditions (38) and (39), are locally Lipschitz continuous from the space H_Ω to H_Ω, i. e. for any $R > 0$ there exist $M(R)$ and $B(R)$ such that if $\|u\|_{H_\Omega} \leq R$ and $\|\bar{u}\|_{H_\Omega} \leq R$, then

$$\|E(u) - E(\bar{u})\|_{H_\Omega} \;\leq\; M(R)\,\|u - \bar{u}\|_{H_\Omega}\,, \tag{46}$$

$$\|F(u) - F(\bar{u})\|_{H_\Omega} \;\leq\; B(R)\,\|u - \bar{u}\|_{H_\Omega}\,, \tag{47}$$

(see Lemma 1 in [4] for a complete proof).

We note however that the proof of quasi m-accretiveness of A essentially works under the hypotheses that E and F are Lipschitz continuous on H_Ω. Consequently, in Section 5 we will prove existence of a solution to problem (36)-(37), under the hypothesis that E and F are Lipschitz continuous functions on H_Ω. After then, in Theorem 5.2, we will apply a method presented in [2] in order to treat the case of E and F locally Lipschitz continuous.

We first prove the quasi-accretiveness.

Lemma 4.1. *Assume (11)-(16) and suppose that E and F are globally Lipschitz continuous functions, i. e. there exist constants $M > 0$ and $B > 0$ such that for $u, \bar{u} \in H_\Omega$*

$$\|E(u) - E(\bar{u})\|_{H_\Omega} \leq M\,\|u - \bar{u}\|_{H_\Omega}\,, \tag{48}$$

$$\|F(u) - F(\bar{u})\|_{H_\Omega} \leq B\,\|u - \bar{u}\|_{H_\Omega}\,. \tag{49}$$

Then, the operator A is quasi accretive on H_Ω.

Proof. Let $u, \bar{u} \in D(A)$, then, using (48) and (49), we have

$$((\lambda I + A)u - (\lambda I + A)\bar{u}, u - \bar{u})_{H_\Omega} = \lambda\,\|u - \bar{u}\|^2_{H_\Omega}$$

$$+ \int_\Omega \{(u_a - \bar{u}_a)(u - \bar{u}) + (E(u) - E(\bar{u}))(u - \bar{u})\}dyda$$

$$+ \int_\Omega K(a,y)(u_y - \bar{u}_y)^2 dyda \geq$$

$$\geq \lambda\,\|u - \bar{u}\|^2_{H_\Omega} + \frac{1}{2}\int_{y_0}^{y_n} (u - \bar{u})^2\big|_{a=a_\dagger}\,dy - \frac{1}{2}\int_{y_0}^{y_n} (u - \bar{u})^2\big|_{a=0}\,dy$$

$$- \left|\int_\Omega (E(u) - E(\bar{u}))(u - \bar{u})dyda\right| + K_0\,\|u_y - \bar{u}_y\|^2_{H_\Omega} \geq$$

$$\geq \left(\lambda - \frac{B^2 a_\dagger}{2} - M\right)\|u - \bar{u}\|^2_{H_\Omega} + K_0\,\|u_y - \bar{u}_y\|^2_{H_\Omega} \geq 0$$

for λ large enough, $\lambda \geq \dfrac{B^2 a_\dagger}{2} + M$. □

The following lemma states m-accretiveness

Lemma 4.2. *Assume the same conditions as in Lemma 4.1. Then A is quasi m-accretive on H_Ω.*

Proof. Since A is quasi accretive, it remains to show that

$$\textit{Range}\ (\lambda I + A) = H_\Omega \tag{50}$$

for λ sufficiently large, i.e., for any $f \in H_\Omega$ we should find $u \in L^2(0, a_\dagger; V)$ with $u_a \in L^2(0, a_\dagger; V')$ such that:

$$\lambda u + Au = f, \tag{51}$$

$$u(0, y) = \int_0^{a_\dagger} F(u)(a, y) da. \tag{52}$$

To come to this end we study the associated Cauchy problem

$$u_a + (\lambda I + A_V(a))u = f - E(\omega), \tag{53}$$

$$u(0, y) = \int_0^{a_\dagger} F(\omega)(a, y) da \tag{54}$$

where $\omega \in H_\Omega$ is fixed and $A_V(a) : V \to V'$ is defined by

$$< A_V(a)v, \psi >_{V',V} = \int_\Omega K(a, y) v_y \psi_y dy,\ \forall \psi \in V.$$

One can prove that the linear operator $\lambda I + A_V(a)$ is coercive and bounded. Since A_V depends on a and it is continuous, the function $a \mapsto A_V(a)v$ is also measurable from $[0, a_\dagger]$ to V'.

Moreover, for ω fixed in H_Ω, one has that $u(0, \cdot) \in H$ and $f - E(\omega) \in H_\Omega \equiv L^2(0, a_\dagger; H)$; so that the hypotheses of Lions theorem (see [6]) are verified and we can conclude that the Cauchy problem (53)-(54) has a unique solution $u \in L^2(0, a_\dagger; V)$, $u_a \in L^2(0, a_\dagger; V')$.

Now it remains to prove that the mapping $\mathcal{P} : H_\Omega \to H_\Omega$, that associates to $\omega \in H_\Omega$ the corresponding solution u to the problem (53)-(54), is a contraction on H_Ω for λ sufficiently large.

Using (54) and the fact that E and F are Lipschitz continuous, we have, for u and $\overline{u}$ solutions corresponding to ω and $\overline{\omega}$,

$$\lambda \left\| u - \bar{u} \right\|_{H_\Omega}^2 \leq \left(B^2 a_\dagger + \frac{M^2}{\lambda} \right) \left\| \omega - \bar{\omega} \right\|_{H_\Omega}^2 .$$

This means that for λ sufficiently large the function $\mathcal{P}(\cdot)$ is a contraction on H_Ω, so that the equation $u = \mathcal{P}(u)$ has a solution. Consequently (51)-(52) has a solution $u \in L^2(0, a_\dagger; V)$, with $u_a \in L^2(0, a_\dagger; V')$ and since $f \in H_\Omega$ we still obtain that $Au = f - \lambda u \in H_\Omega$, proving that $u \in D(A)$. Thus it follows that A is m-accretive in H_Ω. □

Now we are ready to discuss existence of a solution for problem (36)-(37).

5. Existence and properties of the solution

The previous Section is the main step for the analysis of Problem (36)-(37). Actually the first step is to prove existence results and the properties of the solution, under global Lipschitz conditions. These results are formulated and proved in the following theorem

Theorem 5.1. *Assume (11)-(16) and (48)-(49) and let*

$$f \in W^{1,1}(0, T; H_\Omega), \tag{55}$$

$$p_0 \in D(A). \tag{56}$$

Then the problem (36)-(37) has a unique strong solution $p \in C([0,T]; H_\Omega)$

$$p \in W^{1,\infty}(0,T; H_\Omega) \cap L^\infty(0,T; D(A)) \tag{57}$$

which satisfies the estimates

$$\|p(t) - \bar{p}(t)\|^2_{H_\Omega} \leq \left(\|p_0 - \bar{p}_0\|^2_{H_\Omega} + \int_0^T \left\|f(\tau) - \bar{f}(\tau)\right\|^2_{H_\Omega} d\tau \right) e^{\alpha_0 t}, \tag{58}$$

$$\begin{aligned} &\|p(t)\|^2_{H_\Omega} + \int_0^t \|p(\tau, a_\dagger)\|^2_H + \int_0^t \int_0^{a_\dagger} \|p(\tau,a)\|^2_V \, d\tau da \leq \\ &\leq \frac{1}{K_{\min}} \left(\|p_0\|^2_{H_\Omega} + \int_0^T \|f(\tau)\|^2_{H_\Omega} d\tau \right) \exp[(\beta_+^2 a_\dagger + 2)t] < \infty, \end{aligned} \tag{59}$$

for any $t \in [0,T]$*, where* $\alpha_0 = (B^2 a_\dagger + 2M + 1)$ *and* $K_{\min} = \min(1, K_0)$*.*

Proof. The first part and (57) follow immediately from the fundamental results concerning the existence for evolution equations with m-accretive operators in Hilbert spaces (see e.g. [1]). Also estimate (58) is a direct consequence of quasi-accretiveness. In fact, let us consider two solutions p and $\bar{p}$ corresponding to the initial data p_0, f and respectively $\bar{p}_0, \bar{f}$ and multiply the equation

$$\frac{d}{dt}(p - \bar{p}) + Ap - A\bar{p} = f - \bar{f}$$

by $(p - \bar{p})$ to obtain

$$\frac{d}{dt} \|p(t) - \bar{p}(t)\|^2_{H_\Omega} + 2\left(Ap(t) - A\bar{p}(t), p(t) - \bar{p}(t)\right)_{H_\Omega} = 2\left(f(t) - \bar{f}(t), p(t) - \bar{p}(t)\right)_{H_\Omega}.$$

Then, using quasi-accretiveness of A with $\lambda = \dfrac{B^2 a_\dagger}{2} + M$ (see Lemma 4.1) and integrating over $(0,t)$ for $t \in [0,T]$, we get

$$\begin{aligned} \|p(t) - \bar{p}(t)\|^2_{H_\Omega} &\leq \|p_0 - \bar{p}_0\|^2_{H_\Omega} + \\ &+ \alpha_0 \int_0^t \|p(\tau) - \bar{p}(\tau)\|^2_{H_\Omega} d\tau + \int_0^t \left\|f(\tau) - \bar{f}(\tau)\right\|^2_{H_\Omega} d\tau \end{aligned}$$

and, applying Gronwall's lemma, we obtain (58).

In order to obtain the estimate (59), that is actually independent of the Lipschitz constants B and M, we multiply (36) by p, we integrate over $(0,t)$ and we use an estimate for $(Au,u)_{H_\Omega}$ which can be obtained using (13), (14) and (16), then we get

$$\begin{aligned} &\|p(t)\|^2_{H_\Omega} + \int_0^t \|p(\tau)(a_\dagger)\|^2_H d\tau + K_0 \int_0^t \int_0^{a_\dagger} \|p_y(\tau)(a)\|^2_H \, da d\tau + \int_0^t \|p(\tau)\|^2_{H_\Omega} d\tau \leq \\ &\leq \|p_0\|^2_{H_\Omega} + \int_0^t \|f(\tau)\|^2_{H_\Omega} d\tau + (\beta_+^2 a_\dagger + 2) \int_0^t \|p(\tau)\|^2_{H_\Omega} d\tau. \end{aligned}$$

Then, first by Gronwall's lemma we obtain that for any $0 \leq t \leq T < \infty$

$$\|p(t)\|^2_{H_\Omega} \leq \left(\|p_0\|^2_{H_\Omega} + \int_0^t \|f(\tau)\|^2_{H_\Omega} d\tau \right) \exp[(\beta_+^2 a_\dagger + 2)t] < \infty,$$

and, using this in the previous inequality, we have

$$K_{\min}\left(\|p(t)\|^2_{H_\Omega}+\int_0^t\|p(\tau)(a_\dagger)\|^2_H+\int_0^t\int_0^{a_\dagger}\|p(\tau)(a)\|^2_V\,d\tau da\right)\le$$
$$\le\left(\|p_0\|^2_{H_\Omega}+\int_0^t\|f(\tau)\|^2_{H_\Omega}\,d\tau\right)\exp[(\beta_+^2a_\dagger+2)t],$$
where $K_{\min}=\min(1,K_0)$. This implies (59) as claimed, thereby completing the proof. □

For a later use, it is useful to notice that $p_a\in L^\infty(0,T;L^2(0,a_\dagger;V'))$.

A further result concerns non-negativeness of the solution, according with the biological meaning of the problem.

Lemma 5.1. *Assume the conditions of Theorem 5.1 and*
$$f\ge 0\quad a.e.\ in\quad (0,T)\times\Omega, \tag{60}$$
$$p_0\ge 0\quad a.e.\ in\quad \Omega. \tag{61}$$
Then the solution p to problem (36)-(37) satisfies
$$p(t)\ge 0\quad a.e.\ in\quad \Omega\quad for\ each\quad t\in[0,T]. \tag{62}$$

Proof. Under the assumptions of Theorem 5.1 there exists a solution to problem (36)-(37) with the properties specified in (57)-(59). We have to show that the negative part of this solution $p^-(t)=0$ a.e. in Ω for each $t\in[0,T]$.

We multiply equation (36) by $p^-(\tau)$ and integrate over $(0,t)\times\Omega$, for any $t\in[0,T]$. We have, by Stampacchia's lemma that
$$-\frac{1}{2}\int_0^t\int_\Omega\frac{\partial(p^-)^2}{\partial\tau}(\tau,a,y)dydad\tau-\frac{1}{2}\int_0^t\int_{y_0}^{y_n}(p^-)^2(\tau,a_\dagger,y)dyd\tau$$
$$-\int_0^t\int_\Omega K(a,y)(p_y^-)^2(\tau,a,y)dydad\tau=-\frac{1}{2}\int_0^t\int_{y_0}^{y_n}(p^-)^2(\tau,0,y)dyd\tau$$
$$+\int_0^t\int_\Omega\mu(a,y,S(y))(p^-)^2(\tau,a,y)dydad\tau+\int_0^t\int_\Omega f(\tau)(a,y)p^-(\tau,a,y)dydad\tau.$$
Then we obtain, that
$$\frac{1}{2}\left\|p^-(t)\right\|^2_{H_\Omega}-\frac{1}{2}\left\|p^-(0))\right\|^2_{H_\Omega}\le\frac{1}{2}\int_0^t\int_{y_0}^{y_n}(p^-)^2(\tau,0,y)d\tau dy\le$$
$$\le\frac{\beta_+^2a_\dagger}{2}\int_0^t\left\|p^-(\tau)\right\|^2_{H_\Omega}d\tau.$$
where we used (13), (14) and (60). We have to specify that in the above calculations we took into account that if $\phi(t)=\phi_1(t)-\phi_2(t)$, with $\phi_1(t)\ge 0$ and $\phi_2(t)\ge 0$, then it follows that $\phi^-(t)\le\phi_2(t)$.

Hence, since we can write
$$p(\tau,0,y)=\int_0^{a_\dagger}\beta(a,y,S(\tau,y))p^+(\tau,a,y)da-\int_0^{a_\dagger}\beta(a,y,S(\tau,y))p^-(\tau,a,y)da,$$
and $\beta(a,y,S(t,y))\ge 0$; we have
$$p^-(\tau,0,y)\le\int_0^{a_\dagger}\beta(a,y,S)p^-(\tau,a,y)da\le\beta_+\int_0^{a_\dagger}p^-(\tau,a,y)da\le$$
$$\le\beta_+\sqrt{a_\dagger}\left(\int_0^{a_\dagger}(p^-)^2(\tau,a,y)da\right)^{1/2}$$

and consequently we get

$$\int_0^t \int_{y_0}^{y_n} (p^-)^2(\tau, 0, y) dy d\tau \le \beta_+^2 a_\dagger \int_0^t \left\| p^-(\tau) \right\|_{H_\Omega}^2 d\tau.$$

Applying Gronwall's lemma and using (61) we finally conclude that $\|p^-(t)\|_{H_\Omega}^2 = 0$, $\forall t \in [0, T]$, hence $p(t) \ge 0$ a.e. on Ω for each $t \in [0, T]$. □

Once finished this basic part, we can pass to the existence proof under the main assumptions specified before by (11)-(16). In fact we may use Theorem 5.1, in particular estimate (59) which is independent of the Lipschitz constants M and B. We have

Theorem 5.2. *Assume the conditions (11)-(16) and (55)-(56). Then the problem (36)-(37) has a unique strong solution $p \in C([0, T]; H_\Omega)$ such that*

$$p \in W^{1,\infty}(0, T; H_\Omega) \cap L^\infty(0, T; D(A)).$$

and

$$\|p(t)\|_{H_\Omega}^2 \le \frac{1}{K_{\min}} \left(\|p_0\|_{H_\Omega}^2 + \int_0^t \|f(\tau)\|_{H_\Omega}^2 d\tau \right) \exp[(\beta_+^2 a_\dagger + 2)t].$$

Proof. We have already noticed that, by the assumption of the theorem, we have (38)-(43). Moreover we know that, the functions (44) and (45) are locally Lipschitz from H_Ω to H_Ω. We shall reduce the problem to the previous case for which these functions are Lipschitz continuous. To this end we shall approximate $E(\cdot)$ and $F(\cdot)$ (see [2]) for each $N \ge 1$ setting

$$E_N(u) = \begin{cases} E(u) & \textit{for} \quad \|u\|_{H_\Omega} \le N \\ E\left(\dfrac{Nu}{\|u\|_{H_\Omega}}\right) & \textit{for} \quad \|u\|_{H_\Omega} > N \end{cases} \tag{63}$$

and

$$F_N(u) = \begin{cases} F(u) & \textit{for} \quad \|u\|_{H_\Omega} \le N \\ F\left(\dfrac{Nu}{\|u\|_{H_\Omega}}\right) & \textit{for} \quad \|u\|_{H_\Omega} > N \end{cases} \tag{64}$$

Actually these truncated functions are Lipschitz continuous on H_Ω (for each N fixed). Therefore, we consider the approximating problem

$$\frac{dp_N}{dt} + A_N p_N = f, \tag{65}$$

$$p_N(0) = p_0, \tag{66}$$

where A_N is defined by (33)-(35) in which $E(\cdot)$ and $F(\cdot)$ are replaced by $E_N(\cdot)$ and $F_N(\cdot)$ respectively. Since for each N, the assumptions of Theorem 5.1 are fulfilled, we find that for $f \in W^{1,1}(0, T; H_\Omega)$ and $p_0 \in H_\Omega$ problem (65)-(66) has, for each N, a solution

$$p_N \in C([0, T]; H_\Omega) \cap L^\infty(0, T; V), \ (p_N)_a \in L^\infty(0, T; V'), \tag{67}$$

$$p_N(t, 0, y) = \int_0^{a_\dagger} \beta_N(a, y, S_N(t, y)) p_N(t, a, y) da, \ A_N p_N \in H_\Omega. \tag{68}$$

This solution satisfies (59) which is independent of N. Namely, for each T we have

$$\|p_N(t)\|_{H_\Omega}^2 \le R \quad \textit{for} \quad t \in [0, T]$$

with

$$R = \frac{1}{K_{\min}} \left(\|p_0\|^2_{H_\Omega} + \int_0^T \|f(\tau)\|^2_{H_\Omega} \, d\tau \right) \exp[(\beta_+^2 a_\dagger + 2)T] < \infty \tag{69}$$

independent of N. In conclusion, for N sufficiently large, $N > R$, we get that

$$A_N p_N(t) = A p_N(t), \qquad p_N(t, 0, y) = \int_0^{a_\dagger} F(p_N(t))(a, y) da,$$

so that $p_N(t)$ is actually a solution to problem (36)-(37).

To prove the uniqueness we consider that there exist two solutions p and $\bar{p}$ corresponding to the same data f and p_0. Then, by the previous proof we have that if

$$N > \sup_{t \in [0,T]} \|p(t)\|_{H_\Omega} + \sup_{t \in [0,T]} \|\bar{p}(t)\|_{H_\Omega},$$

then $p(t) = p_N(t)$ and $\bar{p}(t) = p_N(t)$, where p_N is the solution to (65)-(66). This proves that the solution is unique.

Finally, estimate (5.2) is a direct consequence of (69). □

6. Analytical solution in the two layer linear case

In this section we present a particular case of the model presented in Section 2, namely we consider the case of two layers of the same thickness h with $y_0 = 0$ and then $y_1 = h$ and $y_2 = 2h$, moreover we suppose that fertility and mortality are the same in the two layers and they are functions of age only. Finally we suppose that the diffusion coefficients K_1 and K_2 do not depend on age. Thus we have to deal with the following linear problem ($j = 1, 2$)

$$\frac{\partial p_j}{\partial t}(t, a, y) + \frac{\partial p_j}{\partial a}(t, a, y) + \mu(a) p_j(t, a, y) - K_j \frac{\partial^2 p_j}{\partial y^2}(t, a, y) = 0, \tag{70}$$

$$p_j(0, a, y) = p_{j0}(a, y), \tag{71}$$

$$p_j(t, 0, y) = \int_0^\infty \beta(a) p_j(t, a, y) \, da, \tag{72}$$

$$\frac{\partial p_1}{\partial y}(t, a, 0) = 0, \qquad \frac{\partial p_2}{\partial y}(t, a, 2h) = 0, \tag{73}$$

$$K_1 \frac{\partial p_1}{\partial y}(t, a, h^-) = K_2 \frac{\partial p_2}{\partial y}(t, a, h^+), \tag{74}$$

$$p_1(t, a, h^-) = p_2(t, a, h^+). \tag{75}$$

In order to treat this problem, first of all we introduce the usual semplification by replacing $p_j(t, a, y) = \widetilde{p}_j(t, a, y)\Pi(a)$ with $\Pi(a) = e^{-\int_0^a \mu(s) ds}$. Thus problem (70)-(75) becomes the following one for the functions $\widetilde{p}_j$, however we do not indicate the symbol $\sim$ in order to simplify the notation,

$$\frac{\partial p_j}{\partial t}(t, a, y) + \frac{\partial p_j}{\partial a}(t, a, y) - K_j \frac{\partial^2 p_j}{\partial y^2}(t, a, y) = 0, \tag{76}$$

$$p_j(0, a, y) = p_{j0}(a, y), \tag{77}$$

$$p_j(t, 0, y) = \int_0^\infty m(a) p_j(t, a, y) \, da, \tag{78}$$

$$\Pi(a)\frac{\partial p_1}{\partial y}(t,a,0)=0, \qquad \Pi(a)\frac{\partial p_2}{\partial y}(t,a,2h)=0, \tag{79}$$

$$K_1\Pi(a)\frac{\partial p_1}{\partial y}(t,a,h^-)=K_2\Pi(a)\frac{\partial p_2}{\partial y}(t,a,h^+), \tag{80}$$

$$p_1(t,a,h^-)=p_2(t,a,h^+), \tag{81}$$

where $m(a)$ is the maternity function $m(a)=\beta(a)\Pi(a)$.

In order to study problem (76)-(81) we consider the corresponding problem for the Laplace transforms of the functions $p_j(t,a,y)$ with respect to the variable t, namely define

$$u_j(\lambda,a,y)=\int_0^\infty e^{-\lambda\sigma}p_j(\sigma,a,y)\ da,$$

the problem for u_j turns out to be

$$\lambda u_j(\lambda,a,y)+\frac{\partial u_j}{\partial a}(\lambda,a,y)-K_j\frac{\partial^2 u_j}{\partial y^2}(\lambda,a,y)=p_{j0}(a,y), \qquad j=1,2, \tag{82}$$

$$u_j(\lambda,0,y)=\int_0^\infty m(a)u_j(\lambda,a,y)\ da, \qquad j=1,2, \tag{83}$$

$$K_1\Pi(a)\frac{\partial u_1}{\partial y}(\lambda,a,h^-)=K_2\Pi(a)\frac{\partial u_2}{\partial y}(\lambda,a,h^+), \tag{84}$$

$$u_1(\lambda,a,h^-)=u_2(\lambda,a,h^+). \tag{85}$$

The idea is to study the problem in the single layer. First of all we introduce a new variable $q(\lambda,a)$, representing the flux at the interface between the two layers rescaled by the survival probability, namely

$$q(\lambda,a)=K_1\Pi(a)\frac{\partial u_1}{\partial y}(\lambda,a,h^-)=K_2\Pi(a)\frac{\partial u_2}{\partial y}(\lambda,a,h^+).$$

One solves analytically the problem (82)-(85) by looking for

$$u_1(\lambda,a,y)=U_1(\lambda,a,y)+\frac{y^2}{2h}\frac{q(\lambda,a)}{K_1\Pi(a)}$$

$$u_2(\lambda,a,y)=U_2(\lambda,a,y)-\frac{(y-2h)^2}{2h}\frac{q(\lambda,a)}{K_2\Pi(a)}$$

where U_1 (respectively U_2) is the solution of the problem in the first (respectively the second) layer with homogeneous boundary conditions. These homogeneous boundary problems can be solved via Fourier expansion. After obtaining the representations for U_1 and U_2, one obtains expressions for u_1 and u_2, which explicitely depend on the new variable q, via the following function

$$w(\lambda,a)=\frac{\partial}{\partial a}\left(\frac{q(\lambda,a)}{K_1\Pi(a)}\right)+\lambda\frac{q(\lambda,a)}{K_1\Pi(a)}.$$

Once obtained these expressions we have to impose the continuity condition of the solution on the boundary between the two layers, i. e. condition (85). Thus it turns out that $w(\lambda,a)$ has to solve an integral equation containing both a Volterra-type term and a Fredholm-type term, namely

$$\int_0^a V(\lambda,a-\tau)w(\lambda,\tau)\ d\tau+\int_0^\infty F(\lambda,a,\tau)w(\lambda,\tau)\ d\tau=H(\lambda,a), \tag{86}$$

where

$$\begin{aligned} V(\lambda,\alpha) = \quad & he^{-\lambda\alpha}\left[\frac{1}{6}\left(1+\frac{K_1}{K_2}\right)\right. \\ & \left. -\frac{2}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{e^{-\frac{n^2\pi^2}{h^2}K_1\alpha}}{n^2}+\frac{K_1}{K_2}\sum_{n=1}^{\infty}\frac{e^{-\frac{n^2\pi^2}{h^2}K_2\alpha}}{n^2}\right)\right], \end{aligned} \tag{87}$$

$$\begin{aligned} F(\lambda,a,\tau) = \quad & he^{-\lambda a}\left[\frac{1}{6}\left(1+\frac{K_1}{K_2}\right)\gamma(\lambda,\tau)\right. \\ & -\frac{2}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{e^{-\frac{n^2\pi^2}{h^2}K_1 a}}{n^2}\gamma_1^{(n)}(\lambda,\tau)\right. \\ & \left.\left. +\frac{K_1}{K_2}\sum_{n=1}^{\infty}\frac{e^{-\frac{n^2\pi^2}{h^2}K_2 a}}{n^2}\gamma_2^{(n)}(\lambda,\tau)\right)\right], \end{aligned} \tag{88}$$

and

$$\begin{aligned} H(\lambda,a) \quad = & \sqrt{\frac{2}{h}}e^{-\lambda a}\left[\int_0^a e^{\lambda\tau}\sum_{n=0}^{\infty}\left(e^{-\frac{n^2\pi^2}{h^2}K_2(a-\tau)}p_{20}^{(n)}(\tau)\right.\right. \\ & \left. -(-1)^n e^{-\frac{n^2\pi^2}{h^2}K_1(a-\tau)}p_{10}^{(n)}(\tau)\right)\ d\tau \\ & +\int_0^{\infty}\sum_{n=0}^{\infty}\left(e^{-\frac{n^2\pi^2}{h^2}K_2 a}\gamma_2^{(n)}(\lambda,\tau)p_{20}^{(n)}(\tau)\right. \\ & \left.\left. -(-1)^n e^{-\frac{n^2\pi^2}{h^2}K_1 a}\gamma_1^{(n)}(\lambda,\tau)p_{10}^{(n)}(\tau)\right)\ d\tau\right]. \end{aligned} \tag{89}$$

where $\gamma(\lambda,\tau)$, $\gamma_1^{(n)}(\lambda,\tau)$, $\gamma_2^{(n)}(\lambda,\tau)$ are known functions depending on the maternity function m.

The integral equation (86) is then the key ingredient for the analytical solution of problem (70)-(75), in fact once we have the function w we can come back to u_1 and u_2 and then to p_1 and p_2.

Moreover we may notice that the rappresentation in terms of w, which is the solution of a Volterra integral equation, points out the fact that the age-structure is strictly related to a delay.

References

[1] Barbu, V. *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishing, Leyden (1976).

[2] Barbu, V., Iannelli, M. *The semigroup approach to non-linear age-structured equations*, Rend. Istit. Mat. Univ. Trieste **(Suppl. Vol. XXVIII)** (1997) 59-71.

[3] Brezis H., Pazy A. *Convergence and approximation of semigroups of nonlinear operators in Banach spces*, J. Funct. Anal., **(9)** (1972) 63–74.

[4] Cusulin, C., Iannelli, M., Marinoschi, G. *Age-structured diffusion in a multi-layer environment*, Nonlinear Anal. Real World Appl. **(6-1)** (2005) 207-223.

[5] Cusulin, C., Iannelli, M., Marinoschi, G. *Analytical solution for the linear age-structured multi-layer model*, in preparation.

[6] Lions, J. L. *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, (1969).
[7] Trotter, H. F. *Approximation of semigroups of operators*, Pacific J. Math. 8 (1958) 887-919.

(Caterina Cusulin) Dipartimento di Matematica,
Università di Trento,
38050 Povo (Trento), Italy
E-mail address: cusulin@science.unitn.it