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Hesitant Fuzzy Hyperideals of Ordered Semihypergroups

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Abstract. Hesitant fuzzy sets introduced by Torra and Narukawa present a powerful tool for study of hesitancy in decision making. In this paper, we introduce the notions of hesitant fuzzy product, characteristic hesitant fuzzy set, hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals of ordered semihypergroups. We characterize regular and intra-regular ordered semihypergroups by the properties of their hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals. We introduce the notion of hesitant fuzzy simple ordered semihypergroups. Furthermore, some characterizations of hesitant fuzzy simple ordered semihypergroups by means of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals are provided.

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1. Introduction

A fuzzy subset f of a given set S (or a fuzzy set in S) is described as an arbitrary function $f : S \longrightarrow [0,1]$, where $[0,1]$ is the usual closed interval of real numbers. This fundamental concept of a fuzzy set, was first introduced by Zadeh in his pioneering paper [19]. There has been a rapid growth worldwide in the interest of fuzzy set theory and its applications from the past several years. Evidence of this can be found in the increasing number of high-quality research articles on fuzzy sets and related topics that have been published in a variety of international journals, symposia, workshops, and international conferences held every year. Hesitant fuzzy set is a novel and recent extension of fuzzy sets that aims to model the uncertainty originated by the hesitation that might arise in the assignment of membership degrees of the elements to a fuzzy set. Despite the previous extensions overcome in different ways the managing of simultaneous sources of vagueness, Torra [17], introduced a useful generalization of the fuzzy set that is designed for situations in which it is difficult to determine the membership of an element to a set owing to ambiguity between a few different values. The hesitant fuzzy set permits the membership degree of an element to a set to be represented by a set of possible values between 0 and 1 (see [17, 18]). The hesitant fuzzy set therefore provides a more accurate representation of peoples hesitancy in stating their preferences over objects than the fuzzy set or its classical extensions. Hesitant fuzzy sets are very useful to deal with group decision making problems when experts have hesitation among several possible memberships for an element to a set.

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During the evaluating process in practice, however, these possible memberships may not be only crisp values in $[0, 1]$, but also interval values. Hesitant fuzzy sets have attracted the attention of many researchers in a short period of time because hesitant situations are very common in different real world problems and this new approach facilitates the management of uncertainty provoked by hesitation. A deep revision of the specialized literature shows the quick growth and applicability of hesitant fuzzy sets which have been extended from different points of view, (see [1, 2, 5, 6, 12]).
Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were originally proposed in 1934 by a French mathematician Marty [9], at the 8^{th} Congress of Scandinavian Mathematicians. One of the main reason which attracts researches towards hyperstructures is its unique property that in hyperstructures composition of two elements is a set, while in classical algebraic structures the composition of two elements is an element. Thus algebraic hyperstructures are natural extension of classical algebraic structures. Since then, hyperstructures are widely investigated from the theoretical point of view and for their applications to many branches of pure and applied mathematics. Especially, semihypergroups are the simplest algebraic hyperstructures which possess the properties of closure and associativity. Nowadays many researchers have studied different aspects of semihypergroups (see [3, 4, 8, 10, 11, 13, 14, 15]).

In this paper, we apply the notion of hesitant fuzzy sets to ordered semihypergroups and introduce the notions of hesitant fuzzy left (resp. right) hyperideals and hesitant fuzzy interior hyperideals of ordered semihypergroups, investigate several properties and present some related examples of these concepts. We show that hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals coincide in regular ordered semihypergroups and intra-regular ordered semihypergroups. We characterize ordered semihypergroups in terms of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals. We introduce the concept of hesitant fuzzy simple ordered semihypergroups. Moreover we characterize hesitant fuzzy simple ordered semihypergroups in terms of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals.

2. Preliminaries

Definition 2.1. [9] A map $\circ : H \times H \to \mathcal{P}^*(H)$ is termed as *hyperoperation* or *join operation* on the set H, where H is a non-empty set and $\mathcal{P}^*(H) = \mathcal{P}(H)\backslash\{\emptyset\}$ denotes the set of all non-empty subsets of H. A *hypergroupoid* is a pair $(H, \circ)$ where $\circ$ is a binary hyperoperation on the set H.

Let $* : \wp^*(H) \times \wp^*(H) \longrightarrow \wp^*(H) | (A, B) \longmapsto A * B = \bigcup\limits_{(a,b)\in A\times B} (a \circ b)$, for every $A, B \in \wp^*(H)$. This operation is well defined [7].

Definition 2.2. [7] A hypergroupoid $(H, \circ)$ is called a semihypergroup if for all x, y, z of H, we have $(x \circ y) * \{z\} = \{x\} * (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v$$

It is clear that $\{x\} * \{y\} = x \circ y$, for every $x, y \in H$.

A non-empty subset B of a semihypergroup H is termed as *sub-semihypergroup* of H if $B * B \subseteq B$ and H is termed as in this case *super-semihypergroup* of B. Let

$(H, \circ)$ be a semihypergroup. For the sake of simplicity, in the following writing $a * H$ we mean $\{a\} * H$.

An algebraic hyperstructure $(S, \circ, \leq)$ is called an ordered semihypergroup (also called po-semihypergroup) if:

(1) $(S, \circ)$ is a semihypergroup.

(2) $(S, \leq)$ is a partially ordered set such that the monotone condition holds, that is, if $a \leq b$ implies that $x \circ a \leq x \circ b$ and $a \circ x \leq b \circ x$ for all $x, a, b \in S$, where, if $A, B \in \mathcal{P}^*(S)$,then we say that $A \preceq B$ if for every $a \in A$ there exists $b \in B$ such that $a \leq b$. If $A = \{a\}$, then we write $a \preceq B$ instead of $\{a\} \preceq B$.

A non-empty subset A of an ordered semihypergroup S is called a subsemihypergroup of S if $A * A \subseteq A$. A non-empty subset A of S is called a left (resp. right) hyperideal of S if: (1) $S * A \subseteq A$ (resp. $A * S \subseteq A$) and (2) $a \in A, b \in S$ and $b \leq a$, implying $b \in A$. By a two sided hyperideal or simply a hyperideal of S we mean a non-empty subset of S which is both a left hyperideal and a right hyperideal of S. A subsemihypergroup A of an ordered semihypergroup S is called an interior hyperideal of S if: (1) $S * A * S \subseteq A$ and (2) $a \in A, b \in S$ and $b \leq a$ imply $b \in A$. An ordered semihypergroup $(S, \circ, \leq)$ is called regular if for every $a \in S$ there exists $x \in S$ such that $a \leq (a \circ x) * a$. An ordered semihypergroup S is called intra-regular if for every $a \in S$, there exist $x, y \in S$ such that $a \leq ((x \circ a) * a) * y$. For $x \in S$, we define $A_x = \{(y, z) \in S \times S \mid x \leq y \circ z\}$.

3. Hesitant fuzzy set

Let S be a reference set. Then we define hesitant fuzzy set of S in terms of a function $\mathcal{H}$ that when applied to S returns a subset of $[0,1]$. A hesitant fuzzy set $\mathcal{H}$ can also be viewed as the following mathematical representation:

$$\mathcal{H} = \{(x, \mathcal{H}(x)) \mid x \in S\}.$$

Note that throughout this paper, the set of all subsets of $[0,1]$ will be denoted by $\mathcal{P}([0,1])$.

Let $\mathcal{G}$ and $\mathcal{H}$ be two hesitant fuzzy sets of S. Then the hesitant union $\mathcal{G} \sqcup \mathcal{H}$ and hesitant intersection $\mathcal{G} \sqcap \mathcal{H}$ of $\mathcal{G}$ and $\mathcal{H}$ are defined to be hesitant fuzzy sets on S as follows: $\mathcal{G} \sqcup \mathcal{H} :\longrightarrow \mathcal{P}([0,1])$, $x \longrightarrow \mathcal{G}(x) \cup \mathcal{H}(x)$ and $\mathcal{G} \sqcap \mathcal{H} :\longrightarrow \mathcal{P}([0,1])$, $x \longrightarrow \mathcal{G}(x) \cap \mathcal{H}(x)$, and also we define $\mathcal{G} \sqsubseteq \mathcal{H}$ if $\mathcal{G}(x) \subseteq \mathcal{H}(x)$ for all $x \in S$.

The hesitant fuzzy product of two hesitant fuzzy sets $\mathcal{G}$ and $\mathcal{H}$ of an ordered semihypergroup S is defined to be a hesitant fuzzy set $\mathcal{G} \circledcirc \mathcal{H}$ of S which given by

$$(\mathcal{G} \circledcirc \mathcal{H})(x) = \begin{cases} \bigcup\limits_{(y,z) \in A_x} \{\mathcal{G}(y) \cap \mathcal{H}(z)\}, & \text{if } A_x \neq \emptyset, \\ \emptyset, & \text{if } A_x = \emptyset, \end{cases}$$

For a non-empty subset A of S, defined a map $\mathcal{H}_A : S \longrightarrow \mathcal{P}([0,1])$,

$$x \longrightarrow \begin{cases} [0,1], & \text{if } x \in A \\ \emptyset, & \text{if } x \notin A \end{cases}$$

Then $\mathcal{H}_A$ is a hesitant fuzzy set of S and is called the characteristic hesitant fuzzy set of S. For an ordered semihypergroup S, the hesitant fuzzy set $\mathcal{H}_S$ is defined as follows:

$$\mathcal{H}_S : S \longrightarrow \mathcal{P}([0,1]),\ x \longmapsto \mathcal{H}_S(x) = [0,1] \text{ for all } x \in S.$$

The hesitant fuzzy set $\mathcal{H}_S$ is called identity hesitant fuzzy set of S.

For a hesitant fuzzy set $\mathcal{H}$ of S a subset δ of $[0,1]$, the set

$$U(\mathcal{H},\delta) = \{x \in S \mid \mathcal{H}(x) \supseteq \delta\},$$

is called the hesitant level set of $\mathcal{H}$.

Theorem 3.1. *Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be hesitant fuzzy sets of an ordered semihypergroup S, where A and B are nonempty subsets S. Then the following properties hold:*

(1) *If $A \subseteq B$ then $\mathcal{H}_A \sqsubseteq \mathcal{H}_B$.*
(2) $\mathcal{H}_A \sqcap \mathcal{H}_B = \mathcal{H}_{A\cap B}$.
(3) $\mathcal{H}_A \sqcup \mathcal{H}_B = \mathcal{H}_{A\cup B}$.
(4) $\mathcal{H}_A \circledcirc \mathcal{H}_B = \mathcal{H}_{(A*B]}$.

Proof. (1). It is obvious.

(2). Let $x \in S$. If $x \in A\cap B$, then $x \in A$ and $x \in B$. Thus we have $(\mathcal{H}_A \sqcap \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cap \mathcal{H}_B(x) = [0,1] = \mathcal{H}_{A\cap B}(x)$.

If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Thus we have $(\mathcal{H}_A \sqcap \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cap \mathcal{H}_B(x) = \emptyset = \mathcal{H}_{A\cap B}(x)$. Therefore $\mathcal{H}_A \sqcap \mathcal{H}_B = \mathcal{H}_{A\cap B}$.

(3). Let $x \in S$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. Thus we have $(\mathcal{H}_A \sqcup \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cup \mathcal{H}_B(x) = [0,1] = \mathcal{H}_{A\cup B}(x)$. If $x \notin A \cup B$, then $x \notin A$ and $x \notin B$. Hence we have $(\mathcal{H}_A \sqcup \mathcal{H}_B)(x) = \mathcal{H}_A(x) \cup \mathcal{H}_B(x) = \emptyset = \mathcal{H}_{A\cup B}$. Therefore $\mathcal{H}_A \sqcup \mathcal{H}_B = \mathcal{H}_{A\cup B}$.

(4). For any $x \in S$, suppose $x \in (A * B]$. Then $x \leq a \circ b$ for some $a \in A$ and $b \in B$, and so $(a,b) \in A_x$. Thus we have $(\mathcal{H}_A \circledcirc \mathcal{H}_B)(x) = \bigcup_{(y,z)\in A_x} \{\mathcal{H}_A(y) \cap \mathcal{H}_B(z)\} \supseteq \mathcal{H}_A(a) \cap \mathcal{H}_B(b) = [0,1]$, and so $(\mathcal{H}_A \circledcirc \mathcal{H}_B)(x) = [0,1]$. Since $x \in (A * B]$, then we get $\mathcal{H}_{(A*B]}(x) = [0,1]$. Suppose $x \notin (A * B]$. Then $\mathcal{H}_{(A*B]}(x) = \emptyset$. If $A_x = \emptyset$, then $(\mathcal{H}_A \circledcirc \mathcal{H}_B)(x) = \emptyset$ and $(\mathcal{H}_A \circledcirc \mathcal{H}_B)(x) = \mathcal{H}_{(A*B]}(x)$. Assume that $A_x \neq \emptyset$. Then $(\mathcal{H}_A \circledcirc \mathcal{H}_B)(x) = \bigcup_{(y,z)\in A_x} \{\mathcal{H}_A(y) \cap \mathcal{H}_B(z)\}$. We now prove that $\mathcal{H}_A(y) \cap \mathcal{H}_B(z) = \emptyset$ for all $(y,z) \in A_x$. Let $(y,z) \in A_x$ then $x \leq y \circ z$. If $y \in A$ and $z \in B$, then $y \circ z \subseteq A * B$ and so $x \in (A * B]$. This is impossible. Thus we have $y \notin A$ or $z \notin B$. If $y \notin A$ then $\mathcal{H}_A(y) = \emptyset$ and so $\mathcal{H}_A(y) \cap \mathcal{H}_B(z) = \emptyset$. Similarly, if $z \notin B$, then $\mathcal{H}_A(y) \cap \mathcal{H}_B(z) = \emptyset$. In any case $\mathcal{H}_A \circledcirc \mathcal{H}_B = \mathcal{H}_{(A*B]}$. □

4. Hesitant fuzzy hyperideals

In this section, we introduce the notion of hesitant fuzzy hyperideals of ordered semihypergroups.

Definition 4.1. A hesitant fuzzy set $\mathcal{H}$ of an ordered semihypergroup is called a hesitant fuzzy subsemihypergroup of S if:

$$\sqcap_{\alpha\in x\circ y}\mathcal{H}(\alpha) \supseteq \mathcal{H}(x) \cap \mathcal{H}(y).$$

Example 4.1. Let $(S, \circ, \leq)$ be an ordered semihypergroup, where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

Let $\mathcal{H}$ be a hesitant fuzzy set of S defined as follows

$$\mathcal{H} : S \longrightarrow \mathcal{P}([0,1]),\ x \longrightarrow \begin{cases} [0.1,0.9] & \text{if } x=a \\ [0.2,0.8] & \text{if } x=b \\ (0.3,0.6] & \text{if } x=c \\ [0.4,0.6] & \text{if } x=d \end{cases}$$

Then $\mathcal{H}$ is a hesitant fuzzy subsemihypergroup of S.

Definition 4.2. A hesitant fuzzy set $\mathcal{H}$ of an ordered semihypergroup is called a hesitant fuzzy left (resp. right) hyperideal of S if it satisfies the following conditions:

(1) $\sqcap_{\alpha\in x\circ y}\mathcal{H}(\alpha) \supseteq \mathcal{H}(y)$ (resp. $\sqcap_{\alpha\in x\circ y}\mathcal{H}(\alpha) \supseteq \mathcal{H}(x)$).

(2) $(\forall x,y\in S)\ x\leq y \Longrightarrow \mathcal{H}(x)\supseteq\mathcal{H}(y)$.

A hesitant fuzzy set $\mathcal{H}$ of an ordered semihypergroup S is called a hesitant fuzzy hyperideal *(* or hesitant fuzzy two-sided hyperideal) of S if it is both a hesitant fuzzy left hyperideal and a hesitant fuzzy right hyperideal of S.

Example 4.2. Let $(S,\circ,\leq)$ be an ordered semihypergroup, where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
d	$\{a\}$	$\{a,d\}$	$\{a\}$	$\{a\}$

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b),(a,c),(a,d)\}.$$

Let $\mathcal{H}$ be a hesitant fuzzy set of S defined as follows

$$\mathcal{H} : S \longrightarrow \mathcal{P}([0,1]),\ x \longrightarrow \begin{cases} (0,1] & \text{if } x=a \\ [0.2,0.8] & \text{if } x=b \\ [0.4,0.7] & \text{if } x=c \\ [0.2,0.9] & \text{if } x=d \end{cases}$$

Then $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S. This is not a hesitant fuzzy right hyperideal as $\sqcap_{\alpha\in b\circ c=\{a,c\}}\mathcal{H}(\alpha) = \mathcal{H}(a)\cap\mathcal{H}(c) = [0.4,0.7] \not\supseteq [0.2,0.8] = \mathcal{H}(b)$.

Example 4.3. Let $(S,\circ,\leq)$ be an ordered semihypergroup, where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,b)\}.$$

Let $\mathcal{H}$ be a hesitant fuzzy set of S defined as follows

$$\mathcal{H} : S \longrightarrow \mathcal{P}([0,1]),\ x \longrightarrow \begin{cases} (0.1,1] & \text{if } x = a \\ [0.2,0.8] & \text{if } x = b \\ [0.4,0.6] & \text{if } x = c \\ (0.4,0.5] & \text{if } x = d \end{cases}$$

Then $\mathcal{H}$ is a hesitant fuzzy hyperideal of S.

Theorem 4.1. *Let $(S,\circ,\leq)$ be an ordered semihypergroup and $\emptyset \neq A \subseteq S$. Then A is a left (resp. right) hyperideal of S if and only if the characteristic hesitant fuzzy set $\mathcal{H}_A$ of A is a is a hesitant fuzzy left (resp. right) hyperideal of S.*

Proof. Suppose that A is a left hyperideal of S. Let $x, y \in S$. Then $\sqcap_{\alpha \in x \circ y}\mathcal{H}_A(\alpha) \supseteq \mathcal{H}_A(y)$. Indeed, if $y \notin A$ then $\mathcal{H}_A(y) = \emptyset$. Since $\mathcal{H}_A(x) \supseteq \emptyset$ for all $x \in S$, then we have $\sqcap_{\alpha \in x \circ y}\mathcal{H}_A(\alpha) \supseteq \emptyset = \mathcal{H}_A(y)$. Let $y \in A$. Since A is a left hyperideal of S and $x \in S$, then we have $x \circ y \subseteq S * A \subseteq A$. Thus in this case $\mathcal{H}_A(\alpha) = [0,1]$ for any $\alpha \in x \circ y$. Hence $\sqcap_{\alpha \in x \circ y}\mathcal{H}_A(\alpha) = [0,1] \supseteq \mathcal{H}_A(y)$. Let now $x, y \in S$, $x \leq y$. Then $\mathcal{H}_A(x) \supseteq \mathcal{H}_A(y)$. In fact, if $y \in A$, then $\mathcal{H}_A(y) = [0,1]$. Since $S \ni x \leq y \in A$, by hypothesis we have $x \in A$, then $\mathcal{H}_A(x) = [0,1]$. Thus $\mathcal{H}_A(x) = [0,1] \supseteq \mathcal{H}_A(y)$. If $y \notin A$, then $\mathcal{H}_A(y) = \emptyset$. Since $x \in S$, then we have $\mathcal{H}_A(x) \supseteq \emptyset = \mathcal{H}_A(y)$. Consequently, $\mathcal{H}_A$ is a hesitant fuzzy left hyperideal of S.

Conversely, let A be a non-empty subset of S such that $\mathcal{H}_A$ is a hesitant fuzzy left hyperideal of S. We claim that $S * A \subseteq A$. To prove our claim, let $x \in S$ and $y \in A$. By hypothesis, $\sqcap_{\alpha \in x \circ y}\mathcal{H}_A(\alpha) \supseteq \mathcal{H}_A(y) = [0,1]$. It thus implies that $\sqcap_{\alpha \in x \circ y}\mathcal{H}_A(\alpha) = [0,1]$. Hence for any $\alpha \in x \circ y$, $\mathcal{H}_A(\alpha) = [0,1]$, i.e., $\alpha \in A$. It thus follows that $S \circ A \subseteq A$. Furthermore, let $x \in A$, $S \ni y \leq x$. Then $y \in A$. Indeed, it is enough to prove that $\mathcal{H}_A(y) = [0,1]$. By $x \in A$, we have $\mathcal{H}_A(x) = [0,1]$. Since $\mathcal{H}_A$ is a hesitant fuzzy left hyperideal of S and $y \leq x$, then we have $\mathcal{H}_A(y) \supseteq \mathcal{H}_A(x) = [0,1]$. Thus $\mathcal{H}_A(y) = [0,1]$. Therefore A is a left hyperideal of S.

Similarly we can show that $\mathcal{H}_A$ is a hesitant fuzzy right hyperideal of S, if and only if A is a right hyperideal of S. □

Corollary 4.2. *Let $(S,\circ,\leq)$ be an ordered semihypergroup and $\emptyset \neq A \subseteq S$. Then A is a hyperideal of S if and only if the characteristic hesitant fuzzy set $\mathcal{H}_A$ of A is a is a hesitant fuzzy hyperideal of S.*

Theorem 4.3. *Let $\mathcal{H}$ be a hesitant fuzzy set of an ordered semihypergroup and $\delta \in \mathcal{P}([0,1])$. Then $\mathcal{H}$ is a hesitant hyperideal of S if and only if the nonempty level set $U(\mathcal{H},\delta)$ of $\mathcal{H}$ is a hyperideal of S.*

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy hyperideal of S. Let $x \in U(\mathcal{H},\delta)$ and $y \in S$. Then $\mathcal{H}(x) \supseteq \delta$. It follows from Definition 4, that

$$\sqcap_{\alpha \in x \circ y}\mathcal{H}(\alpha) \supseteq \mathcal{H}(x) \supseteq \delta$$

and

$$\sqcap_{\alpha \in y \circ x}\mathcal{H}(\alpha) \supseteq \mathcal{H}(x) \supseteq \delta.$$

Hence, we can deduce that $\sqcap_{\alpha \in x \circ y}\mathcal{H}(\alpha) \supseteq \delta$ and $\sqcap_{\alpha \in y \circ x}\mathcal{H}(\alpha) \supseteq \delta$. Thus it can be easily shown that $x \circ y \subseteq U(\mathcal{H},\delta)$ and $y \circ x \subseteq U(\mathcal{H},\delta)$. Furthermore, let $x \in U(\mathcal{H},\delta)$, $S \ni y \leq x$. Then $y \in U(\mathcal{H},\delta)$. Indeed, since $x \in U(\mathcal{H},\delta)$, $\mathcal{H}(x) \supseteq \delta$ and $\mathcal{H}$ is a hesitant fuzzy hyperideal of S, then we have $\mathcal{H}(y) \supseteq \mathcal{H}(x) \supseteq \delta$. We have $\mathcal{H}(y) \supseteq \delta$, i.e., $y \in U(\mathcal{H}, \delta)$. Therefore $U(\mathcal{H}, \delta)$ is a hyperideal of S.

Conversely, let $U(\mathcal{H}, \delta) \neq \emptyset$ be a hyperideal of S. If there exist $x_1, y_1 \in S$ such that $\sqcap_{\alpha \in x_1 \circ y_1} \mathcal{H}(\alpha) \subset \mathcal{H}(y_1)$, then there exists $\delta \in \mathcal{P}([0,1])$ such that $\sqcap_{\alpha \in x_1 \circ y_1} \mathcal{H}(\alpha) \subset \delta \subseteq \mathcal{H}(y_1)$, and we have $\mathcal{H}(y_1) \supseteq \delta$ and $\sqcap_{\alpha \in x_1 \circ y_1} \mathcal{H}(\alpha) \subset \delta$. Thus $y_1 \in U(\mathcal{H}, \delta)$ and $x_1 \circ y_1 \nsubseteq U(\mathcal{H}, \delta)$, which is a contradiction. Hence $\sqcap_{\alpha \in x \circ y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(y)$ for all $x, y \in S$. Moreover if $x \leq y$ then $\mathcal{H}(x) \supseteq \mathcal{H}(y)$. Indeed, if there exist $x_1, y_1 \in S$ such that $x_1 \leq y_1$ and $\mathcal{H}(x_1) \subset \mathcal{H}(y_1)$ then there exists $\delta \in \mathcal{P}([0,1])$ such that $\mathcal{H}(x_1) \subset \delta \subseteq \mathcal{H}(y_1)$ and we have $\mathcal{H}(y_1) \supseteq \delta$ and $\mathcal{H}(x_1) \subset \delta$. Then $y_1 \in U(\mathcal{H}, \delta)$ and $x_1 \notin U(\mathcal{H}, \delta)$. This is a contradiction that $U(\mathcal{H}, \delta)$ is a hyperideal of S. Therefore $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S. In a similar way we can show that $\mathcal{H}$ is a hesitant fuzzy right hyperideal of S and thus $\mathcal{H}$ is a hesitant fuzzy hyperideal of S. This completes the proof. □

Theorem 4.4. *Let S be an ordered semihypergroup and $\mathcal{H}$ is a hesitant fuzzy set of S. Then $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S if and only if $\mathcal{H}$ satisfies the following conditions:*

(1) $\mathcal{H}_S \circledcirc \mathcal{H} \sqsubseteq \mathcal{H}$.

(2) $x \leq y \Longrightarrow \mathcal{H}(x) \supseteq \mathcal{H}(y)$ *for all* $x, y \in S$.

Proof. Suppose that $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S. Then by Definition 4, condition (2) holds. To prove the condition (1) holds, it is enough to prove that $(\mathcal{H}_S \circledcirc \mathcal{H})(x) \subseteq \mathcal{H}(x)$ for any $x \in S$. Indeed, let $x \in S$. If $A_x = \emptyset$, then $(\mathcal{H}_S \circledcirc \mathcal{H})(x) = \emptyset \subseteq \mathcal{H}(x)$. Let $A_x \neq \emptyset$. Then there exist $y, z \in S$ such that $x \leq y \circ z$ and there exists $v \in y \circ z$ such that $x \leq v$. Since $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S, then we have $\mathcal{H}(z) \subseteq \sqcap_{v \in y \circ z} \mathcal{H}(v) \subseteq \sqcap_{\substack{v \in y \circ z \\ x \leq v}} \mathcal{H}(v) \subseteq \sqcap_{\substack{v \in y \circ z \\ x \leq v}} \mathcal{H}(x) = \mathcal{H}(x)$. Hence $\mathcal{H}(z) \subseteq \mathcal{H}(x)$ for any $x \leq y \circ z$. Thus

$$\begin{aligned}
(\mathcal{H}_S \circledcirc \mathcal{H})(x) &= \bigcup_{(y,z) \in A_x} \{\mathcal{H}_S(y) \cap \mathcal{H}(z)\} \\
&= \bigcup_{(y,z) \in A_x} \{[0,1] \cap \mathcal{H}(z)\} = \bigcup_{(y,z) \in A_x} \{\mathcal{H}(z)\} \\
&\subseteq \mathcal{H}(x).
\end{aligned}$$

Thus $(\mathcal{H}_S \circledcirc \mathcal{H})(x) \subseteq \mathcal{H}(x)$ for all $x \in S$.

Conversely, assume that the conditions (1) and (2) hold. Let $y, z \in S$. Then we can prove that $\mathcal{H}(x) \supseteq \mathcal{H}(z)$ for any $x \in y \circ z$. In fact, since $x \in y \circ z$, $x \leq x$, then we have $x \leq y \circ z$. Thus by hypothesis, we have

$$\begin{aligned}
\mathcal{H}(x) \supseteq (\mathcal{H}_S \circledcirc \mathcal{H})(x) &= \bigcup_{(p,q) \in A_x} \{\mathcal{H}_S(p) \cap \mathcal{H}(q)\} \\
&\supseteq \{\mathcal{H}_S(y) \cap \mathcal{H}(z)\} \\
&= \{[0,1] \cap \mathcal{H}(z)\} \\
&= \mathcal{H}(z).
\end{aligned}$$

Hence $\sqcap_{x \in y \circ z} \mathcal{H}(x) \supseteq \mathcal{H}(z)$ for any $x \in y \circ z$. Hence $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S. □

Similarly we can prove the following Theorem.

Theorem 4.5. *Let S be an ordered semihypergroup and $\mathcal{H}$ be a hesitant fuzzy of S. Then $\mathcal{H}$ is a hesitant fuzzy right hyperideal of S if and only if $\mathcal{H}$ satisfies the following conditions:*

(1) $\mathcal{H} \circledcirc \mathcal{H}_S \sqsubseteq \mathcal{H}$.

(2) $x \leq y \Longrightarrow \mathcal{H}(x) \supseteq \mathcal{H}(y)$ *for all* $x, y \in S$.

5. Hesitant fuzzy interior hyperideals

Definition 5.1. Let $\mathcal{H}$ be a hesitant fuzzy set of an ordered semihypergroup S. Then $\mathcal{H}$ is called a hesitant fuzzy interior hyperideal of S if it satisfies the following conditions:

(1) $(\forall x, y \in S)\ \sqcap_{\alpha \in x \circ y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(x) \cap \mathcal{H}(y)$.

(2) $(\forall x, a, y \in S)\ \sqcap_{\alpha \in (x \circ a) * y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(a)$.

(3) $(\forall x, y \in S)\ x \leq y \Longrightarrow \mathcal{H}(x) \supseteq \mathcal{H}(y)$.

Example 5.1. Let $(S, \circ, \leq)$ be an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a, d\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,d)\}.$$

Let $\mathcal{H}$ be a hesitant fuzzy set of S defined as follows

$$\mathcal{H} : S \longrightarrow \mathcal{P}([0,1]),\ x \longrightarrow \begin{cases} (0.1, 0.9] & \text{if } x \in \{a, b\} \\ [0.4, 0.8] & \text{if } x \in \{c, d\} \end{cases}$$

Then $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S.

Theorem 5.1. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and A be a nonempty subset of S. Then A is an interior hyperideal of S if and only if the the characteristic hesitant fuzzy set $\mathcal{H}_A$ of A is a hesitant fuzzy interior hyperideal of S.*

Proof. Suppose that A is an interior hyperideal of S. Let x, y and a be any elements of S. Then $\sqcap_{\alpha \in (x \circ a) * y} \mathcal{H}_A(\alpha) \supseteq \mathcal{H}_A(a)$. Indeed, If $a \in A$, then $\mathcal{H}_A(a) = [0,1]$. Since A is an interior hyperideal of S, then we have $\alpha \in (x \circ a) * y \subseteq (S * A) * S \subseteq A$ we have $\mathcal{H}_A(\alpha) = [0,1]$. Thus $\sqcap_{\alpha \in (x \circ a) * y} \mathcal{H}_A(\alpha) = [0,1] \supseteq \mathcal{H}_A(a)$. If $a \notin A$, then $\mathcal{H}_A(a) = \emptyset$. Since $\mathcal{H}_A(x) \supseteq \emptyset$ for all $x \in S$, then $\sqcap_{\alpha \in (x \circ a) * y} \mathcal{H}_A(\alpha) \supseteq \emptyset = \mathcal{H}_A(a)$. Let $x, y \in S$ with $x \leq y$. Then $\mathcal{H}_A(x) \supseteq \mathcal{H}_A(y)$. Indeed, if $y \notin A$ then $\mathcal{H}_A(y) = \emptyset$ and $\mathcal{H}_A(x) \supseteq \emptyset = \mathcal{H}_A(y)$. If $y \in A$, then $\mathcal{H}_A(y) = [0,1]$. Since $x \leq y$ and A is an interior hyperideal of S, then we have $x \in A$ and thus $\mathcal{H}_A(x) = [0,1] \supseteq \mathcal{H}_A(y)$. Since A is an interior hyperideal of S, then A is a subsemihypergroup of S. Let $x, y \in S$. Then we have, $\sqcap_{\alpha \in x \circ y} \mathcal{H}_A(\alpha) \supseteq \mathcal{H}_A(x) \cap \mathcal{H}_A(y)$. Indeed, if $x \circ y \nsubseteq A$, then there exists $\alpha \in x \circ y$ such that $\alpha \notin A$, and we have $\sqcap_{\alpha \in x \circ y} \mathcal{H}_A(\alpha) = \emptyset$. Besides that $x \circ y \nsubseteq A$ implies that $x \notin A$ or $y \notin A$. Then $\mathcal{H}_A(x) = \emptyset$ or $\mathcal{H}_A(y) = \emptyset$ and hence $\sqcap_{\alpha \in x \circ y} \mathcal{H}_A(\alpha) \supseteq \emptyset = \mathcal{H}_A(x) \cap \mathcal{H}_A(y)$. Let $x \circ y \subseteq A$. Then $\mathcal{H}_A(\alpha) = [0,1]$ for any $\alpha \in x \circ y$. It implies that $\sqcap_{\alpha \in x \circ y} \mathcal{H}_A(\alpha) = [0,1]$. Since we have $\mathcal{H}_A(x) \subseteq [0,1]$

for any $x \in A$, then $\sqcap_{\alpha\in x\circ y}\mathcal{H}_A(\alpha) = [0,1] \supseteq \mathcal{H}_A(x) \cap \mathcal{H}_A(y)$. Therefore $\mathcal{H}_A$ is a hesitant fuzzy interior hyperideal of S .

Conversely, let $\emptyset \neq A \subseteq S$ such that $\mathcal{H}_A$ is a hesitant fuzzy interior hyperideal of S. We claim that $A * A \subseteq A$. To prove the claim, let $x, y \in A$. By hypothesis, $\sqcap_{\alpha\in x\circ y}\mathcal{H}_A(\alpha) \supseteq \mathcal{H}_A(x) \cap \mathcal{H}_A(y) = [0,1]$ which implies that $\sqcap_{\alpha\in x\circ y}\mathcal{H}_A(\alpha) \supseteq [0,1]$. On the other hand $\mathcal{H}_A(x) \subseteq [0,1]$ for all $x \in S$. Thus for any $\alpha \in x\circ y$, $\mathcal{H}_A(\alpha) = [0,1]$ implies that $\alpha \in A$. It thus follows that $A * A \subseteq A$. Let $\alpha \in (S * A) * S$, then there exist $x, y \in S$ and $a \in A$ such that $\alpha \in (x \circ a) * y$. Since $\sqcap_{\alpha\in(x\circ a)*y}\mathcal{H}_A(\alpha) \supseteq \mathcal{H}_A(a)$ and $a \in A$, then we have $\mathcal{H}_A(a) = [0,1]$. Hence for each $\alpha \in (S * A) * S$, we have $\sqcap_{\alpha\in(x\circ a)*y}\mathcal{H}_A(\alpha) \supseteq [0,1]$. On the other hand $\mathcal{H}_A(x) \subseteq [0,1]$ for all $x \in S$. Thus for any $\alpha \in (x \circ a) * y$, $\mathcal{H}_A(\alpha) = [0,1]$ implies that $\alpha \in A$. Thus $(S * A) * S \subseteq A$. Furthermore, let $x \in A$, $S \ni y \leq x$. Then $y \in A$. Indeed, it is enough to prove that $\mathcal{H}_A(y) = [0,1]$. By $x \in A$ we have $\mathcal{H}_A(x) = [0,1]$. Since $\mathcal{H}_A$ is a hesitant fuzzy interior hyperideal of S and $y \leq x$, then we have $\mathcal{H}_A(y) \supseteq \mathcal{H}_A(x) = [0,1]$. Hence we conclude that $\mathcal{H}_A(y) \supseteq \emptyset$. Thus $\mathcal{H}_A(y) = [0,1]$. Therefore A is a interior hyperideal of S. □

Theorem 5.2. *Let $\mathcal{H}$ be a hesitant fuzzy set of an ordered semihypergroup S and $\delta \in \mathcal{P}([0,1])$. Then $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S if and only if each nonempty level set $U(\mathcal{H},\delta)$ of $\mathcal{H}$ is an interior hyperideal of S.*

Proof. Assume that $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S and $U(\mathcal{H},\delta) \neq \emptyset$. Let $x, y \in U(\mathcal{H},\delta)$. Then $\mathcal{H}(x) \supseteq \delta$ and $\mathcal{H}(y) \supseteq \delta$. By hypothesis, we have $\sqcap_{\alpha\in x\circ y}\mathcal{H}(\alpha) \supseteq \mathcal{H}(x) \cap \mathcal{H}(y) \supseteq \delta \cap \delta = \delta$. So we can write as $\sqcap_{\alpha\in x\circ y}\mathcal{H}(\alpha) \supseteq \delta$. Thus for any $\alpha \in x \circ y$, we have $\mathcal{H}(\alpha) \supseteq \delta$, implies that $\alpha \in U(\mathcal{H},\delta)$. It follows that $x \circ y \subseteq U(\mathcal{H},\delta)$. Hence $U(\mathcal{H},\delta)$ is a subsemihypergroup of S. Let $y \in U(\mathcal{H},\delta)$ and $x, z \in S$. Then $\mathcal{H}(y) \supseteq \delta$. Since $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S, then $\sqcap_{w\in(x\circ y)*z}\mathcal{H}(w) \supseteq \mathcal{H}(y) \supseteq \delta$. So we can write as $\sqcap_{w\in(x\circ y)*z}\mathcal{H}(w) \supseteq \delta$. Hence $\mathcal{H}(w) \supseteq \delta$ for any $w \in (x\circ y)*z$ implies that $w \in U(\mathcal{H},\delta)$. Thus $S * U(\mathcal{H},\delta) * S \subseteq U(\mathcal{H},\delta)$. Furthermore, let $x \in U(\mathcal{H},\delta)$, $S \ni y \leq x$. Then $y \in U(\mathcal{H},\delta)$. Indeed, since $x \in U(\mathcal{H},\delta)$, $\mathcal{H}(x) \supseteq \delta$ and $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S, then we have $\mathcal{H}(y) \supseteq \mathcal{H}(x) \supseteq \delta$, we have $\mathcal{H}(y) \supseteq \delta$, i.e., $y \in U(\mathcal{H},\delta)$. Therefore $U(\mathcal{H},\delta)$ is an interior hyperideal of S.

Conversely, suppose that $U(\mathcal{H},\delta) \neq \emptyset$ is an interior hyperideal of S. If there exist $x_1, y_1 \in S$ such that $\sqcap_{\alpha\in x_1\circ y_1}\mathcal{H}(\alpha) \subset \mathcal{H}(x_1) \cap \mathcal{H}(y_1)$. Then there exists $\delta \in \mathcal{P}([0,1])$ such that $\sqcap_{\alpha\in x_1\circ y_1}\mathcal{H}(\alpha) \subset \delta \subseteq \mathcal{H}(x_1) \cap \mathcal{H}(y_1)$, and we have $\mathcal{H}(x_1) \supseteq \delta$, $\mathcal{H}(y_1) \supseteq \delta$ and $\sqcap_{\alpha\in x_1\circ y_1}\mathcal{H}(\alpha) \subset \delta$ which implies that $x_1, y_1 \in U(\mathcal{H},\delta)$ and $x_1 \circ y_1 \nsubseteq U(\mathcal{H},\delta)$. It contradicts the fact that $U(\mathcal{H},\delta)$ is an interior hyperideal of S. Consequently, $\sqcap_{\alpha\in x\circ y}\mathcal{H}(\alpha) \supseteq \mathcal{H}(x) \cap \mathcal{H}(y)$ for all $x, y \in S$. Next we show that $\sqcap_{\alpha\in(x\circ a)*y}\mathcal{H}(\alpha) \supseteq \mathcal{H}(a)$ for all $x, a, y \in S$. If there exist x_1, a_1, y_1 such that $\sqcap_{\alpha\in(x_1\circ a_1)*y_1}\mathcal{H}(\alpha) \subset \mathcal{H}(a_1)$, and $\delta \in \mathcal{P}([0,1])$ such that $\sqcap_{\alpha\in(x_1\circ a_1)*y_1}\mathcal{H}(\alpha) \subset \delta \subseteq \mathcal{H}(a_1)$, so $\mathcal{H}(a_1) \supseteq \delta$ and $\sqcap_{\alpha\in(x_1\circ a_1)*y_1}\mathcal{H}(\alpha) \subset \delta$ then $a_1 \in U(\mathcal{H},\delta)$ and $(x_1 \circ a_1) * y_1 \nsubseteq U(\mathcal{H},\delta)$. This is a contradiction that $U(\mathcal{H},\delta)$ is an interior hyperideal of S. Moreover if $x \leq y$ then $\mathcal{H}(x) \supseteq \mathcal{H}(y)$. Indeed, if there exist $x_1, y_1 \in S$ such that $x_1 \leq y_1$ and $\mathcal{H}(x_1) \subset \mathcal{H}(y_1)$, and $\delta \in \mathcal{P}([0,1])$ such that $\mathcal{H}(x_1) \subset \delta \subseteq \mathcal{H}(y_1)$ and we have $\mathcal{H}(y_1) \supseteq \delta$ and $\mathcal{H}(x_1) \subset \delta$. Then $y_1 \in U(\mathcal{H},\delta)$ and $x_1 \notin U(\mathcal{H},\delta)$. This is a contradiction that $U(\mathcal{H},\delta)$ is an interior hyperideal of S. Thus if $x \leq y$, then $\mathcal{H}(x) \supseteq \mathcal{H}(y)$. □

Theorem 5.3. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $\mathcal{H}$ be a hesitant fuzzy hyperideal of S. Then $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S.*

Proof. Suppose that $\mathcal{H}$ is a hesitant fuzzy hyperideal of S. Let $x, y \in S$. Then by hypothesis $\sqcap_{\alpha \in x \circ y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(x) \supseteq \mathcal{H}(x) \cap \mathcal{H}(y)$. Let $x, a, y \in S$. Since $\mathcal{H}$ is a hesitant fuzzy hyperideal of S, then for any $\alpha \in (x \circ a) * y$, we have $\sqcap_{\alpha \in (x \circ a) * y} \mathcal{H}(\alpha) = \sqcap_{\substack{\alpha \in x \circ \beta \\ \beta \in a \circ y}} \mathcal{H}(\alpha) \supseteq \mathcal{H}(\beta) \supseteq \sqcap_{\beta \in a \circ y} \mathcal{H}(\beta) \supseteq \mathcal{H}(a)$. Thus $\sqcap_{\alpha \in (x \circ a) * y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(a)$. Therefore $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S. □

The converse of above Theorem is not true in general. We can illustrate it by the following example.

Example 5.2. Let $(S, \circ, \leq)$ be an ordered semihypergroup, where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a\}$

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b), (a,c), (a,d), (d,b), (d,c)\}.$$

Let $\mathcal{H}$ be a hesitant fuzzy set of S defined as follows

$$\mathcal{H} : S \longrightarrow \mathcal{P}([0,1]), \ x \longrightarrow \begin{cases} (0, 0.9] & \text{if } x = a \\ [0.2, 0.7] & \text{if } x = b \\ [0.3, 0.5] & \text{if } x = c \\ [0.2, 0.8] & \text{if } x = d \end{cases}$$

Then $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S. This is not a hesitant fuzzy left hyperideal of S as $\sqcap_{\alpha \in c \circ d = \{a,b\}} \mathcal{H}(\alpha) = \mathcal{H}(a) \cap \mathcal{H}(b) = [0.2, 0.7] \not\supseteq [0.2, 0.8] = \mathcal{H}(d)$.

Theorem 5.4. *Let $(S, \circ, \leq)$ be a regular ordered semihypergroup and $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S. Then $\mathcal{H}$ is a hesitant fuzzy hyperideal of S.*

Proof. Let $x, y \in S$. Since $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S, then $\sqcap_{\alpha \in x \circ y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(x)$. Indeed, since S is regular and $x \in S$, then there exists $p \in S$ such that $x \leq (x \circ p) * x$. Then we have $x \circ y \leq ((x \circ p) * x) * y = (x \circ p) * (x \circ y)$. So there exist $\alpha \in x \circ y$, $v \in x \circ p$ and $\beta \in (v \circ x) * y$ such that $\alpha \leq \beta$. So $\mathcal{H}(\alpha) \supseteq \mathcal{H}(\beta)$. Since $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S, then we have $\mathcal{H}(\alpha) \supseteq \mathcal{H}(\beta) \supseteq \sqcap_{\beta \in (v \circ x) * y} \mathcal{H}(\beta) \supseteq \mathcal{H}(x)$. Thus

$$\sqcap_{\alpha \in x \circ y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(x).$$

Therefore $\mathcal{H}$ is a hesitant fuzzy right hyperideal of S. In a similar way we prove that $\mathcal{H}$ is a hesitant left hyperideal of S. □

By Theorems 5.3 and 5.4 we have the following:

Theorem 5.5. *In regular ordered semihypergroups the concepts of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals coincide.*

Theorem 5.6. *Let $(S, \circ, \leq)$ be an intra-regular ordered semihypergroup and $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S. Then $\mathcal{H}$ is a hesitant fuzzy hyperideal of S.*

Proof. Let $a, b \in S$. Then $\sqcap_{u \in a \circ b} \mathcal{H}(u) \supseteq \mathcal{H}(a)$. Indeed, since S is intra-regular and $a \in S$, then there exist $x, y \in S$ such that $a \leq ((x \circ a) * a) * y$. Then $a \circ b \leq (((x \circ a) * a) * y) * b = (x \circ a) * ((a \circ y) * b)$. So there exist $u \in a \circ b$, $v \in (a \circ y) * b$ and $\alpha \in (x \circ a) * v$ such that $u \leq \alpha$. So $\mathcal{H}(u) \supseteq \mathcal{H}(\alpha)$. Since $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S, then we have $\mathcal{H}(u) \supseteq \mathcal{H}(\alpha) \supseteq \sqcap_{\alpha \in (x \circ a) * v} \mathcal{H}(\alpha) \supseteq \mathcal{H}(a)$. Thus

$$\sqcap_{u \in a \circ b} \mathcal{H}(u) \supseteq \mathcal{H}(a).$$

Hence $\mathcal{H}$ is a hesitant fuzzy right hyperideal of S. Similarly we can prove that $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S. Therefore $\mathcal{H}$ is a hesitant fuzzy hyperideal of S. □

By Theorems 5.3 and 5.6 we have the following:

Theorem 5.7. *In intra-regular ordered semihypergroups the concepts of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals coincide.*

6. Characterizations of hesitant fuzzy simple ordered semihypergroups in terms of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals

In this section, we introduce the concept of hesitant fuzzy simple ordered semihypergroups and characterize this type of ordered semihypergroups in terms of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals.

Definition 6.1. (see [16]). An ordered semihypergroup $(S, \circ, \leq)$ is called simple if it has no a proper hyperideal, that is for any hyperideal $A \neq \emptyset$ of S we have $A = S$.

Lemma 6.1. *(see [16]). An ordered semihypergroup $(S, \circ, \leq)$ is a simple ordered semihypergroup if and only if for every $a \in S$, $((S \circ a) * S] = S$.*

Definition 6.2. An ordered semihypergroup $(S, \circ, \leq)$ is called hesitant fuzzy simple if for any hesitant fuzzy hyperideal $\mathcal{H}$ of S, we have $\mathcal{H}(a) \supseteq \mathcal{H}(b)$ for all $a, b \in S$.

Theorem 6.2. *Let be $(S, \circ, \leq)$ an ordered semihypergroup. Then S is hesitant fuzzy simple if and only if for any hesitant fuzzy hyperideal $\mathcal{H}$ of S, we have $U(\mathcal{H}, \delta) = S$ for all $\delta \in \mathcal{P}([0,1])$ if $U(\mathcal{H}, \delta) \neq \emptyset$.*

Proof. Suppose that S is a hesitant fuzzy simple ordered semihypergroup and $\mathcal{H}$ is a hesitant fuzzy hyperideal of S. Let $\delta \in \mathcal{P}([0,1])$ be such that $U(\mathcal{H}, \delta) \neq \emptyset$. We need to prove that $x \in U(\mathcal{H}, \delta)$ for all $x \in S$. Since $U(\mathcal{H}, \delta) \neq \emptyset$, then we can suppose that there exits $y \in U(\mathcal{H}, \delta)$ i.e., $\mathcal{H}(y) \supseteq \delta$. Hence $\mathcal{H}(x) \supseteq \mathcal{H}(y) \supseteq \delta$, we can conclude that $\mathcal{H}(x) \supseteq \delta$, which implies that $x \in U(\mathcal{H}, \delta)$.

Conversely, for any hesitant fuzzy hyperideal $\mathcal{H}$ of S, suppose that $U(\mathcal{H}, \delta) = S$ for all $\delta \in \mathcal{P}([0,1])$ if $U(\mathcal{H}, \delta) \neq \emptyset$. We claim that $\mathcal{H}(a) \supseteq \mathcal{H}(b)$ for all $a, b \in S$. If there exist $x, y \in S$ such that $\mathcal{H}(x) \subset \mathcal{H}(y)$. Then we have $\mathcal{H}(x) \subset \delta \subseteq \mathcal{H}(y)$ for some $\delta \in \mathcal{P}([0,1])$. Thus $\mathcal{H}(x) \subset \delta$, i.e., $x \notin U(\mathcal{H}, \delta) = S$, which is a contradiction. Therefore $\mathcal{H}(a) \supseteq \mathcal{H}(b)$ holds for all $a, b \in S$. Thus S is hesitant fuzzy simple. □

Let $(S, \circ, \leq)$ be an ordered semihypergroup and $a \in S$, and $\mathcal{H}$ be a hesitant fuzzy of S, we denote by I_a the subset of S defines as follows:

$$I_a = \{b \in S \mid \mathcal{H}(b) \supseteq \mathcal{H}(a)\}.$$

Clearly $I_a \neq \emptyset$, since $a \in I_a$.

Theorem 6.3. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $\mathcal{H}$ is a hesitant fuzzy left hyperideals of S. Then the set I_a is a left hyperideal of S for every $a \in S$.*

Proof. Suppose that $\mathcal{H}$ is a hesitant fuzzy left hyperideals of S. Let $b \in I_a$ and $s \in S$. Then $s \circ b \subseteq I_a$. Indeed, since $\mathcal{H}$ is a hesitant fuzzy left hyperideal of S and $b, s \in S$, then we have $\sqcap_{\alpha \in s \circ b} \mathcal{H}(\alpha) \supseteq \mathcal{H}(b)$. Since $b \in I_a$, then we have $\mathcal{H}(b) \supseteq \mathcal{H}(a)$. Thus $\mathcal{H}(\alpha) \supseteq \sqcap_{\alpha \in s \circ b} \mathcal{H}(\alpha) \supseteq \mathcal{H}(b) \supseteq \mathcal{H}(a)$. Thus $\alpha \in I_a$ and hence $s \circ b \subseteq I_a$. Let $b \in I_a$ and $S \ni s \leq b$. Then $s \in I_a$. Indeed, since $\mathcal{H}$ is a hesitant fuzzy left hyperideals of S, $b, s \in S$ and $s \leq b$, then we have $\mathcal{H}(s) \supseteq \mathcal{H}(b)$. Since $b \in I_a$, then we have $\mathcal{H}(b) \supseteq \mathcal{H}(a)$. Then $\mathcal{H}(s) \supseteq \mathcal{H}(a)$, so $s \in I_a$. □

In a similar way we prove the following:

Theorem 6.4. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $\mathcal{H}$ is a hesitant fuzzy right hyperideals of S. Then the set I_a is a right hyperideal of S for every $a \in S$.*

By Theorems 6.2 and 6.3 we have the following:

Theorem 6.5. *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $\mathcal{H}$ is a hesitant fuzzy hyperideals of S. Then the set I_a is a hyperideal of S for every $a \in S$.*

Theorem 6.6. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then S is simple if and only if it is hesitant fuzzy simple.*

Proof. Assume that S is a simple ordered semihypergroup. Let $\mathcal{H}$ is a hesitant fuzzy hyperideal of S and $a, b \in S$. By Theorem 6.4, we obtain I_a is a hyperideal of S. Since S is simple, then $I_a = S$. Therefore $b \in I_a$, that is $\mathcal{H}(b) \supseteq \mathcal{H}(a)$. Therefore S is hesitant fuzzy simple.

Conversely, suppose that S is hesitant fuzzy simple. Let I be a hyperideal of S. By Corollary 4.1, we obtain the characteristic hesitant fuzzy function $\mathcal{H}_{\mathcal{I}}$ is a hesitant fuzzy hyperideal of S. We claim that $I = S$. To prove our claim, let $x \in S$. Since S is hesitant fuzzy simple, then $\mathcal{H}_{\mathcal{I}}(x) \supseteq \mathcal{H}_{\mathcal{I}}(y)$ for all $y \in S$. Since $I \neq \emptyset$, let $a \in I$. Then $\mathcal{H}_{\mathcal{I}}(x) \supseteq \mathcal{H}_{\mathcal{I}}(a) = [0, 1]$. We conclude that $\mathcal{H}_{\mathcal{I}}(x) = [0, 1]$, i.e., $x \in I$. Thus we have shown that $S \subseteq I$, and so, $S = I$. Hence S is simple. □

Theorem 6.7. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then S is a simple if and only if for every hesitant fuzzy interior hyperideal $\mathcal{H}$ of S, we have $\mathcal{H}(a) \supseteq \mathcal{H}(b)$ for all $a, b \in S$.*

Proof. Suppose that S is a simple ordered semihypergroup. Let $\mathcal{H}$ be a hesitant fuzzy interior hyperideal of S, and $a, b \in S$. By Lemma 6.1, we have $S = ((S * b) * S]$. Thus by $a \in S$, we have $a \in ((S * b) * S]$. Then there exist $x, y \in S$ such that $a \leq (x \circ b) * y$. Then $a \leq \alpha$ for some $\alpha \in (x \circ b) * y$. Since $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S, then we have $\mathcal{H}(a) \supseteq \mathcal{H}(\alpha)$. Also since $\sqcap_{\alpha \in (x \circ b) * y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(b)$, then $\mathcal{H}(a) \supseteq \mathcal{H}(\alpha) \supseteq \sqcap_{\alpha \in (x \circ b) * y} \mathcal{H}(\alpha) \supseteq \mathcal{H}(b)$.

Conversely, assume that for every hesitant fuzzy interior hyperideal $\mathcal{H}$ of S , we have $\mathcal{H}(a) \supseteq \mathcal{H}(b)$ for all $a, b \in S$. Let $\mathcal{H}$ be any hesitant fuzzy hyperideal of S. Then by Theorem 5.3, $\mathcal{H}$ is a hesitant fuzzy interior hyperideal of S. Hence S is hesitant fuzzy simple by Definition 6.1. It thus follows from Theorem 6.5 that S is a simple ordered semihypergroup. □

As a consequence of Lemma 6.1, Theorem 6.5, and Theorem 6.6, we provide some characterizations of a simple ordered semihypergroup in the following theorem.

Theorem 6.8. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then the following statements are equivalent:*

(1) *S is a simple ordered semihypergroup.*

(2) *$S = ((S * a) * S]$ for every $a \in S$.*

(3) *S is hesitant fuzzy simple.*

(4) *For every hesitant fuzzy interior hyperideal of S, we have $\mathcal{H}(a) \supseteq \mathcal{H}(b)$ for all $a, b \in S$.*

7. Conclusion

In this paper, we introduced and studied the notions of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals of ordered semihypergroups. Moreover, we introduced the notion of hesitant fuzzy simple ordered semihypergroup. Some characterizations of hesitant fuzzy simple ordered semihypergroups by means of hesitant fuzzy hyperideals and hesitant fuzzy interior hyperideals were provided. This study is just at the begining and it can be extended in many directions:

(1) To do some further work on the properties of hesitant fuzzy hyperideals, which may be useful to characterize the structure of ordered semihypergroups.

(2) To apply the hesitant fuzzy set theory of ordered semihypergroups to some applied fields, such as decision making, data analysis and forecasting and so on.

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