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Bifurcations and asymptotics for singular elliptic problems

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Abstract. We consider a class of singular elliptic problems in bounded domains. The main feature here is the presence of the singular and a sublinear nonlinearities combined with a sub-quadratic convection term. We establish existence, nonexistence, bifurcation and asymptotic behavior of the solution. We also describe the decay rate of the solution as well as a blow-up result around the bifurcation point.

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Singular elliptic problems have been intensively studied in the last decades. Problems of this type arise in the study of non-Newtonian fluids, chemical heterogenous catalysts, in the theory of heat conduction in electrically conducting materials. For instance, problems of this type characterize some reaction-diffusion processes where the condition $u \geq 0$ is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, where no reaction takes place (see [1] for the study of a single, irreversible steady-state reaction).

Nonlinear singular elliptic equations are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents (see [4] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence). For more details we also refer to [6, 4, 14, 16] and the references therein.

We are first concerned with a sublinear singular elliptic problems with two parameters

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (P_{\lambda,\mu})$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ is a smooth bounded domain, $K, h \in C^{0,\gamma}(\overline{\Omega})$, with $h > 0$ on Ω and λ, μ are positive real numbers. We suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is positive on $\overline{\Omega} \times (0,\infty)$. We also assume that f is nondecreasing with respect to the second variable and is sublinear, that is,

$$(f1) \qquad \text{the mapping } (0,\infty) \ni s \longmapsto \frac{f(x,s)}{s} \quad \text{is nonincreasing for all } x \in \overline{\Omega};$$

$$(f2) \qquad \lim_{s \searrow 0} \frac{f(x,s)}{s} = +\infty \quad \text{and} \quad \lim_{s \to \infty} \frac{f(x,s)}{s} = 0, \quad \text{uniformly for } x \in \overline{\Omega}.$$

We assume that $g \in C^{0,\gamma}(0,\infty)$ is a nonnegative and nonincreasing function satisfying

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(g1) $\lim_{s\searrow 0} g(s) = +\infty$;

(g2) there exist $C, \delta_0 > 0$ and $\alpha \in (0,1)$ such that $g(s) \le Cs^{-\alpha}$ for all $s \in (0, \delta_0)$.

Obviously, hypothesis (g2) implies the following Keller-Osserman type condition around the origin

(g3) $$\int_0^1 \left(\int_0^t g(s)ds \right)^{-1/2} dt < \infty.$$

As proved by Bénilan, Brezis and Crandall [2], condition (g3) is equivalent to the *property of compact support*, that is, for every $h \in L^1(\mathbf{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbf{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbf{R}^N)$ and

$$-\Delta u + g(u) = h \qquad \text{a.e. in } \mathbf{R}^N.$$

Due to the singular character of $(P_{\lambda,\mu})$, we can not expect to find solutions in $C^2(\overline{\Omega})$. However, under the above assumptions we will show that $(P_{\lambda,\mu})$ has at least one solution in the class $\mathcal{E} := \{ u \in C^2(\Omega);\ \Delta u \in L^1(\Omega)\}$ for λ, μ belonging to a certain range.

A fundamental role will be played in our analysis by the numbers

$$K^* = \max_{x\in\overline{\Omega}} K(x),\ K_* = \min_{x\in\overline{\Omega}} K(x).$$

Our main results are the following.

Theorem 1. *Assume that $K_* > 0$, f satisfies $(f1)-(f2)$ and g satisfy $(g1)$. If $\int_0^1 g(s)ds = +\infty$, then $(P_{\lambda,\mu})$ has no solution in $\mathcal{E}$ for any $\lambda, \mu > 0$.*

Theorem 2. *Assume that $K_* > 0$, f satisfies $(f1)-(f2)$ and g satisfies $(g1)-(g2)$. Then there exist $\lambda_*, \mu_* > 0$ such that*

$(P_{\lambda,\mu})$ has at least one solution in $\mathcal{E}$ if $\lambda > \lambda_*$ or $\mu > \mu_*$.

$(P_{\lambda,\mu})$ has no solution in $\mathcal{E}$ if $\lambda < \lambda_*$ and $\mu < \mu_*$.

Moreover, if $\lambda > \lambda_*$ or $\mu > \mu_*$, then $(P_{\lambda,\mu})$ has a maximal solution in $\mathcal{E}$ which is increasing with respect to λ and μ.

Theorem 3. *Assume that $K^* \le 0$, f satisfies $(f1)-(f2)$ and g satisfies $(g1)-(g2)$. Then $(P_{\lambda,\mu})$ has a unique solution $u_{\lambda,\mu} \in \mathcal{E}$ for any $\lambda, \mu > 0$. Moreover, $u_{\lambda,\mu}$ is increasing with respect to λ and μ.*

Theorems 2 and 3 also show the role played by the sublinear term f and the sign of $K(x)$. Indeed, if f becomes linear then the situation changes radically. First, by the results in [8], the problem

$$\begin{cases} -\Delta u - u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a solution, for any $\alpha > 0$. Next, as showed in [3], the problem

$$\begin{cases} -\Delta u + u^{-\alpha} = u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

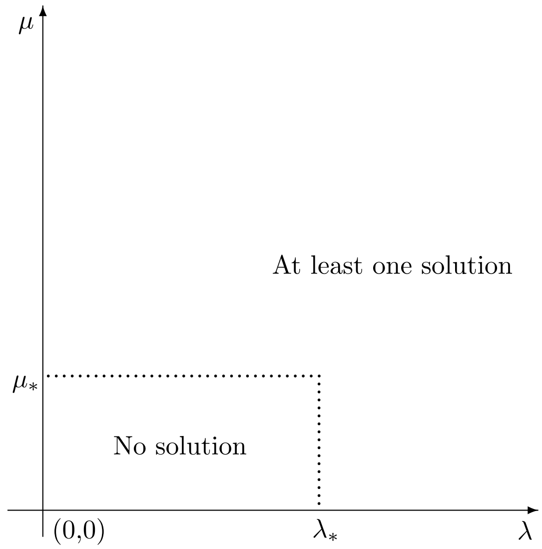

FIGURE 1. The dependence on λ and μ in Theorem 2.

has no solution, provided $0 < \alpha < 1$ and $\lambda_1 \geq 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

As it was pointed out in [5], problems related to multiplicity or to uniqueness become difficult even in simple cases. In this sense we also refer to [17], where it is studied the existence of radial symmetric solutions of the problem

$$\begin{cases} \Delta u + \lambda(u^p - u^{-\alpha}) = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1}$$

where $0 < \alpha, p < 1$, $\lambda > 0$ and B_1 is the unit ball in $\mathbf{R}^N$. Using a bifurcation theorem of Crandall and Rabinowitz, it has been shown in [17] that there exist $\lambda_1 > \lambda_0 > 0$ such that (1) has no solutions for $\lambda < \lambda_0$, one solution for $\lambda = \lambda_0$ or $\lambda > \lambda_1$, two solutions for $\lambda_1 \geq \lambda > \lambda_0$.

We are next concerned with the following singular elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbf{R}^N$ $(N \geq 2)$ is a bounded domain and $\lambda \in \mathbf{R}$ is a real parameter. We suppose here that $0 < f \in C^{0,\gamma}[0,\infty)$ is nondecreasing on $(0,\infty)$ while $f(s)/s$ is nonincreasing for $s > 0$; and $0 \leq g \in C^{0,\gamma}(0,\infty)$ $(0 < \beta < 1)$ fulfills the hypotheses $(g1), (g2)$. We first observe that there exists

$$m := \lim_{s \to \infty} \frac{f(s)}{s} \in [0, \infty).$$

This number plays a crucial role in our analysis. More precisely, the existence of the solutions to (P_λ) will be separately discussed for $m > 0$ and $m = 0$. Let $a_* = \min_{x \in \overline{\Omega}} a(x)$.

Our first result is

Theorem 4. *Assume $(g1)$, $(g2)$ and $m = 0$. If $a_* > 0$ (resp., $a_* = 0$), then (P_λ) has a unique solution $u_\lambda \in \mathcal{E} \cap C^{1,1-\alpha}(\overline{\Omega})$ for all $\lambda \in \mathbf{R}$ (resp., $\lambda \geq 0$) with the properties:*
(i) *u_λ is strictly increasing with respect to λ.*
(ii) *there exist two positive constant $c_1, c_2 > 0$ depending on λ such that $c_1 d(x) \leq u_\lambda \leq c_2 d(x)$ in Ω.*

The bifurcation diagram in the "sublinear" case $m = 0$ is depicted in Figure 2.

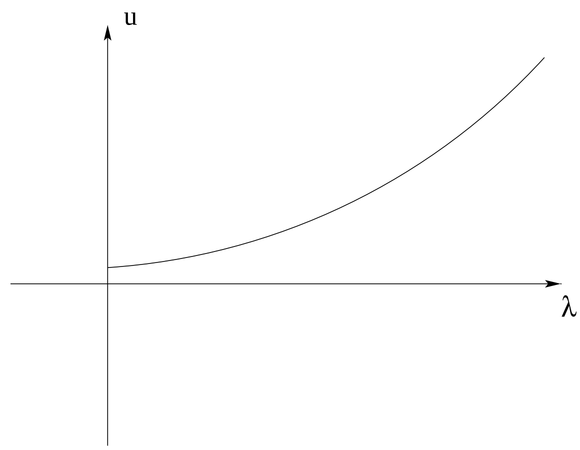

FIGURE 2. The "sublinear" case $m = 0$.

We now consider the case $m > 0$. The results in this case are different from those presented in Theorem 4. A careful examination of (P_λ) reveals the fact that the singular term $g(u)$ is not significant. Actually, the conclusions are close to those established in [15, Theorem A], where an elliptic problem associated to an asymptotically linear function is studied.

Let λ_1 be the first Dirichlet eigenvalue of $(-\Delta)$ in Ω and $\lambda^* = \dfrac{\lambda_1}{m}$. Our result in this case is the following.

Theorem 5. *Assume $(g1)$, $(g2)$ and $m > 0$. Then the following hold.*
(i) *If $\lambda \geq \lambda^*$, then (P_λ) has no solutions in $\mathcal{E}$.*
(ii) *If $a_* > 0$ (resp. $a_* = 0$) then (P_λ) has a unique solution $u_\lambda \in \mathcal{E} \cap C^{1,1-\alpha}(\overline{\Omega})$ for all $-\infty < \lambda < \lambda^*$ (resp. $0 < \lambda < \lambda^*$) with the properties:*
(ii1) *u_λ is strictly increasing with respect to λ;*
(ii2) *there exists two positive constants $c_1, c_2 > 0$ depending on λ such that $c_1 d(x) \leq u_\lambda \leq c_2 d(x)$ in Ω;*
(ii3) *$\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω.*

The bifurcation diagram in the "linear" case $m > 0$ is depicted in Figure 3.

The proofs rely on the sub and super-solution method and can be found in [7, 10, 9]. One of the difficulty in the treatment of $(P_{\lambda,\mu})$ or (P_λ) is the lack of the usual maximal principle. The following result which is due to Shi and Yao [21] gives a comparison principle that applies to singular elliptic equations.

Lemma 1. *Let $F : \overline{\Omega} \times [0, \infty) \to \mathbf{R}$ be a continuous function such that the mapping $(0, \infty) \ni s \to \frac{F(x,s)}{s}$ is strictly decreasing at each $x \in \Omega$. Assume $v, w \in C^2(\Omega) \cap C(\overline{\Omega})$ and*

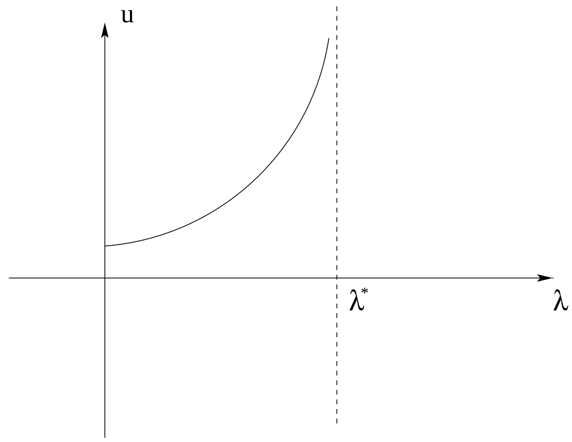

FIGURE 3. The "linear" case $m > 0$.

(*a*) $\Delta w + F(x,w) \leq 0 \leq \Delta v + F(x,v)$ *in* Ω;
(*b*) $v, w > 0$ *in* Ω *and* $v \leq w$ *on* $\partial\Omega$;
(*c*) $\Delta v \in L^1(\Omega)$.
Then $v \leq w$ *in* Ω.

We next consider the following singular elliptic problem (see [11, 12])

$$\begin{cases} -\Delta u = g(u) + \lambda|\nabla u|^p + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $0 < p \leq 2$ and $\lambda, \mu \geq 0$. As remarked in [13], the requirement that the nonlinearity grows at most quadratically in $|\nabla u|$ is natural in order to apply the maximum principle.

We suppose that $g : (0,\infty) \to (0,\infty)$ is a Hölder continuous function which is nonincreasing and $\lim_{s\searrow 0} g(s) = +\infty$. We assume that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and is positive on $\overline{\Omega} \times (0,\infty)$. We also need the following assumptions on f

($A1$) there exists $c > 0$ such that $f(s) \geq cs$ for all $s \in \overline{\Omega}$;

($A2$) the mapping $(0,\infty) \ni s \longmapsto \dfrac{f(s)}{s}$ is nondecreasing;

($A3$) the mapping $(0,\infty) \ni s \longmapsto \dfrac{f(s)}{s}$ is nonincreasing;

($A4$) $\displaystyle\lim_{s\to\infty} \frac{f(s)}{s} = 0$.

Problems of this type arise in the study of guided modes of an electromagnetic field in a nonlinear medium, satisfying adequate constitutive hypotheses. The following two examples illustrate situations of this type: (i) if $f(u) = u^3(1+\gamma u^2)^{-1}$ ($\gamma > 0$) then problem (2) describes the variation of the dielectric constant of gas vapors where a laser beam propagates (see [18, 19]); (ii) nonlinearities of the type $f(u) = (1-e^{-\gamma u^2})u$ arise in the context of laser beams in plasmas (see [20]).

By the monotony of g, there exists

$$a = \lim_{s\to\infty} g(s) \in [0,\infty).$$

The first result concerns the case $\lambda = 1$ and $1 < p \leq 2$. In the statement of the following result we do not need assumptions $(A1) - (A4)$; we just require that f is a Hölder continuous function which is nondecreasing with respect to the second variable and is positive on $\overline{\Omega} \times (0, \infty)$.

Theorem 6. *Assume $\lambda = 1$ and $1 < p \leq 2$.*
(i) *If $p = 2$ and $a \geq \lambda_1$, then (2) has no solutions;*
(ii) *If $p = 2$ and $a < \lambda_1$ or $1 < p < 2$, then there exists $\mu^* > 0$ such that (2) has at least one classical solution for $\mu < \mu^*$ and no solutions exist if $\mu > \mu^*$.*

If $\lambda = 1$ and $0 < p \leq 1$, the study of existence is closely related to the asymptotic behavior of the nonlinear term $f(x, u)$. In this case we prove

Theorem 7. *Assume $\lambda = 1$ and $0 < p \leq 1$.*
(i) *If f satisfies $(A1)$ or $(A2)$, then there exists $\mu^* > 0$ such that (2) has at least one classical solution for $\mu < \mu^*$ and no solutions exist if $\mu > \mu^*$;*
(ii) *If $0 < p < 1$ and f satisfies $(f3) - (f4)$, then (2) has at least one solution for all $\mu \geq 0$.*

Next we are concerned with the case $\mu = 1$. Our result is the following

Theorem 8. *Assume $\mu = 1$ and f satisfies assumptions $(A3)$ and $(A4)$. Then the following properties hold true.*
(i) *If $0 < p < 1$, then (2) has at least one classical solution for all $\lambda \geq 0$;*
(ii) *If $1 \leq p \leq 2$, then there exists $\lambda^* \in (0, \infty]$ such that (2) has at least one classical solution for $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$. Moreover, if $1 < p \leq 2$, then λ^* is finite.*

Related to the above result we raise the following **open problem:** if $p = 1$ and $\mu = 1$, is λ^* a finite number? Theorem 8 shows the importance of the convection term $\lambda|\nabla u|^p$ in (2). Indeed, according to Theorem 3 (see [9, Theorem 1.3]) and for any $\mu > 0$, the boundary value problem

$$\begin{cases} -\Delta u = u^{-\alpha} + \lambda|\nabla u|^p + \mu u^\beta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

has a unique solution, provided $\lambda = 0$ and $\alpha, \beta \in (0, 1)$. The above theorem shows that if λ is not necessarily 0, then the following situations may occur : (i) problem (3) has solutions if $p \in (0, 1)$ and for all $\lambda \geq 0$; (ii) if $p \in (1, 2)$ then there exists $\lambda^* > 0$ such that problem (3) has a solution for any $\lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$.

To see the dependence between λ and μ in (2), we consider the special case $f \equiv 1$ and $p = 2$. In this case we can say more about the problem (2). More precisely we have

Theorem 9. *Assume that $p = 2$ and $f \equiv 1$. Then the following properties hold.*
(i) *The problem (2) has a solution if and only if $\lambda(a + \mu) < \lambda_1$;*
(ii) *Assume $\mu > 0$ is fixed, g is decreasing and let $\lambda^* = \dfrac{\lambda_1}{a + \mu}$. Then (2) has a unique solution u_λ for every $\lambda < \lambda^*$ and the sequence $(u_\lambda)_{\lambda<\lambda^*}$ is increasing with respect to λ.*
Moreover, if $\limsup_{s \searrow 0} s^\alpha g(s) < +\infty$, for some $\alpha \in (0, 1)$, then the sequence of solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ has the following properties

(ii1) *For all $0 < \lambda < \lambda^*$ there exist two positive constants c_1, c_2 depending on λ such that $c_1 \operatorname{dist}(x, \partial\Omega) \leq u_\lambda \leq c_2 \operatorname{dist}(x, \partial\Omega)$ in Ω;*

(ii2) $u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;
(ii3) $u_\lambda \longrightarrow +\infty$ *as* $\lambda \nearrow \lambda^*$, *uniformly on compact subsets of* Ω.

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