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Optimality conditions in multiobjective control problems with generalized invexity

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Abstract. In this paper, we formulate optimality conditions for a multiobjective control problem under generalized invexity assumptions. Hereby, we extend the results obtained by Preda in [8].

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1. Introduction

The problem of optimal control with equality and inequality restrictions was considered by many authors: Mond and Smart [6], Bhatia and Kumar [1], Nahak and Nanda [7], or, more recently, Zhian and Qingkai [11] and Chen [4].

In Mond and Hanson [5], some duality theorems for control problems are given under convexity assumptions. Mond and Smart [6] have defined some invexity conditions and established duality and sufficiency results based on these invexity conditions. Preda [8] has generalized the results of Mond and Smart under generalized ρ-invexity assumptions for scalar control problems.

Bhatia and Kumar have studied in [1] multiobjective control problems under ρ-pseudoinvexity, ρ-strictly pseudoinvexity, ρ-quasiinvexity, ρ-strictly quasiinvexity assumptions. Nahak and Nanda have studied in [7] the efficiency and duality for multiobjective control problems under $(F-\rho)$ convexity.

Zhian and Qingkai have considered in [11] the duality for similar multiobjective control problems as in [1], but using the invexity defined in [6].

Chen has generalized in [4] some of the results obtained by Mond and Smart in [6].

In this paper, we consider a control problem similar to the one considered in [11]. For this problem, we define the mixed dual problem that can be particularized to dual problems of both Wolfe and Mond-Weir types. Details on this constructions can be found in [10].

2. Preliminaries

Consider $I=[a,b]$. Let:

$$f_i : I \times I\!R^n \times I\!R^m \to I\!R,\ i=\overline{1,p}$$
$$g_j : I \times I\!R^n \times I\!R^m \to I\!R,\ j=\overline{1,l}$$
$$h_k : I \times I\!R^n \times I\!R^m \to I\!R,\ k=\overline{1,n}$$

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be continuously differentiable functions.

Denote by X the space of piecewise smooth functions $x : I \to I\!R^n$ with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by:

$$u = Dx \quad \Longleftrightarrow \quad x(t) = \int_a^t u(s)ds.$$

Denote by U the space of piecewise continuous control functions $u : I \to I\!R^m$, with norm $\|.\|_\infty$.

Consider the following multiobjective control problem (see [1, 10, 11]), denoted by (VCP):

$$\min \int_a^b f(t,x,u)dt = \min \left(\int_a^b f_1(t,x,u), \ldots, \int_a^b f_p(t,x,u) \right),$$

with restrictions:

$$u(a) = a_0, u(b) = b_0 \tag{1}$$

$$\dot{x} = h(t,x,u), t \in I \tag{2}$$

$$g(t,x,u) \leqq 0, t \in I \tag{3}$$

Denote by $\mathcal{S}$ the set of feasible solutions of (VCP), i.e.:

$$\mathcal{S} = \{(x,u) \mid x \in X, u \in U \text{ verifying } (1), (2), (3)\}$$

Remark 2.1. *If $p = 1$, the problem* (VCP) *is the control problem studied in* [8].

Definition 2.1. *A feasible solution (x^*, u^*) of the problem* (VCP) *is called an* **efficient solution** *of* (VCP) *iff for all feasible solutions (x,u) of* (VCP) *such that :*

$$\int_a^b f(t,x,u)dt \leqq \int_a^b f(t,x^*,u^*)dt,$$

we have

$$\int_a^b f(t,x,u)dt = \int_a^b f(t,x^*,u^*)dt.$$

The following definitions introduce the concept of ρ-invexity and generalized ρ-invexity for the functional $\phi : X \times X \times U \to I\!R$,

$$\phi(x,\dot{x},u) = \int_a^b \Phi(t,x,\dot{x},u)dt$$

where $\Phi : I \times X \times X \times U \to I\!R$.

3. Necessary optimality conditions

In this section, we write necessary optimality conditions for the multiobjective control problem (VCP), using the relationship between the efficient solution of the problem (VCP) and the optimal solution of the associated scalar control problem. The method was used by Bhatia and Mehra in [2] for a multiobjective continuous programming problem.

First, we write necessary optimality conditions for a scalar control problem.

Theorem 3.1 (Kuhn-Tucker conditions)**.** *Let (x°, y°) be a normal optimal solution of the scalar control problem (CP):*

$$\min \int_a^b f(t,x,u)dt,$$

with restrictions:

$$\dot{x} = h(t,x,u)$$
$$g(t,x,u) \leqq 0$$

If the Fréchet derivative $[D - H_x(x^\circ, y^\circ)]$ is surjective, then there exist the piecewise smooth functions $\mu^\circ : I \to I\!R^l$ and $\gamma^\circ : I \to I\!R^n$ that satisfy the following conditions:

$$f_x(t,x^\circ,u^\circ) + \mu^\circ(t)^T g_x(t,x^\circ,u^\circ) + \gamma^\circ(t)^T h_x(t,x^\circ,u^\circ) + \dot{\gamma}^\circ(t) = 0$$
$$f_u(t,x^\circ,u^\circ) + \mu^\circ(t)^T g_u(t,x^\circ,u^\circ) + \gamma^\circ(t)^T h_u(t,x^\circ,u^\circ) = 0$$
$$\mu^\circ(t)^T g(t,x^\circ,u^\circ) = 0$$
$$\mu^\circ(t) \geqq 0$$

The following theorem makes an association between the multiobjective control problem (VCP) and a set of p scalar control problems and it establishes the connection between the efficient solution of the multiobjective control problem (VCP) and the optimal solution of the associated scalar control problems.

Theorem 3.2 ([3])**.** *(x°, y°) is the efficient solution of the problem (VCP) if and only if (x°, y°) is the optimal solution of the scalar control problems $P_k(x^\circ, y^\circ)$, for any $k = \overline{1,p}$, where $P_k(x^\circ, y^\circ)$ is defined by:*

$$\min \int_a^b f_k(t,x,u)dt,$$

with restrictions:

$$x(a) = a_0, x(b) = b_0$$
$$\dot{x} = h(t,x,u)$$
$$g(t,x,u) \leqq 0$$
$$\int_a^b f_j(t,x,u)dt \leq \int_a^b f_j(t,x^\circ,u^\circ)dt, \quad \forall j = \overline{1,p}, j \neq k.$$

Using Theorems 3.1 and 3.2, we can write now the necessary optimality conditions for the problem (VCP).

Theorem 3.3. *Let $(x^*, u^*) \in \mathcal{S}$ be a proper efficient solution of the problem* (VCP) *that is a normal solution of the problem $P_k(x^*, u^*)$, for any $k = \overline{1,p}$. Then, there exist $\lambda^* \in I\!R^p$ and the piecewise smooth functions $\mu^* : I \to I\!R^l$ and $\nu^* : I \to I\!R^n$ that satisfy the conditions:*

$$\lambda^{*T} f_x(t,x^*,u^*) + \mu^*(t)^T g_x(t,x^*,u^*) + \nu^*(t)^T h_x(t,x^*,u^*) + \dot{\nu}^*(t) = 0$$
$$\lambda^{*T} f_u(t,x^*,u^*) + \mu^*(t)^T g_u(t,x^*,u^*) + \nu^*(t)^T h_u(t,x^*,u^*) = 0$$
$$\mu^*(t)^T g(t,x^*,u^*) = 0, \forall t \in I$$
$$\mu^*(t) \geqq 0, \forall t \in I$$
$$\lambda^* \geqq 0$$

Proof. As a proper efficient solution of the problem (VCP), (x^*, u^*) is also an efficient solution and from Theorem 3.2, it is an optimal solution of the scalar problem $P_k(x^*, u^*)$, for any $k = \overline{1,p}$. Since (x^*, u^*) is also a normal solution of $P_k(x^*, u^*)$, for any $k = \overline{1,p}$, it follows that the conditions from Theorem 3.1 hold. This implies that for any $k = \overline{1,p}$, there exist $\lambda_{1k}, \dots, \lambda_{pk}$, with $\lambda_{kk} = 1$ and the piecewise smooth functions μ_k and ν_k such that :

$$
\begin{aligned}
& f_{kx}(t,x^*,u^*) + \sum_{i=1, i\neq k}^{p} \lambda_{ik}^T f_{ix}(t,x^*,u^*) + \mu_k(t)^T g_x(t,x^*,u^*) + \\
& \qquad +\nu_k(t)^T h_x(t,x^*,u^*) + \dot{\nu}_k(t) = 0 \\
& f_{ku}(t,x^*,u^*) + \sum_{i=1, i\neq k}^{p} \lambda_{ik}^T f_{iu}(t,x^*,u^*) + \mu_k(t)^T g_u(t,x^*,u^*) + \\
& \qquad +\nu_k(t)^T h_u(t,x^*,u^*) = 0 \\
& \mu_k(t)^T g(t,x^*,u^*) = 0, \forall t \in I \\
& \mu_k(t) \geqq 0, \forall t \in I \\
& \lambda_{ik} \geqq 0, \forall i = \overline{1,p}, i \neq k
\end{aligned}
$$

By the addition of the above relations over $k = \overline{1,p}$ and by taking

$$
\lambda_i^* = \sum_{k=1}^{p} \lambda_{ik}, i = \overline{1,p}
$$

$$
\mu^*(t) = \sum_{k=1}^{p} \mu_k(t)
$$

$$
\nu^*(t) = \sum_{k=1}^{p} \nu_k(t)
$$

we obtain the conclusion of the theorem.

□

3.1. Sufficient optimality conditions.

In this section, we establish a series of efficient conditions for the problem (VCP), by using generalized ρ-invexity on the related functions f, g and h.

The following definitions are similar to those from [9] and generalize the definitions given in Section 2.

Let $\phi : I \times X \times X \times U \to \mathbb{R}^p$ be a vector function and let $\psi : X \times X \times U \to \mathbb{R}^p$ be a real number such that

$$
\psi(x,\dot{x},u) = \int_a^b \phi(t,x(t),\dot{x}(t),u(t))dt.
$$

Definition 3.1. *Consider $x \in X$, $u \in U$ and $\rho \in \mathbb{R}^p$.*

a) *If there exist $\eta(t,x(t),\dot{x}(t),u(t),x^*(t),\dot{x}^*(t),u^*(t)) \in \mathbb{R}^n$ with $\eta = 0$ for $x(t) = x^*(t)$, $t \in I$ and $\xi(t,x(t),\dot{x}(t),u(t),x^*(t),\dot{x}^*(t),u^*(t)) \in \mathbb{R}^m$, $\theta : X \times U \times X \times U \to R^s$ such that for any $(x,u) \neq (x^*,u^*)$:*

$$
\begin{aligned}
& \psi(x,\dot{x},u) - \psi(x^*,\dot{x}^*,u^*) \geqq \int_a^b [\phi_x(t,x^*,\dot{x}^*,u^*)\eta + \\
& +\phi_{\dot{x}}(t,x^*,\dot{x}^*,u^*)(D\eta) + \phi_u(t,x^*,\dot{x}^*,u^*)\xi]\, dt + \rho\|\theta(x,u;x^*,u^*)\|^2,
\end{aligned}
$$

*then ψ is called ρ-**invex** in $(x^*,\dot{x}^*,u^*)$ with respect to η, ξ and θ.*

a′) *If the above inequality takes place with $>$ instead of $\geqq$, we say that ψ is ρ-**strictly invex** in $(x^*,\dot{x}^*,u^*)$ with respect to η, ξ and θ.*

a$''$) *If the above inequality takes place with $\geq$ instead of $\geqq$, we say that ψ is* **strong ρ-invex** *in $(x^*, \dot{x}^*, u^*)$ with respect to η, ξ and θ.*

b) *If there exist η, ξ and θ such that for any $(x, u) \neq (x^*, u^*)$:*

$$\psi(x, \dot{x}, u) \leq \psi(x^*, \dot{x}^*, u^*) \Rightarrow \int_a^b [\phi_x(t, x^*, \dot{x}^*, u^*)\eta +$$

$$+\phi_{\dot{x}}(t, x^*, \dot{x}^*, u^*)(D\eta)+\phi_u(t, x^*, \dot{x}^*, u^*\xi]\,dt<-\rho\|\theta(x, y, z; u, v, w)\|^2$$

then ψ is called **weak ρ-strictly pseudoinvex** *in $(x^*, \dot{x}^*, u^*)$ with respect to η, ξ and θ.*

c) *If*

$$\psi(x, \dot{x}, u) \leq \psi(x^*, \dot{x}^*, u^*) \Rightarrow \int_a^b [\phi_x(t, x^*, \dot{x}^*, u^*)\eta +$$

$$+\phi_{\dot{x}}(t, x^*, \dot{x}^*, u^*)(D\eta)+\phi_u(t, x^*, \dot{x}^*, u^*)\xi]\,dt\leq-\rho\|\theta(x, y, z; u, v, w)\|^2$$

then ψ is called **strong ρ-pseudoinvex** *in $(x^*, \dot{x}^*, u^*)$ with respect to η, ξ and θ.*

d) *If*

$$\psi(x, \dot{x}, u)\leqq\psi(x^*, \dot{x}^*, u^*) \Rightarrow \int_a^b [\phi_x(t, x^*, \dot{x}^*, u^*)\eta +$$

$$+\phi_{\dot{x}}(t, x^*, \dot{x}^*, u^*)(D\eta)+\phi_u(t, x^*, \dot{x}^*, u^*)\xi]\,dt\leqq- \rho\|\theta(x, y, z; u, v, w)\|^2$$

then ψ is called **weak ρ-quasiinvex** *in $(x^*, \dot{x}^*, u^*)$ with respect to η, ξ and θ.*

e) *If*

$$\psi(x, \dot{x}, u)\leqq\psi(x^*, \dot{x}^*, u^*) \Rightarrow \int_a^b [\phi_x(t, x^*, \dot{x}^*, u^*)\eta +$$

$$+\phi_{\dot{x}}(t, x^*, \dot{x}^*, u^*)(D\eta)+\phi_u(t, x^*, \dot{x}^*, u^*)\xi]\,dt\leq-\rho\|\theta(x, y, z; u, v, w)\|^2$$

then ψ is called **ρ-substrictly pseudoinvex** *in $(x^*, \dot{x}^*, u^*)$ with respect to η, ξ and θ.*

Theorem 3.4. *Let (x^*, u^*) be a feasible solution of the problem* (VCP)*, let $\lambda^* \in I\!R^p$ be a real number and let $\mu^* : I \to I\!R^l$ and $\nu^* : I \to I\!R^n$ be the piecewise smooth functions given by* Theorem 3.3.

If $\lambda^* > 0$ and:

i_1) $\int_a^b f(t, x, u)dt$ *strong ρ-pseudoinvex in (x^*, u^*) with respect to η, ξ and θ.*

i_2) $\int_a^b \mu^*(t)^T g(t, x, u)dt$ *weak α-quasiinvex in (x^*, u^*) with respect to η, ξ and θ.*

i_3) $\int_a^b \nu^*(t)^T [h(t, x, u) - \dot{x}]dt$ *weak β-quasiinvex in (x^*, u^*) with respect to η, ξ and θ.*

i_4) $\lambda^{*T}\rho + \alpha + \beta\geqq0$

then (x^*, u^*) is an efficient solution of the problem (VCP).

Proof. Suppose toward a contradiction that (x^*, u^*) is not an efficient solution. Then, there exists (x, u) a feasible solution of (VCP) such that :

$$\int_a^b f(t, x, u)dt \leq \int_a^b f(t, x^*, u^*)dt$$

From condition i_1) and definition, we have:

$$\int_a^b [f_x(t,x^*,u^*)\eta + f_u(t,x^*,u^*)\xi]dt \leq -\rho\|\theta(x,x^*,u,u^*)\|^2 \tag{4}$$

On the other hand, $\mu^*(t)\geqq 0$, $g(t,x,u)\leqq 0$, with (x,u) a feasible solution of (VCP), hence:

$$\int_a^b \mu^*(t)^T g(t,x,u)dt \leqq \int_a^b \mu^*(t)^T g(t,x^*,u^*)dt = 0 \tag{5}$$

Using condition i_2), it follows:

$$\int_a^b [\mu^*(t)^T g_x(t,x^*,u^*)\eta + \mu^*(t)^T g_u(t,x^*,u^*)\xi]dt \leq -\alpha\|\theta(x,x^*,u,u^*)\|^2 \tag{6}$$

Since (x,u) and (x^*,u^*) are feasible solutions of the problem (VCP), we have $\dot{x} = h(t,x,u)$, respectively $\dot{x}^* = h(t,x^*,u^*)$, thus:

$$\int_a^b \nu^*(t)^T[h(t,x,u) - \dot{x}]dt = \int_a^b \nu^*(t)^T[h(t,x^*,u^*) - \dot{x}^*]dt$$

Condition i_3) leads us to:

$$\int_a^b [\nu^*(t)^T h_x(t,x^*,u^*)\eta + \nu^*(t)^T h_u(t,x^*,u^*)\xi]dt - \int_a^b \nu^*(t)^T E_{n\times n}(D\eta) \leq$$
$$\leq -\beta\|\theta(x,x^*,u,u^*)\|^2$$

Using the initial conditions, the above relation becomes:

$$\begin{aligned}&\int_a^b \{[\nu^*(t)^T h_x(t,x^*,u^*) + \dot{\nu}^*(t)]\eta + \nu^*(t)^T h_u(t,x^*,u^*)\xi\}dt \leq \\ &\leq -\beta\|\theta(x,x^*,u,u^*)\|^2\end{aligned} \tag{7}$$

Multiplying the relation (4) by $\lambda^{*T} > 0$ and then adding it to the relations (6) and (7), we obtain:

$$\begin{aligned}&\int_a^b \{[\lambda^{*T} f_x(t,x^*,u^*) + \mu^*(t)^T g_x(t,x^*,u^*) + \nu^*(t)^T h_x(t,x^*,u^*) + \dot{\nu}^*(t)]\eta + \\ &+[\lambda^{*T} f_u(t,x^*,u^*) + \mu^*(t)^T g_u(t,x^*,u^*) + \nu^*(t)^T h_u(t,x^*,u^*)\xi]\}dt \leq \\ &\leq -(\lambda^{*T}\rho + \alpha + \beta)\|\theta(x,x^*,u,u^*)\|^2\end{aligned} \tag{8}$$

Finally, using the first two relations from Theorem 3.3, we obtain a contradiction with hypothesis i_4).

□

The condition $\lambda^* > 0$ is absolutely necessary in the above proof. In the following theorem, we replace this condition with a weaker one, $\lambda^* \geq 0$; while strengthening the invexity hypothesis on the objective function.

Theorem 3.5. *Let* (x^*,u^*), $\lambda^* \in I\!R^p$, μ^* *and* ν^* *be like in* Theorem 3.4. *If* $\lambda^* \geq 0$ *and:*

j_1) $\int_a^b f(t,x,u)dt$ *weak* ρ*-strictly pseudoinvex in* (x^*,u^*) *with respect to* η, ξ *and* θ.

j_2) $\int_a^b \mu^*(t)^T g(t,x,u)dt$ *weak* α*-quasiinvex in* (x^*,u^*) *with respect to* η, ξ *and* θ.

j_3) $\int_a^b \nu^*(t)^T[h(t,x,u) - \dot{x}]dt$ *weak* β*-quasiinvex in* (x^*,u^*) *with respect to* η, ξ *and* θ.

j_4) $\lambda^{*T}\rho + \alpha + \beta \geqq 0$

then (x^*, u^*) *is an efficient solution of the problem* (VCP).

Proof. Suppose toward a contradiction that (x^*, u^*) is not an efficient soltuion. By using condition j_1), we obtain:

$$\int_a^b [f_x(t, x^*, u^*)\eta + f_u(t, x^*, u^*)\xi]dt < -\rho\|\theta(x, x^*, u, u^*)\|^2 \tag{9}$$

Then, like in the proof of Theorem 3.4, we obtain relations (6) and (7). Multiplying the relation (4) by $\lambda^{*T} \geq 0$ and adding it to relations (6) and (7), we obtain the relation (8), from which we obtain the contradiction with hypothesis j_4).

□

Theorem 3.6. *Let* (x^*, u^*), $\lambda^* \in I\!R^p$, μ^* *and* ν^* *be like in* Theorem 3.4. *If* $\lambda^* \geq 0$ *and:*

k_1) $\int_a^b f(t, x, u)dt$ *weak* ρ*-quasiinvex in* (x^*, u^*) *with respect to* η, ξ *and* θ.

k_2) *One of the functionals* $\int_a^b \mu^*(t)^T g(t, x, u)dt$ *or* $\int_a^b \nu^*(t)^T[h(t, x, u) - \dot{x}]dt$ *is weak* α*-quasiinvex, and the other one is* β*-substrictly pseudoinvex in* (x^*, u^*) *with respect to* η, ξ *and* θ.

k_3) $\lambda^{*T}\rho + \alpha + \beta \geqq 0$

then (x^*, u^*) *is an efficient solution of* (VCP).

Proof. Suppose toward a contradicition that (x^*, u^*) is not an efficient solution. By using condition k_1), we obtain:

$$\int_a^b [f_x(t, x^*, u^*)\eta + f_u(t, x^*, u^*)\xi]dt \leqq -\rho\|\theta(x, x^*, u, u^*)\|^2$$

hence, multiplying by $\lambda^* \geq 0$, it follows:

$$\int_a^b [\lambda^{*T} f_x(t, x^*, u^*)\eta + \lambda^{*T} f_u(t, x^*, u^*)\xi]dt \leqq -\lambda^{*T}\rho\|\theta(x, x^*, u, u^*)\|^2$$

Hypothesis k_2) leads us to the following inequalities:

$$\int_a^b [\mu^*(t)^T g_x(t, x^*, u^*)\eta + \mu^*(t)^T g_u(t, x^*, u^*)\xi]dt \leq -\alpha\|\theta(x, x^*, u, u^*)\|^2$$

and

$$\int_a^b [\nu^*(t)^T h_x(t, x^*, u^*)\eta + \nu^*(t)^T h_u(t, x^*, u^*)\xi]dt - \int_a^b \nu^*(t)^T E_{n\times n}(D\eta) \leqq$$
$$\leqq -\beta\|\theta(x, x^*, u, u^*)\|^2,$$

respectively (or viceversa).

By adding the above relations, we obtain the inequality (8), which leads us to a contradiction with hypothesis k_3). □

References

[1] D. Bhatia, P. Kumar, Multiobjective control problem with generalized invexity, *J.Math.Anal.Appl.*, 189(1995), 676-692.

[2] D. Bhatia, A. Mehra, Optimality conditions and duality for multiobjective variational problems with generalized B-invexity, *J.Math. Anal. Appl.*, 234(1999), 341-360.

[3] V. Chankong, Y.Y. Haimes, *Multiobjective decision making: theory and methodology*, North-Holland, New-York, 1983.

[4] X. Chen, Duality for a class of multiobjective control problems, *J.Math. Anal.Appl.*, 267(2002), 377-394.
[5] B. Mond, M.A. Hanson, Duality for variational problems, *J.Math.Anal. Appl.*, 18(1967), 355-364.
[6] B. Mond, I. Smart, Duality and sufficiency in control problems with invexity, *J.Math.Anal.Appl.*, 136(1988), 325-333.
[7] C. Nahak, S. Nanda, On efficiency and duality for multiobjective variational control problems with (F, ρ)-convexity, *J.Math.Anal.Appl.*, 209(1997), 415-434.
[8] V. Preda,On duality and suffciency in control problems with general invexity, *Bull. Math. de la Soc. Sci. Math de Roumanie*, 35(1991), 271-280.
[9] V. Preda, S. Gramatovici, Some sufficient optimality conditions for a class of multiobjective variational problems, *An. Univ. Buc.*, LI(2002), 33-43.
[10] S. Gramatovici, Multiobjective Continous Programming Problems and Applications, PhD thesis, Bucharest, 2003 (in Romanian).
[11] L. Zhian, Y. Qingkai, Duality for a class of multiobjective control problems with generalized invexity, *J.Math.Anal.Appl.*, 256(2001), 446-461.

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