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Some Harmonic and Biharmonic Problems on the Tangent Bundle with a Berger-Type Deformed Sasaki Metric

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Abstract. Let $\psi : (M_{2k}, \phi, g) \longrightarrow (N_{2k'}, \phi', h)$ be a smooth map between almost anti-paraHermitian manifolds. The map ψ induces the tangent map $\Psi : (TM, g^{BS}) \longrightarrow (TN, h^{BS})$. In this paper, we deal with the harmonicity of a tangent map Ψ and the biharmonicity of the identity map in the case where the tangent bundles TM, TN are endowed with the Berger type deformed Sasaki metric g^{BS}, h^{BS}.

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1. Introduction

Let M be a $2k$-dimensional Riemannian manifold with a Riemannian metric g. Throughout the paper, manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞.

An almost paracomplex manifold is an almost product manifold (M_{2k}, ϕ), $\phi^2 = id$, such that the two eigenbundles T^+M and T^-M associated to the two eigenvalues $+1$ and -1 of ϕ, respectively, have the same rank. The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor:

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + [X, Y]$$

A paracomplex structure is an integrable almost paracomplex structure.

Let (M_{2k}, ϕ) be an almost paracomplex manifold. A Riemannian metric g is said to be an anti-paraHermitian metric if

$$g(\phi X, \phi Y) = g(X, Y)$$

or equivalently

$$g(X, \phi Y) = g(\phi X, Y)$$

for any vector fields X , Y on M_{2k}. If (M_{2k}, ϕ) is an almost paracomplex manifold with an anti-paraHermitian metric g, then the triple (M_{2k}, ϕ, g) is said to be an almost anti-paraHermitian manifold. Moreover,(M_{2k}, ϕ, g) is said to be anti-paraKähler if ϕ is parallel with respect to the Levi-Civita connection ∇^g of g. As is well known, the anti-paraKähler condition ($\nabla^g \phi = 0$) is equivalent to paraholomorphicity of the

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anti-paraHermitian metric g, that is,$\Phi_\phi g = 0$, where Φ_ϕ is the Tachibana operator [28].

In [1], the authors defined a new metric, which is called a Berger type deformed Sasaki metric, on the tangent bundle over an anti-paraKähler manifold. They studied the geodesics and curvature properties of the tangent bundle with Berger type deformed Sasaki metric and gave the conditions for some almost anti-paraHermitian structures to be anti-paraKähler and quasi-anti-paraKähler on this setting.

Motivated by the results presented in [2], we think up the paper. Clearly, in the present paper, we again consider the tangent bundle with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. First, we study the map $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, h^S)$ and find conditions under which it is totally geodesic and we also study the biharmonicity of the identity map $I : (TM, g^S) \longrightarrow (TM, \widetilde{\phi}, g^{BS})$. In the second part we give the conditions for the map $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ to be totally geodesic and we also study the biharmonicity of the identity map $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$. And finally we take $\psi : (M_{2k}, \phi, g) \longrightarrow (N_{2k'}, \phi', h)$ to be a smooth map between almost anti-paraHermitian manifolds. The map ψ induces the tangent map $\Psi : (TM, g^{BS}) \longrightarrow (TN, h^{BS})$ between the tangent bundles of M and N. The motivation of this final part of the paper is to study the harmonicity of the tangent map $\Psi : (TM, g^{BS}) \longrightarrow (TN, h^{BS})$.

1.1. Harmonic maps. Consider a smooth map $\phi : (M^n, g) \longrightarrow (N^n, h)$ between two Riemannian manifolds,then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g \tag{1}$$

(or over any compact subset$K \subset M$).
A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$). For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)|_{t=0} = -\int_M h(\tau(\phi), V)dv_g \tag{2}$$

where

$$\tau(\phi) = tr_g \nabla d\phi \tag{3}$$

is the tension field of ϕ. Then we have

Theorem 1.1. *A smooth map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is harmonic if and only if*

$$\tau(\phi) = 0 \tag{4}$$

If $(x^i)_{1\le i\le m}$ and $(y^\alpha)_{1\le\alpha\le n}$ denote local coordinates on M and N respectively then equation 4 takes the form

$$\tau(\phi)^\alpha = (\Delta\phi^\alpha + g^{ij}\, \overset{N}{\Gamma}{}^\alpha_{\beta\gamma}\, \frac{\partial\phi^\beta}{\partial x^i}\frac{\partial\phi^\gamma}{\partial x^j}) = 0 \tag{5}$$

where $\Delta\phi^\alpha = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}g^{ij}\frac{\partial\phi^\alpha}{\partial x^j})$ is the Laplace operator on (M^m, g) and $\overset{N}{\Gamma}{}^\alpha_{\beta\gamma}$ are the Christoffel symbols on N.

1.2. Biharmonic maps.

Definition 1.1. A map $\phi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds is called biharmonic if is a critical point of bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau(\phi)|^2 dv^g \tag{6}$$

we have

$$\frac{d}{dt}E_2(\phi_t)|_{t=0} = -\int_M h(\tau_2(\phi), V)dv_g \tag{7}$$

The Euler-Lagrange equation attached to bienergy is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J_\phi(\tau(\phi)) = -(\Delta^\phi\tau(\phi) + tr_g R^N(\tau(\phi), d\phi)d\phi) \tag{8}$$

where J_ϕ is the jacobi operator defined by

$$\begin{array}{rcl} J_\phi : \Gamma(\varphi^{-1}(TN)) & \longrightarrow & \Gamma(\varphi^{-1}(TN)) \\ V & \longmapsto & \Delta^\phi V + tr_g R^N(V, d\phi)d\phi \end{array} \tag{9}$$

The biharmonic map introduced by Eelles and Sampson in 1964, which is a generalization of harmonic maps. One can refer to [19], [21] and [25] for background on harmonic and biharmonic maps.

2. The Berger type deformed Sasaki metric on the tangent bundle

Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\begin{array}{rcl} \mathcal{V}_{(x,u)} & = & \ker(d\pi_{(x,u)}) \\ & = & \{a^i \frac{\partial}{\partial y^i}{}_{|(x,u)}; a^i \in \mathbb{R}\}, \\ \mathcal{H}_{(x,u)} & = & \{\frac{\partial}{\partial x^i}{}_{|(x,u)} - a^i u^j \Gamma^k_{ij}\frac{\partial}{\partial y^k}{}_{|(x,u)}; a^i \in \mathbb{R}\}, \end{array}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.
Let $X = X^i\frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i\frac{\partial}{\partial y^i}, \tag{10}$$

$$X^H = X^i\frac{\delta}{\delta x^i} = X^i\Big\{\frac{\partial}{\partial x^i} - y^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}\Big\}. \tag{11}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1..n}$ is a local adapted frame in TTM.

Definition 2.1. The Sasaki metric g^S on the tangent bundle TM of M is given by
(1) $g^S(X^H, Y^H) = g(X, Y) \circ \pi$.
(2) $g^S(X^H, Y^V) = 0$.

(3) $g^S(X^V, Y^V) = g(X,Y)\circ\pi$.
for all vector fields $X, Y \in \Gamma(TM)$.

Proposition 2.1. **([20]).** Let (M, g) be a Riemannian manifold and $\widehat{\nabla}$ be the levi-Civita connection of the tangent bundle (TM, g^s) equipped with the Sasaki metric. Then

$$\begin{aligned}
(\widehat{\nabla}_{X^H}Y^H)_{(x,u)} &= (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V,\\
(\widehat{\nabla}_{X^H}Y^V)_{(x,u)} &= (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(u,Y)X)^H,\\
(\widehat{\nabla}_{X^V}Y^H)_{(x,u)} &= \frac{1}{2}(R_x(u,X)Y)^H,\\
(\widehat{\nabla}_{X^V}Y^V)_{(x,u)} &= 0,
\end{aligned}$$

for all vector fields $X, Y \in \Gamma(TM)$ and $(x,u) \in TM$.

Definition 2.2. Let (M_{2k}, ϕ, g) be an almost anti-paraHermitian manifold and TM be its tangent bundle.The Berger type deformed Sasaki metric on TM is defined by
(1) $g^{BS}(X^H, Y^H) = g(X,Y)$,
(2) $g^{BS}(X^H, Y^V) = 0$,
(3) $g^{BS}(X^V, Y^V) = g(X,Y) + \delta^2 g(X,\phi u)g(Y,\phi u)$,
for all vector fields $X, Y \in \Gamma(TM)$, where δ is some constant. If $\delta = 0$ then g^{BS} is called the Sasaki metric.

A direct consequence of usual calculations using the Koszul formula gives the following result

Proposition 2.2. ([1]) Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM be its tangent bundle. The Levi-Civita connection of the Berger type deformed Sasaki metric g^{BS} on TM satisfies the following properties:

$$\begin{aligned}
(\widetilde{\nabla}_{X^H}Y^H)_{(x,u)} &= (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V,\\
(\widetilde{\nabla}_{X^H}Y^V)_{(x,u)} &= (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(u,Y)X)^H,\\
(\widetilde{\nabla}_{X^V}Y^H)_{(x,u)} &= \frac{1}{2}(R_x(u,X)Y)^H,\\
(\widetilde{\nabla}_{X^V}Y^V)_{(x,u)} &= \frac{\delta^2}{1+\delta^2\alpha} g(X,\phi Y)(\phi u)^V,
\end{aligned}$$

where ∇ is the Levi-Civita connection, R is its Riemannian curvature tensor and $\alpha = g(u,u)$.

Definition 2.1. Let (M, g) be a Riemannian manifold and $F : TM \to TTM$ be a smooth bundle endomorphism of the tangent bundle TM. Then we define the vertical and horizontal lifts $F^V : TM \to TTM$, $F^H : TM \to TTM$ of F by

$$F^V(\eta) = \sum_{i=1}^{m} \eta_i F(\partial i)^V$$

and

$$F^H(\eta) = \sum_{i=1}^{m} \eta_i F(\partial i)^H$$

where $\sum_{i=1}^{m} \eta_i \partial i \in \pi^{-1}(V)$ is a local representation of $\eta \in \mathcal{C}^\infty(TM)$

From Definition 2.1 and Theorem 7.1, we have

Proposition 2.1. *Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM be its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. If $F \in \mathfrak{T}_1^1(M)$ is a tensor of type (1,1), then we have*

$$\begin{aligned}
(\overline{\nabla}_{X^H} HF)_{(x,u)} &= (\nabla_X F)^H_{(x,u)}. \\
(\overline{\nabla}_{X^H} VF)_{(x,u)} &= (\nabla_X F)^V_{(x,u)}. \\
(\overline{\nabla}_{X^V} HF)_{(x,u)} &= (F(X))^H_{(x,u)} + \frac{3}{2}(R_x(u, X_x)F(u))^H. \\
(\overline{\nabla}_{X^V} VF)_{(x,u)} &= (F(X))^V_{(x,u)} + \frac{\delta^2}{1+\delta^2\alpha} g(X, \phi F(u))(\phi u)^V.
\end{aligned}$$

where $(x,u) \in TM$ and $X \in \Gamma(TM)$.

Using Proposition 2.1 and formula of curvature, we have

Theorem 2.2. *(M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM be its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Then the corresponding Riemannian curvature tensor $\overline{R}$ is given by*

$$\begin{aligned}
\overline{R}(X^H, Y^H)Z^H &= \Big[R(X,Y)Z + \frac{3}{2}R(u, R(X,Y)u)Z\Big]^H \\
\overline{R}(X^H, Y^H)Z^V &= \Big[R(X,Y))Z + \frac{\delta^2}{1+\alpha\delta^2} g(R(X,Y)u, \phi Z)(\phi u\Big]^V \\
\overline{R}(X^H, Y^V)Z^H &= \Big[\frac{3}{2}(\nabla_X R)(u,Y)Z\Big]^H. \\
\overline{R}(X^H, Y^V)Z^V &= 0. \\
\overline{R}(X^V, Y^V)Z^H &= \frac{3}{4}\Big[4R(X,Y)Z + 3R(u,X)R(u,Y)Z - 3R(u,Y)R(u,X)Z\Big]^V. \\
\overline{R}(X^V, Y^V)Z^V &= \Big(\frac{\delta^2}{1+\delta^2\alpha}\Big)^2 \big[g(X,u)g(Y,\phi Z) - g(Y,u)g(X,\phi Z)\big](\phi u)^V.
\end{aligned}$$

3. Harmonicity of the map $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, h^S)$

In the section, we denote (M_{2k}, ϕ, g) be an anti-paraKähler manifold and (TM, ϕ, g^{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$, (N^n, g) be an n-dimensional Riemannian manifold and (TM, h^S) its tangent bundle equipped with the Sasaki metric h^S.

Lemma 3.1. ([15]). Let $\psi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds. The map ψ induces the tangent map $\Psi = d\psi : TM \longrightarrow TN$. For all vector field $X \in \Gamma(TM)$, we have

$$\begin{aligned}
d\Psi\Big((X)^V\Big) &= \Big(d\psi(X)\Big)^V, \\
d\Psi\Big((X)^H\Big) &= \Big(d\psi(X)\Big)^H + \Big(\nabla d\psi(u, X)\Big)^V,
\end{aligned}$$

Theorem 3.1. Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, then the tension field $\tau(\Psi)$ of ψ is given by

$$\begin{aligned}\tau(\Psi) &= \Big[\tau(\psi) + tr_h R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*)\Big]^H + \Big[div(\nabla d\psi)(u) \\ &\quad - \frac{\delta^2}{1+\alpha\delta^2}\Big(tr_g g(*, \phi(*)) - \frac{\delta^2}{1+\alpha\delta^2} g(u, \phi u) d\psi(\phi(u))\Big)\Big]^V.\end{aligned}$$

Proof. Let $(\psi(x), d\psi(u)) \in TN$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned}\tau(\Psi) =& \widetilde{\nabla}^\Psi_{e_i^H} d\Psi(e_i^H) - d\Psi(\widetilde{\nabla}_{e_i^H} e_i^H) + \widetilde{\nabla}^\Psi_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V} d\Psi(\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) \\ & - d\Psi(\widetilde{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V}\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) + \widetilde{\nabla}^\Psi_{\phi(e_j)^V} d\Psi\phi(e_j)^V \\ & - d\Psi(\widetilde{\nabla}_{\phi(e_j)^V}\phi(e_j)^V) \\ =& \widehat{\nabla}_{d\psi(e_i)^H + \nabla d\psi(u,e_i)^V}(d\psi(e_i)^H + \nabla d\psi(u,e_i)^V) - d\psi(\widetilde{\nabla}_{e_i^H} e_i^H) \\ & + \widehat{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}} d\Psi\phi(e_1)^V}\frac{1}{\sqrt{1+\alpha\delta^2}} d\Psi(\phi(e_1)^V) + \widehat{\nabla}_{d\Psi(\phi(e_j)^V)} d\Psi\phi(e_j)^V \\ & - d\Psi(\widetilde{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V}\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) - d\Psi(\widetilde{\nabla}_{\phi(e_j)^V}\phi(e_j)^V),\end{aligned}$$

from Proposition 2.2, we have

$$\begin{aligned}\tau(\Psi) =& (\nabla_{d\psi(e_i)} d\psi(e_i))^H + (\nabla_{d\psi(e_i)}\nabla d\psi(u,e_i))^V \\ & + (R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i))^H - d\psi(\nabla_{e_i} e_i)^H \\ & - \Big(\frac{\delta}{1+\alpha\delta^2}\Big)^2 g(\phi(e_1), \phi\phi(e_1)) d\psi(\phi(u))^V - g(\phi(e_j), \phi\phi(e_j)) d\psi(\phi(u))^V.\end{aligned}$$

□

Theorem 3.2. Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, then Ψ is harmonic if and only if the following conditions are verified

$$\begin{aligned}&\tau(\Psi) = 0, \quad tr_h R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*) = 0, \quad div(\nabla d\psi)(u) = 0, \\ &tr_g g(*, \phi(*)) - \frac{\delta^2}{1+\alpha\delta^2} g(u, \phi u) d\psi(\phi(u)) = 0.\end{aligned}$$

Corollary 3.1. Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, if ψ is totally geodesic then Ψ is harmonic if and only if

$$tr_g g(*, \phi(*)) = \frac{\delta^2}{1+\alpha\delta^2} g(u, \phi u).$$

Lemma 3.2. Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, then the then the energy density associated to Ψ is given by

$$e(\Psi) = 2e(\psi) + \frac{1}{2} tr_h |(\nabla d\psi(u,*))|^2 - \frac{\delta^2}{2(1+\alpha\delta^2)} |d\psi(\phi(u))|^2. \tag{12}$$

Proof. Let $(x,u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \frac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\left\{e_i^H, \frac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\right\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned}
e(\Psi)_{(\psi(x),d\psi(x))} =& \frac{1}{2}\Big[\big(h^S_{(\psi(x),d\psi(x))}(d\Psi(e_i^H), d\Psi(e_i^H))\big) \\
&+ \frac{1}{1+\alpha\delta^2} h^S_{(\psi(x),d\psi(x))}(d\Psi(\phi(e_1)^V), d\Psi(\phi(e_1)^V)) \\
&+ h^S_{(\psi(x),d\psi(x))}(d\Psi(\phi(e_j)^V), d\Psi(\phi(e_j)^V))\Big] \\
=& \frac{1}{2}\Big[\Big(h^S(d\psi(e_i)^H, d\psi(e_i)^H) + h^S(\nabla d\psi(u,e_i)^V, \nabla d\psi(u,e_i)^V)\Big) \\
&+ \frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1))^V, d\psi(\phi(e_1))^V) \\
&+ h^S(d\psi(\phi(e_j))^V, d\psi(\phi(e_j))^V)\Big] \\
=& \frac{1}{2}\Big[2e(\psi) + tr_h|\nabla d\psi(u,*)|^2 + \frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1)), d\psi(\phi(e_1))) \\
&+ h^S(d\psi\phi(e_i)), d\psi(\phi(e_i))) - h^S(d\psi(\phi(e_1)), d\psi(\phi(e_1)))\Big] \\
=& \frac{1}{2}\Big[4e(\psi) + tr_h||\nabla d\psi(u,*)||^2 + \frac{1}{\alpha(1+\alpha\delta^2)} h(d\psi(\phi(u)), d\psi(\phi(u))) \\
&- \frac{1}{\alpha} h(d\psi(\phi(u)), d\psi(\phi(u)))\Big].
\end{aligned}$$

□

Theorem 3.3. Let TM be a compact tangent bundle and $\Psi : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TN, h^S)$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow (N^n, h)$, then Ψ is harmonic if and only if ψ is totally geodesic and

$$tr_g g(*, \phi(*)) = \frac{\delta^2}{1+\alpha\delta^2} g(u, \phi u).$$

Proof. If ψ is totally geodesic and $tr_g g(*, \phi(*)) = \frac{\delta^2}{1+\alpha\delta^2} g(u, \phi u)$, from Corollary3.1, we deduce that Ψ is harmonic. Inversely.

Let $\omega : I \times M \longrightarrow N$ be a smooth map satisfying for all $t \in I = (-\epsilon, \epsilon), \epsilon > 0$ and all $x \in M$

$$\omega(t,x) = \psi_t(x) = (1+t)\psi(x),$$

and

$$\omega(0,x) = \psi(x).$$

The variation vector field $v \in \Gamma(\psi^{-1}TN)$ associated to the variation $\{\psi_t\}_{t\in I}$ is given for all $x \in M$ by

$$v(x) = d_{(0,x)}\omega(\frac{d}{dt}),$$

From Lemma 3.2, we have

$$e(\Psi_t) = 2(1+t)^2 e(\psi_t) + \frac{(1+t)^2}{2} tr_h |(\nabla d\psi_t(u,*))|^2 - \frac{\delta^2(1+t)^2}{2(1+\alpha\delta^2)} |d\psi_t(\phi(u))|^2.$$

If Ψ is a critical point of the energy functional, from equation (2), we have

$$\begin{aligned} \frac{d}{dt} E(\phi_t)_{t=0} &= 0 \\ &= \int_{TM} 4e(\psi) + tr_h |(\nabla d\psi(u,*))|^2 - \frac{\delta^2}{(1+\alpha\delta^2)} |d\psi(\phi(u))|^2 dv_{g^{BS}} = 0. \end{aligned}$$

If Ψ is harmonic hence $\nabla d\psi = 0$. □

4. Biharmonic identity map $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$

Proposition 4.1. Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$ is the identity map. Then the tension field $\tau(I)$ of I is given by

$$\tau(I) = \frac{\delta^4}{(1+\alpha\delta^2)^2} g(u, \phi(u))(\phi(u))^V - \frac{\delta^2}{1+\alpha\delta^2} tr_g\Big(g(*, \phi(*))(\phi(u))^V\Big) \tag{13}$$

Proof. Let $(x,u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned} \tau(I) =& \nabla^I_{e_i^H} dI(e_i^H) + \nabla^I_{(\phi(e_1))^V} dI((\phi(e_1))^V) + \nabla^I_{(\phi(e_j))^V} dI((\phi(e_j))^V) \\ &- dI(\overline{\nabla}_{e_i^H} e_i^H + \frac{1}{1+\alpha\delta^2} \overline{\nabla}_{(\phi(e_1))^V} (\phi(e_1))^V + \overline{\nabla}_{(\phi(e_j))^V} (\phi(e_j))^V) \end{aligned}$$

From Theorem 7.1, we have

$$\begin{aligned} \tau(I) =& \frac{-\delta^2}{1+\alpha\delta^2} \Big(\frac{1}{1+\alpha\delta^2} g(e_1, \phi(e_1)) + g(e_j, \phi(e_j)) \Big) (\phi(u))^V \\ =& \frac{-\delta^2}{1+\alpha\delta^2} \Big(\frac{-\delta^2}{1+\alpha\delta^2} g(u, \phi(u)) + g(e_i, \phi(e_i)) \Big) (\phi(u))^V \end{aligned}$$

□

Theorem 4.1. Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$ is the identity map. Then the bitension field $\tau_2(I)$ of I is given by

$$\tau_2(I)_{(x,u)} = \Big\{\Delta(\tau(I))\Big\}^V_{(x,u)} + tr_g\Big\{R(u, \nabla_* \tau(I)) * \Big\}^H_{(x,u)}$$

where $\Delta(\tau(I)) = tr_g(\nabla^2_* \tau(I))$.

Proof. Let $(x,u)\in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1=\dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\left\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j=2...n\right\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i}e_i)_x=0$. Then by summing over i, we have:

$$\begin{aligned}
tr_{g^S}(\nabla^2\tau(I))_{(x,u)} =&\left\{\nabla^I_{e_i^H}\nabla^I_{e_i^H}\tau(I)\right\}_{(x,u)}+\frac{1}{1+\alpha\delta^2}\left\{\nabla^I_{(\phi(e_1))^V}\nabla^I_{(\phi(e_1))^V}\tau(I)\right\}_{(x,u)}\\
&+\left\{\nabla^I_{(\phi(e_j))^V}\nabla^I_{(\phi(e_j))^V}\tau(I)\right\}_{(x,u)}-\left\{\nabla^I_{\overline{\nabla}_{e_i^H}e_i^H}\tau(I)\right\}_{(x,u)}\\
&-\frac{1}{1+\gamma\delta^2}\left\{\nabla^I_{\overline{\nabla}_{(\phi(e_1))^V}(\phi(e_1))^V}\tau(I)\right\}_{(x,u)}\\
&-\left\{\nabla^I_{\overline{\nabla}_{(\phi(e_j))^V}(\phi(e_j))^V}\tau(I)\right\}_{(x,u)}
\end{aligned}$$

By using the Levi-Civita connection of Sasaki metric see [20] , we have

$$\begin{aligned}
tr_{g^S}(\nabla^2\tau(I))_{(x,u)} =&\left\{\nabla_{e_i}\nabla_{e_i}\tau(I)-\frac{1}{4}R(e_i,R(u,\tau(I))e_i)u\right\}^V_{(x,u)}\\
&+\frac{1}{2}\left\{R(u,\nabla_{e_i}\tau(I))e_i+\nabla_{e_i}R(u,\tau(I))e_i\right\}^H_{(x,u)}
\end{aligned}$$

By using the Riemannian curvature tensor of Sasaki metric see [20] , we have

$$\begin{aligned}
tr_{g^S}(R(\tau(I),dI)dI)_{(x,u)} =&(R(\tau(I),e_i^H)e_i^H)_{(x,u)}=-(R(e_i^H,\tau(I))e_i^H)_{(x,u)}\\
=&\left\{-\frac{1}{4}R(R(u,\tau(I))e_i,e_i)u-\frac{1}{2}R(e_i,e_i)\tau(I)\right\}^V_{(x,u)}\\
&\left\{-\frac{1}{2}(\nabla_{e_i}R)(u,\tau(I))e_i\right\}^H_{(x,u)}
\end{aligned}$$

Considering the formula 9, we deduce

$$\tau_2(I)_{(x,u)} =\left\{\nabla_{e_i}\nabla_{e_i}\tau(I)\right\}^V_{(x,u)}+\left\{R(u,\nabla_{e_i}\tau(I))e_i\right\}^H_{(x,u)}$$

□

Theorem 4.2. Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I:(TM,\widetilde{\phi},g^{BS})\longrightarrow(TM,g^S)$ is the identity map. Then I is biharmonic if and only if

$$\Delta(\tau(I))=0 \qquad \text{and} \quad tr_g(R(u,\nabla_*\tau(I))*)=0.$$

Corollary 4.1. Let (M_{2k},ϕ,g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. If $\tau(I)$ is a parallel tension field then I is biharmonic.

5. Harmonicity of the map $\Psi:(TN,h^S)\to(TM,\widetilde{\phi},g^{BS})$

In the section, we denote (N^n,g) be an n-dimensional Riemannian manifold and (TM,h^S) its tangent bundle equipped with the Sasaki metric h^S, (M_{2k},ϕ,g) be an anti-paraKähler manifold and (TM,ϕ,g^{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$.

Theorem 5.1. Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, then the tension field $\tau(\Psi)$ of ψ is given by

$$\begin{aligned}\tau(\Psi) \quad = \quad & \Big[\tau(\psi) + tr_g R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*)\Big]^H + \Big[div(\nabla d\psi)(u) \\ & + \frac{\delta^2}{1+\alpha\delta^2} tr_g\Big(g(\nabla d\psi(u,*), \phi\nabla d\psi(u,*)) + g(d\psi(*), \phi d\psi(*))\Big)\phi d\psi(u)\Big]^V.\end{aligned}$$

Proof. Let $(\psi(x), d\psi(y)) \in TN$, and $\{e_i^H, f_i^V\}_{i=1}^m$ be a local orthonormal frame on TM such that $(\nabla_{e_i} e_i)_x = 0$ then by summing over i, we have

$$\begin{aligned}\tau(\Psi) =& \nabla^{\Psi}_{e_i^H} d\Psi(e_i^H) - d\Psi(\widehat{\nabla}_{e_i^H} e_i^H) + \nabla^{\Psi}_{e_i^V} d\Psi(e_i^V) - d\Psi(\widehat{\nabla}_{e_i^V} e_i^V) \\ =& \widetilde{\nabla}_{d\Psi(e_i^H)} d\Psi(e_i^H) - d\Psi(\widehat{\nabla}_{e_i^H} e_i^H) + \widetilde{\nabla}_{d\Psi(e_i^V)} d\Psi(e_i^V) - d\Psi(\widehat{\nabla}_{e_i^V} e_i^V).\end{aligned}$$

From Proposition 2.1 and Proposition 2.2 , we have

$$\begin{aligned}\tau(\Psi) =& \widetilde{\nabla}_{d\psi(e_i)^H + (\nabla d\psi(u,e_i))^V}[d\psi(e_i)^H + (\nabla d\psi(u,e_i))^V] + \widetilde{\nabla}_{d\psi(e_i)^V} d\psi(e_i)^V \\ =& (\nabla_{d\psi(e_i)} d\psi(e_i))^H + (R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i))^H - (d\psi(\nabla_{e_i} e_i))^H \\ & + (\nabla_{d\psi(e_i)} \nabla d\psi(u,e_i))^V - (\nabla d\psi(u, \nabla_{e_i} e_i))^V \\ & + \frac{\delta^2}{1+\delta^2\alpha}\big[g(\nabla d\psi(u,e_i), \phi\nabla d\psi(u,e_i))\phi d\psi(u) + g(d\psi(e_i), \phi d\psi(e_i))\phi d\psi(u)\big]^V \\ =& \Big[\tau(\psi) + (R(d\psi(u), \nabla d\psi(u,e_i))d\psi(e_i))\Big]^H - \Big[div(\nabla d\psi) \\ & + \frac{\delta^2}{1+\delta^2\alpha}\big(g(\nabla d\psi(u,e_i), \phi\nabla d\psi(u,e_i)) + g(d\psi(e_i), \phi d\psi(e_i))\big)\phi d\psi(u)\Big]^V.\end{aligned}$$

□

Theorem 5.2. Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, then Ψ is harmonic if and only if the following conditions are verified

$$\begin{aligned}&0 = \tau(\psi) + tr_h R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*), \\ &0 = div(\nabla d\psi)(u) + tr_g\Big(g(\nabla d\psi(u,*), \phi\nabla d\psi(u,*)) + g(d\psi(*), \phi d\psi(*))\Big)\phi d\psi(u).\end{aligned}$$

Corollary 5.1. Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, if ψ is totally geodesic then Ψ is harmonic if and only if

$$tr_g g(d\psi(*), \phi d\psi(*)) = 0.$$

Lemma 5.1. Let $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow (M_{2k}, \phi, g)$, then the then the energy density associated to Ψ is given by

$$e(\Psi) = 2e(\psi) + \frac{1}{2} tr_g |\nabla d\psi(u,*)|^2 + \frac{\delta^2}{2}(g^2(\nabla d\psi(u,*), \phi d\psi(u)) + g^2(d\psi(*), \phi d\psi(u))). \tag{14}$$

Proof. Let $(x, u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, e_i^V, i = 1...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TN$ at (x, u)

such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned} e(\Psi) =& \frac{1}{2}\Big[g^{BS}(d\Psi(e_i^H), d\Psi(e_i^H)) + g^{BS}(d\Psi(e_i^V), d\Psi(e_i^V))\Big] \\ =& \frac{1}{2}\Big[g^{BS}(d\psi(e_i)^H, d\psi(e_i)^H) + g^{BS}(\nabla d\psi(u, e_i)^V, \nabla d\psi(u, e_i)^V) \\ &+ g^{BS}(d\psi(e_i)^V, d\psi(e_i)^V)\Big]. \end{aligned}$$

From Definition 2.2, we have

$$e(\Psi) = 2e(\psi) + \frac{1}{2}tr_g|\nabla d\psi(u,*)|^2 + \frac{\delta^2}{2}(g^2(\nabla d\psi(u, e_i), \phi d\psi(u)) + g^2(d\psi(e_i), \phi d\psi(u))).$$

□

Theorem 5.3. Let TN be a compact tangent bundle and $\Psi : (TN, h^S) \to (TM, \widetilde{\phi}, g^{BS})$ be a the tangent map of the map $\psi : (N^n, h) \longrightarrow: (M_{2k}, \phi, g)$, then Ψ is harmonic if and only if ψ is totally geodesic and

$$tr_g g(d\psi(*), \phi d\psi(*)) = 0.$$

Proof. If ψ is totally geodesic and $tr_g g(d\psi(*), \phi d\psi(*)) = 0$ from Corollary5.1, we deduce that Ψ is harmonic. Inversely.

Let $\omega : I \times N \longrightarrow M$ be a smooth map satisfying for all $t \in I = (-\epsilon, \epsilon), \epsilon > 0$ and all $x \in N$

$$\omega(t, x) = \psi_t(x) = (1 + t)\psi(x),$$

and

$$\omega(0, x) = \psi(x).$$

The variation vector field $v \in \Gamma(\psi^{-1}TM)$ associated to the variation $\{\psi_t\}_{t\in I}$ is given for all $x \in N$ by

$$v(x) = d_{(0,x)}\omega(\frac{d}{dt}),$$

From Lemma 5.1, we have

$$\begin{aligned} e(\Psi_t) = 2e(\psi) + \frac{(1+t)^2}{2}tr_g|\nabla d\psi(u,*)|^2 + \frac{\delta^2(1+t)^2}{2}(g^2(\nabla d\psi(u, e_i), \phi d\psi(u)) \\ + (1+t)^2 g^2(d\psi(e_i), \phi d\psi(u))). \end{aligned}$$

If Ψ is a critical point of the energy functional, from equation 2, we have

$$\begin{aligned} \frac{d}{dt}E(\phi_t)_{t=0} =& 0 \\ =& \int_{TN} 2e(\psi) + tr_g|\nabla d\psi(u,*)|^2 + \frac{\delta^2}{2}(g^2(\nabla d\psi(u, e_i), \phi d\psi(u)) \\ &+ g^2(d\psi(e_i), \phi d\psi(u))) = 0. \end{aligned}$$

If Ψ is harmonic hence $\nabla d\psi = 0$. □

6. Biharmonic identity map $I : (TM, g^S) \longrightarrow (TM, \widetilde{\phi}, g^{BS})$

Now we investigate the harmonicity of the Berger type deformed Sasaki metric g^{BS} and the Sasaki metric g^S with respect to each other. By using the Levi-Civita connection of these metrics we state the following two propositions (for the Levi-Civita connection of the Sasaki metric see [20]).

Proposition 6.1. Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that $I : (TM, g^S) \longrightarrow (TM, \widetilde{\phi}, g^{BS})$ is the identity map. Then the tension field $\tau(I)$ of I is given by

$$\tau(I) = tr_g\Big(\frac{\delta^2}{1+\alpha\delta^2} g(*, \phi(*))(\phi(u))^V\Big) \tag{15}$$

Theorem 6.1. Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Suppose that TM is a compact tangent bundle, then the identity map $I : (TM, g^S) \longrightarrow (TM, \widetilde{\phi}, g^{BS})$ is biharmonic if and only if is hrarmonic.

Proof. Let I_t be a compactly supported variation of $\tau(I)$ defined by $\tau(I_t) = (1+t)\tau(I)$.

$$\begin{aligned} E_2(\tau(I)_t) =& \frac{1}{2}\int |\tau(I_t)|^2_{g^{BS}} v_g \\ =& \frac{1}{2}\int g(\tau(I_t), \tau(I_t))v_g + \frac{\delta^2}{2}\int g(\tau(I_t), \phi(u))v_g \\ =& \frac{(1+t)^2}{2}\int g(\tau(I), \tau(I)))v_g + \frac{\delta^2(1+t)^2}{2}\int (g(\tau(I), \phi(u))^2 v_g \end{aligned}$$

then

$$0 = \frac{d}{dt}E_2(\tau(I)_t)_{t=0} = \int g(\tau(I), \tau(I)))v_g + \int (g(\tau(I), \phi(u))^2 v_g$$

then, we have

$$\tau(I) = 0.$$

□

Theorem 6.2. Let (M_{2k}, ϕ, g) be an anti-paraKähler manifold and TM its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$. Then the bitension field $\tau_2(I)$ of I is given by

$$\begin{aligned} \tau_2(I)_{(x,u)} =& \Big\{tr_g\Big(-\frac{3}{2}(\nabla_* R)(u, \tau(I))*\Big)\Big\}^H_{(x,u)} + \Big\{tr_g\Big(\triangle(\tau(I)) \\ &+ \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2 g(\tau(I), u)g(*, \phi(*))\phi(u)\Big)\Big\}^V_{(x,u)} \end{aligned}$$

where $\Delta(\tau(I)) = tr_g(\nabla^2_* \tau(I))$.

Proof. Let $(x,u)\in TM$ and $\{e_i^H, e_i^H\}_{i=1}^{2k}$ be a local orthonormal frame on TM such that $(\nabla_{e_i}e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned} tr_{g^S}(\nabla^2\tau(I))_{(x,u)} =& \Big\{\nabla^I_{e_i^H}\nabla^I_{e_i^H}\tau(I)\Big\}_{(x,u)} + \Big\{\nabla^I_{e_i^V}\nabla^I_{e_i^V}\tau(I)\Big\}_{(x,u)} \\ &- \Big\{\nabla^I_{\nabla^S_{e_i^H}e_i^H}\tau(I)\Big\}_{(x,u)} - \Big\{\nabla^I_{\nabla^S_{e_i^V}e_i^V}\tau(I)\Big\}_{(x,u)} \end{aligned}$$

From Theorem 7.1, we have

$$\begin{aligned} tr_{g^S}(\nabla^2\tau(I))_{(x,u)} =& \Big\{\nabla_{e_i}\nabla_{e_i}\tau(I)\Big\}^V_{(x,u)} \\ &+ \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2\Big\{g(e_i,\phi(\tau(I))g(e_i,u)\phi(u)\Big\}^V_{(x,u)} \end{aligned}$$

On the other hand, we have

$$tr_{g^S}(\overline{R}(\tau(I),dI)dI)_{(x,u)} = \overline{R}(\tau(I),e_i^H)e_i^H)_{(x,u)} + \overline{R}(\tau(I),e_i^V)e_i^V)_{(x,u)}$$

From Theorem 2.2, we have

$$\begin{aligned} tr_{g^S}(\overline{R}(\tau(I),dI)dI)_{(x,u)} =& \Big\{-\frac{3}{2}(\nabla_{e_i}R)(u,\tau(I))e_i\Big\}^H_{(x,u)} \\ &+ \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2\Big\{(g(\tau(I),u)g(e_i,\phi(e_i)) \\ &- g(e_i,u)g(\tau(I),\phi(e_i)))\phi(u)\Big\}^V_{(x,u)} \end{aligned}$$

Considering the formula (9), we deduce

$$\begin{aligned} \tau_2(I)_{(x,u)} =& \Big\{-\frac{3}{2}(\nabla_{e_i}R)(u,\tau(I))e_i\Big\}^H_{(x,u)} + \Big\{\triangle(\tau(I)) \\ &+ \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2 g(\tau(I),u)g(e_i,\phi(e_i))\phi(u)\Big\}^V_{(x,u)} \end{aligned}$$

□

From Theorem 5.3, we have

Theorem 6.3. Let $(TM,\widetilde{\phi},g^{BS})$ be a anti-paraKähler manifold. Then the identity map $I:(TM,g^S)\longrightarrow(TM,\widetilde{\phi},g^{BS})$ is biharmonic if and only if

$$\triangle(\tau(I)) + \Big(\frac{\delta^2}{1+\alpha\delta^2}\Big)^2 g(\tau(I),u)g(*,\phi(*))\phi(u) = 0.$$

7. Harmonicity of the map $\Psi:(TM,\widetilde{\phi},g^{BS})\to(TN,\widetilde{\phi'},h^{BS})$

In the section, we denote (M_{2k},ϕ,g) be an anti-paraKähler manifold and $(TM,\widetilde{\phi},g^{BS})$ its tangent bundle equipped with the Berger type deformed Sasaki metric g^{BS} and the paracomplex structure $\widetilde{\phi}$, $(N_{2k'},\phi',h)$ be an anti-paraKähler manifold and $(TN,\widetilde{\phi'},h^{BS})$ its tangent bundle equipped with the Berger type deformed Sasaki metric h^{BS} and the paracomplex structure $\widetilde{\phi'}$.

Theorem 7.1. Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi}', h^{BS})$ be a the tangent map of the map
$\psi : (M_{2k}, \phi, g) \longrightarrow: (N_{2k'}, \phi', h)$, then the tension field $\tau(\Psi)$ of ψ is given by

$$\begin{aligned}\tau(\Psi) =& \Big[\tau(\psi) + tr_h R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*)\Big]^H \\ &+ \Big[div(\nabla d\psi)(u) - \frac{\delta^2}{(1+\alpha\delta^2)^2} tr_g g(*, \phi(*)) d\psi(\phi(u))\Big]^V \\ &- \frac{\delta'^2}{1+\alpha'\delta'^2}\Big[\frac{\alpha\delta^2}{1+\beta\delta^2} tr_h h(d\psi(\phi(*)), \phi'(d\psi(*)) \\ &- tr_h h(\nabla d\psi(u,*), \phi'(\nabla d\psi(u,*)))\phi'(d\psi(u))^V\Big]^V.\end{aligned}$$

Proof. Let $(\psi(x), d\psi(u)) \in TN$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \dfrac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \dfrac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x,u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned}\tau(\Psi) =& \nabla^{\Psi}_{e_i^H} d\Psi(e_i^H) - d\Psi(\widetilde{\nabla}_{e_i^H} e_i^H) + \nabla^{\Psi}_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V} d\Psi(\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) \\ &- d\Psi(\widetilde{\nabla}_{\frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V} \frac{1}{\sqrt{1+\alpha\delta^2}}\phi(e_1)^V) + \nabla^{\Psi}_{\phi(e_j)^V} d\Psi \phi(e_j)^V) \\ &- d\Psi(\widetilde{\nabla}_{\phi(e_j)^V}\phi(e_j)^V).\end{aligned}$$

From Proposition 2.2 , we have

$$\begin{aligned}\tau(\Psi) =& \Big[\nabla^{\psi} d\psi(e_i) + R(d\psi(u), \nabla d\psi(u, e_i))d\psi(e_i)\Big]^H \\ &+ \Big[div(\nabla d\psi)(u) - \frac{\delta^2}{(1+\alpha\delta^2)^2} g(e_i, \phi(e_i)) d\psi(\phi(u))\Big]^V \\ &- \frac{\delta'^2}{1+\alpha'\delta'^2}\Big[\frac{\alpha\delta^2}{1+\beta\delta^2} tr_h h(d\psi(\phi(e_i)), \phi'(d\psi(e_i)) \\ &- tr_h h(\nabla d\psi(u,e_i), \phi'(\nabla d\psi(u,e_i)))\phi'(d\psi(u))^V\Big]^V.\end{aligned}$$

□

Theorem 7.2. Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi}', h^{BS})$ be a the tangent map of the map
$\psi : (M_{2k}, \phi, g) \longrightarrow: (N_{2k'}, \phi', h)$, then Ψ is harmonic if and only if the following conditions are verified

$$\begin{aligned}0 &= \tau(\psi) + tr_h R^N(d\psi(u), \nabla d\psi(u,*))d\psi(*), \\ 0 &= div(\nabla d\psi)(u) + tr_g g(*, \phi(*)), \\ 0 &= \frac{\alpha\delta^2}{1+\beta\delta^2} tr_h h(d\psi(\phi(*)), \phi'(d\psi(*)) - tr_h h(\nabla d\psi(u,*), \phi'(\nabla d\psi(u,*))).\end{aligned}$$

Corollary 7.1. Let $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi'}, h^{BS})$ be a the tangent map of the map $\psi : (M_{2k}, \phi, g) \longrightarrow: (N_{2k'}, \phi', h)$, if ψ is totally geodesic then Ψ is harmonic if and only if

$$tr_g g(*, \phi(*)) = tr_h h(d\psi(\phi(*)), \phi'(d\psi(*)) = 0.$$

Lemma 7.1. Let $(TM, \widetilde{\phi}, g^{BS}) \to (TN, \widetilde{\phi'}, h^{BS})$ be a the tangent map of the map $\psi : (N_{2k'}, \phi', h) \longrightarrow (M_{2k}, \phi, g)$, then the energy density associated to Ψ is given by

$$\begin{aligned} e(\Psi) =& \frac{1}{2}\Big[2e(\psi) + tr_g||\nabla d\psi(u,*)||^2 + (\delta tr_g h(\nabla d\psi(u,*), \phi' d\psi(u)))^2 \\ &+ tr_g||d\psi(\phi(*))||^2 + \delta^2 tr_h h^2(d\psi(\phi(*)), \phi' d\psi(u)) \\ &- \frac{\delta^2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2 + h^2(d\psi(\phi(u)), \phi' d\psi(u))\Big). \end{aligned}$$

Proof. Let $(x, u) \in TM$ and let $\{e_i\}_{i=1}^{2k}$, such that $e_1 = \frac{u}{\|u\|}$ is an orthonormal basis of TM at x. Then $\Big\{e_i^H, \frac{1}{\sqrt{1+\alpha\delta^2}}(\phi(e_1))^V, (\phi(e_j))^V, j = 2...n\Big\}$ is an orthonormal basis of $T_{(x,u)}TM$ at (x, u) such that $(\nabla_{e_i} e_i)_x = 0$. Then by summing over i, we have:

$$\begin{aligned} e(\Psi)_{(\psi(x), d\psi(x))} =& \frac{1}{2}\Big[\big(h(d\Psi(e_i^H), d\Psi(e_i^H))\big) + \frac{1}{1+\alpha\delta^2} h(d\Psi(\phi(e_1)^V), d\Psi(\phi(e_1)^V)) \\ &+ \sum_{j=2}^{n} h(d\Psi(\phi(e_j)^V), d\Psi(\phi(e_j)^V))\Big] \\ =& \frac{1}{2}\Big[\Big(\sum h(d\psi(e_i)^H, d\psi(e_i)^H) + h(\nabla d\psi(u, e_i)^V, \nabla d\psi(u, e_i)^V)\Big) \\ &+ \frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1))^V, d\psi(\phi(e_1))^V) \\ &+ \sum_{j=2}^{n} h(d\psi(\phi(e_j))^V, d\psi(\phi(e_j))^V)\Big] \\ =& \frac{1}{2}\Big[\Big(h(d\psi(e_i), d\psi(e_i)) + h(\nabla d\psi(u, e_i), \nabla d\psi(u, e_i))\Big) \\ &+ (\delta h(\nabla d\psi(u, e_i), \phi' d\psi(u)))^2 + \frac{1}{1+\alpha\delta^2} h(d\psi(\phi(e_1)), d\psi(\phi(e_1))) \\ &+ \frac{\delta^2}{1+\alpha\delta^2} h^2(d\psi(\phi(e_1)), \phi' d\psi(u)) + \sum_{j=2}^{n} h(d\psi(\phi(e_j)), d\psi(\phi(e_j))) \\ &+ (\delta h(d\psi(\phi(e_j)), \phi' d\psi(u)))^2\Big] \\ =& \frac{1}{2}\Big[2e(\psi) + tr_h||\nabla d\psi(u,*)||^2 + (\delta h(\nabla d\psi(u, e_i), \phi' d\psi(u)))^2 \\ &+ \frac{1}{1+\alpha\delta^2}\Big(h(d\psi(\phi(e_1)), d\psi(\phi(e_1))) + (\delta h(d\psi(\phi(e_1)), \phi' d\psi(u)))^2\Big) \\ &+ h(d\psi(\phi(e_i)), d\psi(\phi(e_i))) - h(d\psi(\phi(e_1)), d\psi(\phi(e_1))) \\ &+ \delta^2 h^2(d\psi(\phi(e_i)), \phi' d\psi(u)) - \delta^2 h^2(d\psi(\phi(e_1)), \phi' d\psi(u))\Big] \end{aligned}$$

$$=\frac{1}{2}\Big[2e(\psi)+tr_g||\nabla d\psi(u,*)||^2+(\delta h(\nabla d\psi(u,e_i),\phi' d\psi(u)))^2$$
$$+||d\psi(\phi(e_i))||^2+\delta^2 h^2(d\psi(\phi(e_i)),\phi' d\psi(u))$$
$$-\frac{\delta^2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2+h^2(d\psi(\phi(u)),\phi' d\psi(u))\Big).$$

□

Theorem 7.3. Let TN be a compact tangent bundle and $\Psi:(TM,\widetilde{\phi},g^{BS})\to(TN,\widetilde{\phi}',h^{BS})$ be a the tangent map of the map $\psi:(M_{2k},\phi,g)\longrightarrow(N_{2n},\phi',h)$, then Ψ is harmonic if and only if ψ is totally geodesic and

$$tr_g g(d\psi(*),\phi d\psi(*))=tr_h h(d\psi(\phi(*)),\phi'(d\psi(*)))=0.$$

Proof. If ψ is totally geodesic and $tr_g g(d\psi(*),\phi d\psi(*))=0$ from Corollary 7.1, we deduce that Ψ is harmonic. Inversely:
Let $\omega:I\times M\longrightarrow N$ be a smooth map satisfying for all $t\in I=(-\epsilon,\epsilon),\epsilon>0$ and all $x\in M$

$$\omega(t,x)=\psi_t(x)=(1+t)\psi(x),$$

and

$$\omega(0,x)=\psi(x).$$

The variation vector field $v\in\Gamma(\psi^{-1}TN)$ associated to the variation $\{\psi_t\}_{t\in I}$ is given for all $x\in N$ by

$$v(x)=d_{(0,x)}\omega(\frac{d}{dt}),$$

From Lemma 7.1, we have

$$e(\Psi)=\frac{1}{2}\Big[2e(\psi)+(1+t)^2 tr_g||\nabla d\psi(u,*)||^2+(1+t)^2(\delta tr_g h(\nabla d\psi(u,*),\phi' d\psi(u)))^2$$
$$+(1+t)^2 tr_g||d\psi(\phi(*))||^2+\delta^2(1+t)^2 tr_g h^2(d\psi(\phi(*)),\phi' d\psi(u))$$
$$-\frac{\delta^2(1+t)^2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2+h^2(d\psi(\phi(u)),\phi' d\psi(u))\Big).$$

If Ψ is a critical point of the energy functional, from equation (2), we have

$$\frac{d}{dt}E(\phi_t)_{t=0}=0$$
$$=\int_{TM}\frac{1}{2}\Big[2e(\psi)+2tr_g||\nabla d\psi(u,*)||^2+2tr_g(\delta h(\nabla d\psi(u,*),\phi' d\psi(u)))^2$$
$$+2tr_g||d\psi(\phi(*))||^2+\delta^2 2tr_g h^2(d\psi(\phi(*)),\phi' d\psi(u))$$
$$-\frac{\delta^2 2}{\alpha(1+\alpha\delta^2)}\Big(||d\psi(\phi(u))||^2+2h^2(d\psi(\phi(u)),\phi' d\psi(u))\Big)\Big]dv_{g^{BS}}=0.$$

If Ψ is harmonic hence $\nabla d\psi=0$. □

8. Conclusion

In this research we studied the harmonicity of the map $\Psi : (TM, \widetilde{\phi}, g^{BS}) \to (TN, h^S)$ and find conditions under which it is totally geodesic and we get the following results. Firstly the identity map $I : (TM, g^S) \longrightarrow (TM, \widetilde{\phi}, g^{BS})$ is biharmonic if and only if is harmonic. Secondly the identity map $I : (TM, \widetilde{\phi}, g^{BS}) \longrightarrow (TM, g^S)$ is biharmonic if and only if

$$\Delta(\tau(I)) = 0 \qquad \text{and} \quad tr_g(R(u, \nabla_* \tau(I))*) = 0.$$

References

[1] M. Altunbas, R. Simsek, A. Gezer, A study concerning Berger type deformed Sasaki metric on the tangent bundle, *Journal of Math. Physics, Analysis, Geometry* **15** (2019), no. 4, 435–447.

[2] M. Altunbas, R. Simsek, A. Gezer, Some harmonic problems on the tangent bundle with a berger-type deformed Sasaki metric, *U.P.B. Sci. Bull., Series A* **82** (2020), no. 2, 37–42.

[3] Y. Bahous, L. Belarbi, M. Belarbi, H. Elhendi, f-Biharmonic curves in three dimensional generalized symmetric spaces, *J. Appl. Math. Inform.* **42** (2024), no. 1, 85–92.

[4] C.L. Bejan, M. Benyounes, Harmonic φ morphisms, *Beitrge zur Algebra und Geometry* **44** (2003), no. 2, 309-321.

[5] L. Belarbi, M. Belarbi, H. Elhendi, Legendre curves on Lorentzian Heisenberg Space, *Bull. Transilv. Univ. Brasov SER. III* **13(62)** (2020), 41–50.

[6] L. Belarbi, H. Elhendi, F. Latti, On the geometry of the tangent bundle with vertical rescaled generalized Cheeger-Gromoll metric, *Bull. Transilv. Univ. Brasov SER. III* **12(61)** (2019), no. 2, 247–264.

[7] L. Belarbi, H. El Hendi, Harmonic and biharmonic maps between tangent bundles, *Acta Math. Univ. Comenianae* **88** (2019), no. 2, 187–199.

[8] L. Belarbi, H.Elhendi, Harmonic vector fields on gradient Sasaki metric, *Afr. Mat.* **34** (2023), no. 1, 1–13.

[9] L. Belarbi, H. El hendi, Geometry of twisted Sasaki metric, *J. Geom. Symmetry Phys.* **53** (2019), 1–19.

[10] N. Cengiz, A.A. Salimov, Diagonal lift in the tensor bundle and its applications, *Appl. Math. Comput.* **142** (2003), no. 2-3, 309–319.

[11] M. Djaa, H. EL Hendi, S. Ouakkas, On the Biharmonic vector field, *Turkish J. Math.* **36** (2012), 463-474.

[12] M. Djaa, J. Gancarzewicz, The geometry of tangent bundles of order r, *Boletin Academia, Galega de Ciencias, Espagne* **4** (1985), 147–165.

[13] N.E.H. Djaa, S. Ouakkas, M. Djaa, Harmonic sections on the tangent bundle of order two, *Ann. Math. Inform.* **38** (2011), 15–25.

[14] H. Elhendi, L. Belarbi, Deformed diagonal metrics on tangent bundle of order two and harmonicity, *Panamer. Math. J.* **27** (2017), no. 2, 90–106.

[15] H. Elhendi, L. Belarbi, Naturally harmonic maps between tangent bundles, *Balkan J. Geom. Appl.* **25** (2020), no. 1, 34–46.

[16] H. Elhendi, L. Belarbi, On paraquaternionic submersions of tangent bundle of order two, *Nonlinear Studies* **25** (2018), no. 3, 65–664.

[17] H. Elhendi, L. Belarbi, On the geometry of the tangent bundle with gradient Sasaki metric, *Arab J. Math. Sci.* **29** (2023), no. 1, 14–28.

[18] H. Elhendi, M. Terbeche, M. Djaa, Tangent bundle of order two and biharmonicity, *Acta Math. Univ. Comenianae* **83** (2014), no. 2, 165-179.

[19] J. Eells, J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* **86** (1964), 109-60.

[20] S. Gudmundsson, E. Kappos, On the geometry of the tangent bundles, *Expo. Math.* **20** (2002), no. 1, 1-41.

[21] T. Ishihara, Harmonic sections of tangent bundles, *J. Math. Univ. Tokushima* **13** (1979), 23–27.

[22] G.Y. Jiang, Harmonic maps and their first and second variational formulas, *Chinese Ann. Math. Ser. A* **7** (1986), no. 3, 89–402.
[23] J.J. Konderak, On harmonic vector fields, *Publications Matmatiques* **36** (1992), 217–288.
[24] H. Mazouzi, H. Elhendi, L. Belarbi, On the generalized bi-f-harmonic map equations on singly warped product manifolds, *Comm. Appl. Nonlinear Anal.* **25** (2018), no. 3, 52–76.
[25] V. Oproiu, On harmonic maps between tangent bundles, *Rend. Sem. Mat.* **47** (1989), 47–55.
[26] A.A. Salimov, A. Gezer, K. Akbulut, Geodesics of Sasakian metrics on tensor bundles, *Mediterr. J. Math.* **6** (2009), no. 2, 135–147.
[27] A. Sanini, Applicazioni armoniche tra i fibrati tangenti di varieta riemanniane, *Boll. U.M.I.* **6**(2A) (1983), 55–63.
[28] A.A. Salimov, M. Iscan, F. Etayo, Paraholomorphic B-manifold and its properties, *Topology Appl.* **154** (2007), no. 4, 925–933.
[29] K. Yano, S. Ishihara, *Tangent and Cotangent Bundles*, Marcel Dekker INC, New York, 1973.

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