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Some New Results on Stability and Input to State Stability of Fuzzy Differential Equations

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Abstract. In this paper, we introduce a new notion of stability, namely generalized practical $h-$stability. Both generalized practical $h-$stability and generalized input-to-state practical $h-$stability are considered. Under Lyapunov techniques, some sufficient conditions are given which guarantee generalized practical $h-$stability of fuzzy differential equations. The generalized practical $h-$stability analysis is also accomplished with the help of the scalar $h-$stable functions.

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1. Introduction

In the mathematical modeling theory of real world phenomena, we essentially encounter two inconveniences. The first is due to the excessive complexity of the model. As for the second is du to the indeterminacy caused by our inability to differentiate events in a real situation exactly, and therefore to define instrumental notions in precise form. The development of new fields such as robotics, general systems theory, language theory and artificial intelligence, force us to engage in the specification of imprecise notions. One of the tools which makes it possible to describe vague notions and manipulations with them is the modified set theory known as fuzzy set theory, which is initiated in 1965 by L. A. Zadeh. In this context, we recall that the theory of fuzzy sets is simple and natural. A fuzzy set is simply a function of a set in a lattice or as a special case, into the interval $[0, 1]$. Using this theory, mathematicians are able to model the meaning of vague notions and also certain types of human reasoning. Since the 1970s, fuzzy set theory and its applications have been developed very intensively and there are several papers and books that deal with these aspects. In this context, we recall that fuzzy differential equations are very powerful tools for modeling uncertainties and for processing vague or subjective information in many mathematical models. In addition they have been applied to a huge variety of real-world problems, for example, the engineering applications [8], quantum optics and gravity [7] , medicine [1] and golden mean [3].

In the recent years, the theory of practical stability has been developed very intensively and attracts much attention. The first who proposed the concept of practical stability are LaSalle and Lefschetz in 1961 [16]. In this context, Lakshmikantham et al. presented a systematic study of practical stability in 1990. Since 1987, many authors

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have initiated the study of fuzzy differential equations [3, 6, 11, 12, 14, 15, 23, 26, 27]. In [23], sufficient conditions, in terms of Lyapunov-Like functions are given to guarantee the practical stability, the boundedness and Lagrange stability of fuzzy differential systems which unifies the Lyapunov's second method. Moreover, in [26] practical stability of fuzzy differential equations with the second type of Hukuhara derivative are considered. The practical stability of a class of uncertain T-S fuzzy systems is also investigated in [27]. Sufficient condition for practical stability of dynamic systems is presented and a controller design method for uncertain T-S fuzzy systems is derived.

Recently, the problem of practical $h-$stability [9], which in turn, is a generalization of the concept of $h-$stability [18], has been thoroughly investigated. This notion of stability is introduced with the intention of obtaining results about practical stability for a weakly practically stable system under some perturbations. In addition, it generalizes several types of known stabilities, namely practical stability, practical Lipschitz stability, practical exponential stability and practical asymptotic stability (see [9]).

On the other hand, we recall that the problems of practical asymptotic stability and input to state practical stability of nonlinear time varying systems attracted the attention of several well-known and famous authors [4, 5, 10, 17, 19, 20, 21], in particular Andrew R. Teel and Laurent Praly [22]. In the work [22] an interesting fact about functions of type $\mathcal{KL}$ is proved. More precisely, they proved a lemma involving $\mathcal{KL}$ functions which does not seem to have been noticed in the literature, and which is of independent interest.

Motivated by the existing literature on practical asymptotic stability, practical $h-$stability and input-to-state practical stability, the main contribution of the paper is to introduce and study a new notion of practical stability called generalized practical $h-$stability. That is, we will extend the study of practical asymptotic stability and input-to-state practical stability to a variety of reasonable systems when stabilities weaker than those given by practical asymptotic stability and input-to-state practical stability. For this purpose, we use Lyapunov theory to obtain necessary conditions to ensure the generalized practical $h-$stability and generalized input-to state practical $h-$stability of nonlinear fuzzy systems with control. Another approach to study generalized practical $h-$stability analysis is accomplished with the help of the scalar $h-$stable functions which generalizes the notion of scalar stable functions introduced by [25]. The remainder of this paper is organized as follows: In section 2, basic definitions and some preliminary results about practical asymptotic stability and input-to-state practical stability are presented. In subsection 3.1 of section 3, some sufficient conditions are given, to prove the main theorem about the generalized practical $h-$stability of nonlinear fuzzy system with $u \equiv 0$. However, in subsection 3.2 of section 3 we establishes the problems of generalized input-to-state practical $h-$stability analysis and generalized integral input-to-state practical $h-$stability for nonlinear fuzzy system with control $u \in \mathcal{L}^n_\infty(\mathbf{R}_+)$ not necessarily zero. Finally, in section 4 the conclusion is given.

2. Definitions, notations and hypotheses

Let $x \in \mathbf{R}^n$ and A a nonempty subset of the space $\mathbf{R}^n$. The distance $d(x, A)$ from x to A is defined by the following formula

$$d(x, A) = \inf\{\|x - a\|, a \in A\}$$

with $\|.\|$ denotes a norm in $\mathbf{R}^n$. Now, for two nonempty subsets A and B of $\mathbf{R}^n$, we define the Hausdorff separation of B from A given the formula

$$d_H[A, B] = \sup\{d(a, B), a \in A\}.$$

In general, $d_H[A, B] \neq d_H[B, A]$.

Throughout this paper $K_c(\mathbf{R}^n)$ denote the family of all nonempty, compact, convex subsets of $\mathbf{R}^n$. Let $D[A, B]$ be the Hausdorff distance between the sets $A, B \in K_c(\mathbf{R}^n)$

$$D[A, B] = \max\{d_H[A, B], d_H[B, A]\}.$$

Let $I = [t_0, t_0+a]$, $t_0 \in \mathbf{R}_+$ and $a > 0$ and E^n be the set of all functions $\tilde{u} : \mathbf{R}^n \to [0.1]$ such that $\tilde{u}$ satisfies the following properties

i): $\tilde{u}$ is normal, that is, there exists an $x_0 \in \mathbf{R}^n$ such that $\tilde{u}(x_0) = 1$;

ii): $\tilde{u}$ is fuzzy convex, that is, for $x, y \in \mathbf{R}^n$ and $0 \leq \lambda \leq 1$;

$$\tilde{u}(\lambda x + (1 - \lambda)y \geq \min\{\tilde{u}(x), \tilde{u}(y)\};$$

iii): u is upper semicontinuous;

iv): $[\tilde{u}]^0 = \overline{\{x \in \mathbf{R}^n; \tilde{u}(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, we denote $[\tilde{u}]^\alpha = \{x \in \mathbf{R}^n; \tilde{u}(x) \geq \alpha\}$. Taking into account the conditions i)-iv), the $\alpha-$sets $[\tilde{u}]^\alpha \in K_c(\mathbf{R}^n)$ for $\alpha \in [0, 1]$. For addition and scalar multiplication in fuzzy set space E^n,we have

$$[\tilde{u} + \tilde{v}]^\alpha = [\tilde{u}]^\alpha + [\tilde{v}]^\alpha \text{ and } [\lambda\tilde{u}]^\alpha = \lambda[\tilde{u}]^\alpha,\ \tilde{u}, \tilde{v} \in E^n,\ \lambda \in \mathbf{R} \text{ and } \alpha \in [0, 1].$$

According to the definition of the Hausdorff distance between the sets $A, B \in K_c(\mathbf{R}^n)$, we define the metric function $D_0 : E^n \times E^n \to \mathbf{R}_+$ by

$$D_0[\tilde{u}, \tilde{v}] = \sup_{0 \leq \alpha \leq 1} D[[\tilde{u}]^\alpha, [\tilde{v}]^\alpha].$$

It follows that (E^n, D_0) is a complete metric space. We recall that the metric D_0 satisfies the following properties

(P1): $D_0[\tilde{u} + \tilde{w}, \tilde{v} + \tilde{w}] = D_0[\tilde{u}, \tilde{v}]$ and $D_0[\tilde{u}, \tilde{v}] = D_0[\tilde{v}, \tilde{u}]$, for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^n$.

(P2): $D_0[\lambda\tilde{u}, \lambda\tilde{v}] = |\lambda| D_0[\tilde{u}, \tilde{v}]$, for all $\tilde{u}, \tilde{v} \in E^n$ and $\lambda \in \mathbf{R}$.

(P3): $D_0[\tilde{u}, \tilde{v}] \leq D_0[\tilde{u}, \tilde{w}] + D_0[\tilde{w}, \tilde{v}]$, for all $\tilde{u}, \tilde{v}, \tilde{w} \in E^n$.

Let $\tilde{u}, \tilde{v}, \tilde{w} \in E^n$. If the relation $\tilde{u} = \tilde{v} + \tilde{w}$ is satisfies, then $\tilde{w}$ is called the Hukuhara difference of $\tilde{u}$ and $\tilde{v}$ and is denoted by $\tilde{u} \ominus \tilde{v}$. Generally, we have $\tilde{u} \ominus \tilde{v} \neq \tilde{u} + (-1)\tilde{v}$

Definition 2.1. A mapping $F : I \to E^n$ is differentiable at $t \in I$, if there exists a $D_H F(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t + h) \ominus F(t)}{h} \text{ and } \lim_{h \to 0^+} \frac{F(t) \ominus F(t - h)}{h} \tag{1}$$

exist and are equal to $D_H F(t)$. The above limits are taken in the metric space (E^n, D_0) and at the boundary points we consider only the one-sided derivatives.

Consider the following non-linear time-varying fuzzy system

$$D_H x(t) = f(t, x(t), u) \tag{2}$$

where $f \in \mathcal{C}(\mathbf{R}_+ \times E^n \times E^p, E^n)$, $x \in E^n$ the state and $u \in E^p$ is the control. Along this paper, we define the function $\hat{0} \in E^n$ as follows

$$\hat{0}(s) = \begin{cases} 1, & \text{if } s = 0_{\mathbf{R}^n} \\ 0, & \text{if } s \neq 0_{\mathbf{R}^n} \end{cases}$$

Notation. Throughout this paper, we use the following notation

(1) The solution of system (2) passing through the point $(t_0, x_0) \in \mathbf{R}_+ \times E^n$, is denoted by $x(t, t_0, x_0)$ or simply $x(t)$ which satisfies $x(t_0, t_0, x_0) = x_0$. We suppose that for all initial condition (t_0, x_0) there exists a unique solution $x(t, t_0, x_0)$ of (2) and defined for all $t \geq t_0$.

(2) If $\xi : \mathbf{R}_+ \to E^n$, we use

$$\|\xi\|_{[t_0,t]} = \sup_{\tau \in [t_0,t]} D_0[\xi(\tau), \hat{0}]$$

to denote the truncation of the norm of ξ at t.

(3) we denote

$$\mathcal{L}^n_\infty(\mathbf{R}_+) = \Big\{\xi : \mathbf{R}_+ \to E^n; \sup_{\tau \in \mathbf{R}_+} D_0[\xi(\tau), \hat{0}] < \infty\Big\}.$$

(4) $\mathcal{PC}[\mathbf{R}_+, \mathbf{R}]$ the set of all piecewise continuous function from $\mathbf{R}_+$ to $\mathbf{R}$.

(5) $\mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$ the set of all continuous functions from $\mathbf{R}_+ \times E^n$ to $\mathbf{R}_+$.

(6) $\mathcal{HCB}_+ = \Big\{h : \mathbf{R}_+ \longrightarrow \mathbf{R}^*_+; h \text{ is a continuous, bounded and positive function}\Big\}$

(7) For $\delta \geq 0$, $\mathcal{B}_\delta$ denotes the closed ball of E^n centered at zero, i.e.,

$$\mathcal{B}_\delta := \Big\{\tilde{u} \in E^n : D_0[\tilde{u}, \hat{0}] \leq \delta\Big\}.$$

For readers convenience, some definitions are given in the following.

Definition 2.2. Consider a continuous function $V : \mathbf{R}_+ \times E^n \longrightarrow \mathbf{R}_+$. The function V is said to be globally Lipschitzian with respect to variable x if there exists $L > 0$ such that

$$|V(t, x) - V(t, y)| \leq L D_0[x, y] \text{ for all } t \in \mathbf{R}_+ \text{ and } x, y \in E^n.$$

Definition 2.3. Let $V \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$ be a Lyapunov-like function.

(1) The fuzzy Dini derivative $D^+V_{(2)}(t, x)$ of V is defined as follows

$$D^+V_{(2)}(t, x) = \limsup_{\delta \longrightarrow 0^+} \frac{1}{\delta}\big[V(t + \delta, x + \delta f(t, x, u))) - V(t, x)\big],\ (t, x) \in \mathbf{R}_+ \times E^n.$$

If $u \equiv 0$, the fuzzy Dini derivative $D^+V_{(2)}(t, x)$ is denoted $D^+V_{(2)}(t, x)\Big|_{u \equiv 0}$.

(2) If $x(t)$ is a solution of (2), we define the upper right-hand derivative of $V(t, x(t))$ we denote by $D^+V(t, x(t))$, i.e.,

$$D^+V_{(2)}(t, x(t)) = \limsup_{\delta \longrightarrow 0^+} \frac{1}{\delta}\big[V(t + \delta, x(t + \delta)) - V(t, x(t))\big]. \tag{3}$$

If $x(t)$ is a solution of (2) with $u \equiv 0$, the upper right-hand derivative of $V(t, x(t))$ is denote by

$$D^+V_{(2)}(t, x(t))\Big|_{u \equiv 0}.$$

Note that if $V(t,x)$ is a Lipschitzian function with respect to x, then we have

$$D^+V(t,x(t)) \le D^+V_{(2)}(t,x). \tag{4}$$

Definition 2.4. [13] A scalar continuous function $\alpha : [0,a) \to \mathbf{R}_+$ is said to be of class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it defined for all $r \ge 0$ and $\alpha(r) \to +\infty$ as $r \to +\infty$.

Definition 2.5. [13] A scalar continuous function $\beta : [0,a) \times \mathbf{R}_+ \to \mathbf{R}_+$ is said to belong to class $\mathcal{KL}$ ($\beta \in \mathcal{KL}$) if, for each $s \ge 0$ the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to +\infty$.

We begin by recalling the definition of the notion of practical asymptotic stability of the fuzzy system (2).

Definition 2.6. [5] The non-linear time-varying fuzzy system (2) is said to be

(1) uniformly practically asymptotically stable ($UPAS$) if there exist $\beta \in \mathcal{K}_\infty$, a positive constant c, independent of t_0 and $\delta \ge 0$, such that with the control $u = 0$,

$$D_0[x(t),\hat{0}] \le \beta\Big(D_0[x(t_0),\hat{0}], t-t_0\Big) + \delta, \; \forall\, t \ge t_0, \; \forall\, D_0[x(t_0),\hat{0}] \le c. \tag{5}$$

If $\delta = 0$, the system (2) is said uniformly asymptotically stable (UAS)(see [13]).

(2) globally uniformly practically asymptotically stable ($GUPAS$) if and only if inequality (5) is satisfied for any initial state $x(t_0) \in E^n$. If $\delta = 0$, the system (2) is said globally uniformly asymptotically stable ($GUAS$) (see [13]).

Now, let's recall the notion of input-to-state practical stability of the fuzzy system (2).

Definition 2.7. [17] The non-linear time-varying fuzzy system (2) is said to be

(1) input-to-state practically stable ($ISPS$) if there exist $\alpha, \sigma, \gamma \in \mathcal{K}_\infty$ such that, for any bounded input $u \in \mathcal{L}^n_\infty(\mathbf{R}_+)$

$$D_0[x(t),\hat{0}] \le \beta\Big(D_0[x(t_0),\hat{0}], t-t_0\Big) + \gamma(\|u\|_{[t_0,t]}) + \delta. \tag{6}$$

If $\delta = 0$, the system (2) is said input-to-state stable (ISS) (see [20]).

(2) integral input-to-state practically stable ($iISPS$) if there exist $\alpha, \sigma, \gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that,

$$D_0[x(t),\hat{0}] \le \beta\Big(D_0[x(t_0),\hat{0}], t-t_0\Big) + \gamma_1\Big(\int_{t_0}^{t} \gamma_2(D_0[u(s),\hat{0}])ds\Big) + \delta. \tag{7}$$

If $\delta = 0$, the system (2) is said integral input-to-state stable ($iISS$) (see [21]).

In this article, the main objective is to generalize the notions of practical asymptotic stability and input to state practical stability. More precisely, we will introduce other types of stabilities in order to obtain results on the stability of a weakly PAS system and a weakly $ISPS$ system. The origin of this generalization is due to the following lemma.

Lemma 2.1. [22] *For each class–$\mathcal{KL}$ function β and each number $\lambda > 0$, there exist functions $\tilde{\alpha}_1 \in \mathcal{K}_\infty$ and $\tilde{\alpha}_2 \in \mathcal{K}_\infty$ such that $\tilde{\alpha}_1$ is locally Lipschitz and*

$$\tilde{\alpha}_1(\beta(r,s)) \le \tilde{\alpha}_2(r)e^{-\lambda s} \; \forall\, (r,s) \in \mathbf{R}_+ \times \mathbf{R}_+. \tag{8}$$

The result given by Lemma 2.1 shows that any class$-\mathcal{KL}$ function $\beta(r,s)$ can always be upper bounded by a continuous class$-\mathcal{KL}$ function of the form $\tilde{\alpha}_1^{-1}\Big(\tilde{\alpha}_2(r)e^{-\lambda s}\Big)$ even if we assume minimal continuity properties for the function β.

Remark 2.1. By Lemma 2.1, it should be noted that for each class$-\mathcal{KL}$ function β and each number $\lambda > 0$, there exist functions $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ and a continuous, bounded function $h_0 : \mathbf{R}_+ \longrightarrow \mathbf{R}_+^*$ such that

$$\beta\Big(D_0[x(t_0),\hat{0}], t-t_0\Big) \leq \tilde{\alpha}_1^{-1}\Big(\tilde{\alpha}_2(D_0[x(t_0),\hat{0}])h_0(t)h_0^{-1}(t_0)\Big),\ t \geq t_0 \geq 0,\ x(t_0) \in E^n \tag{9}$$

where $h_0(s) = e^{-\lambda s},\ s \in \mathbf{R}_+$.

On the basis of the last inequality (9), we introduce a new notion of stability which generalizes those cited above of this paper. For this fact let consider a continuous, bounded and positive function $h \in \mathcal{HCB}_+$. We beginning by defining the notion of generalized practical $h-$stability.

Definition 2.8. The non-linear time-varying fuzzy system (2) is said to be

(1) generalized practically $h-$stable ($gPhS$) if there exist $\alpha, \sigma, \gamma \in \mathcal{K}_\infty$ such that, with the control $u = 0$, for all $t \geq t_0 \geq 0$

$$D_0[x(t),\hat{0}] \leq \alpha\Big(K(t_0)\sigma(D_0[x(t_0),\hat{0}])h(t)h(t_0)^{-1}\Big) + \delta$$

where $K(t_0) \geq 1$. If $\delta = 0$, the system (2) is said generalized $h-$stable (ghS). If there exist a constant $M > 0$ such that $\alpha(r) \leq Mr$ and $\sigma(r) \leq Mr$, the system (2) is said practically $h-$stable (PhS)(see [9]).

(2) generalized practically uniformly $h-$stable (gPUhS) if there exist $\alpha, \sigma, \gamma \in \mathcal{K}_\infty$ such that, with the control $u = 0$,

$$D_0[x(t),\hat{0}] \leq \alpha\Big(\sigma(D_0[x(t_0),\hat{0}])h(t)h(t_0)^{-1}\Big) + \delta.$$

If $\delta = 0$, the system (2) is said generalized uniformly $h-$stable. If there exist a constant $M > 0$ such that $\alpha(r) \leq Mr$ and $\sigma(r) \leq Mr$, the system (2) is said uniformly practically $h-$stable (see [9]).

Now, we define the notion of generalized input-to-state practical $h-$stability.

Definition 2.9. The non-linear time-varying fuzzy system (2) is said to be

(1) generalized input-to-state practically $h-$stable (gISPhS) if there exist $\alpha, \sigma, \gamma \in \mathcal{K}_\infty$ such that, for any $u \in \mathcal{L}_\infty^n(\mathbf{R}_+)$

$$D_0[x(t),\hat{0}] \leq \alpha\Big(K(t_0)\sigma(D_0[x(t_0),\hat{0}])h(t)h(t_0)^{-1}\Big) + \delta + \gamma(\|u\|_{[t_0,t]})$$

where $K(t_0) \geq 1$. If $\delta = 0$, the system (2) is said generalized input-to-state $h-$stable (gIShS). If there exist a constant $M > 0$ such that $\alpha(r) \leq Mr$ and $\sigma(r) \leq Mr$, the system (2) is said input-to-state practically $h-$stable (ISPhS).

(2) generalized input-to-state uniformly practically $h-$stable (gISUPhS) if there exist $\alpha, \sigma, \gamma \in \mathcal{K}_\infty$ such that, for any $u \in \mathcal{L}_\infty^n(\mathbf{R}_+)$

$$D_0[x(t),\hat{0}] \leq \alpha\Big(\sigma(D_0[x(t_0),\hat{0}])h(t)h(t_0)^{-1}\Big) + \delta + \gamma(\|u\|_{[t_0,t]}).$$

If $\delta = 0$, the system (2) is said generalized input-to-state uniformly $h-$stable (gISUhS). If there exist a constant $M > 0$ such that $\alpha(r) \leq Mr$ and $\sigma(r) \leq Mr$, the system (2) is said input-to-state uniformly practically $h-$stable (ISUPhS).

Definition 2.10. The non-linear time-varying fuzzy system (2) is said to be

(1) generalized integral input-to-state practically $h-$stable (giISPhS) if there exist $\alpha, \sigma, \gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that, for any $u \in \mathcal{L}^n_\infty(\mathbf{R}_+)$

$$D_0[x(t), \hat{0}] \leq \alpha\Big(K(t_0)\sigma(D_0[x(t_0), \hat{0}])h(t)h(t_0)^{-1}\Big) + \delta + \gamma_1\Big(\int_{t_0}^{t} \gamma_2(D_0[u(s), \hat{0}])ds\Big)$$

where $K(t_0) \geq 1$. If $\delta = 0$, the system (2) is said generalized integral input-to-state $h-$stable (giIShS). If there exist a constant $M > 0$ such that $\alpha(r) \leq Mr$ and $\sigma(r) \leq Mr$, the system (2) is said integral input-to-state practically $h-$stable (iISPhS).

(2) generalized integral input-to-state uniformly practically $h-$stable (giISUPhS) if there exist $\alpha, \sigma, \gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that, for any $u \in \mathcal{L}^n_\infty(\mathbf{R}_+)$

$$D_0[x(t), \hat{0}] \leq \alpha\Big(\sigma(D_0[x(t_0), \hat{0}])h(t)h(t_0)^{-1}\Big) + \delta + \gamma_1\Big(\int_{t_0}^{t} \gamma_2(D_0[u(s), \hat{0}])ds\Big).$$

If $\delta = 0$, the system (2) is said generalized integral input-to-state uniformly $h-$stable (giISUhS). If there exist a constant $M > 0$ such that $\alpha(r) \leq Mr$ and $\sigma(r) \leq Mr$, the system (2) is said integral input-to-state uniformly practically $h-$stable (iISUPhS).

Remark 2.2. Note that, if the system (2) is *PAS* (resp; *ISPS*, *iISPS*) then it is *gPhS* (resp; *gISPhS*, *giISPhS*). However, the converse is false and this leads us to extend the notion of practical asymptotic stability (resp; input-to-state practical stability) to a variety of reasonable systems with stabilities weaker than those given by practical asymptotic stability. Hence, the notion of generalized practical $h-$stability (resp; generalized input-to-state practical $h-$stability) is of significant practical importance but never investigated.

Next, we introduce the concept of $h-$stable functions which generalizes the notion of stable function introduced by [24].

Consider the following scalar linear time-varying (LTV) system:

$$\dot{\omega}(t) = \mu(t)\omega(t) \tag{10}$$

where $\omega : \mathbf{R}_+ \longrightarrow \mathbf{R}$ is the state variable and $\mu \in \mathcal{PC}[\mathbf{R}_+, \mathbf{R}]$. The general solution of system (10) is of the form

$$\omega(t) = \omega(t_0)\Psi(t, t_0),\ t \geq t_0 \geq 0$$

with $\Psi(t, t_0)$ is state transition matrix for system (10) given by

$$\Psi(t, t_0) = \exp\Big(\int_{t_0}^{t} \mu(s)ds\Big),\ t \geq t_0 \geq 0. \tag{11}$$

Definition 2.11. The scalar function μ is said to be

(1) $h-$stable if the scalar (LTV) system (10) is $h-$stable, namely, for all $t_0 \geq 0$ there exist constant $k(t_0) \geq 1$ such that

$$|\omega(t)| \leq k(t_0)|\omega(t_0)|h(t)h^{-1}(t_0),\ t \geq t_0 \geq 0 \tag{12}$$

(2) uniformly $h-$stable if the scalar (LTV) system (10) is uniformly $h-$stable, namely, the constant $k(t_0) \geq 1$ in (12) is independent of t_0.

Remark 2.3. In Definition 2.11, for some special cases of h, the notion of $h-$stable functions coincides with the notion of stable functions. More precisely,

i): if $h(t) = e^{-\lambda t}$ for positive constant λ, then the scalar function μ is exponentially stable (see [24]).

ii): if $h(t)$ is a strictly decreasing function such that $\lim_{t\longrightarrow+\infty} h(t) = 0^+$, then the scalar function μ is asymptotically stable (see [24]).

The following proposition gives a characterization of the $h-$stability of a scalar function $\mu \in \mathcal{PC}[\mathbf{R}_+, \mathbf{R}]$, whose the proof is obvious.

Proposition 2.2. *The scalar function $\mu \in \mathcal{PC}[\mathbf{R}_+, \mathbf{R}]$ is*

i): *$h-$stable if and only if for all $t_0 \geq 0$, there exists $\zeta(t_0) \in \mathbf{R}_+$ such that*

$$\int_{t_0}^{t} \mu(s)ds \leq \ln(h(t)) - \ln(h(t_0)) + \zeta(t_0), \quad t \geq t_0 \geq 0. \tag{13}$$

ii): *uniformly $h-$stable if and only if (13) is satisfied, where $\zeta(t_0) = \zeta$ is independent of t_0.*

Remark 2.4. The $h-$stability characterization given by Proposition 2.2 is a generalization of that given by Lemma 1, [25]

i): if $h(t) = e^{-\lambda t}$ for positive constant λ, then (13) becomes

$$\int_{t_0}^{t} \mu(s)ds \leq -\lambda(t - t_0) + \zeta(t_0), \quad t \geq t_0 \geq 0$$

and we obtain the necessary and sufficient condition of exponential stability of the scalar function μ (see [24], [25]).

ii): if $h(t)$ is a strictly decreasing function such that $\lim_{t\longrightarrow+\infty} h(t) = 0^+$, then from (13) we get

$$\lim_{t\longrightarrow+\infty} \int_{t_0}^{t} \mu(s)ds = -\infty$$

which is equivalent to saying that the scalar function μ is asymptotically stable (see [24], [25]).

The following lemmas will also be required in the investigations of our results of the paper.

Lemma 2.3. [13] *Consider the scalar differential equation*

$$\dot{z} = \varphi(t, z), \; z(t_0) = z_0 \tag{14}$$

where $\varphi(t, z)$ is continuous in t and locally Lipschitz in z, for all $t \geq 0$ and all $z \in J \subset \mathbf{R}$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution $z(t)$, and suppose $z(t) \in J$ for all $t \in [t_0, T)$. Let $v(t)$ be a continuous function whose upper right-hand derivative $D^+v(t)$ satisfies the differential inequality

$$D^+v(t) \leq \varphi(t, v(t)), \; v(t_0) \leq z_0$$

with $v(t) \in J$ *for all* $t \in [t_0, T)$*. Then,* $v(t) \leq z(t)$ *for all* $t \in [t_0, T)$*. In particular, if* $\varphi(t,z) = p(t)z + q(t)$ *where* $p(t)$ *and* $q(t)$ *are two continuous functions, then the function* v *satisfies the inequality*

$$v(t) \leq z(t) = v(t_0)e^{\int_{t_0}^{t} p(s)ds} + \int_{t_0}^{t} e^{\int_s^t p(\tau)d\tau} q(s)ds,\ t \geq t_0.$$

Lemma 2.4. *Let* $a, b \geq 0$*. Then, we have the following inequalities:*
- **a):** *For all* $p \geq 1$, $(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$.
- **b):** *For all* $p \geq 1$, $(a+b)^p \leq 2^{p-1}(a^p + b^p)$.

Lemma 2.5. *If* $\alpha \in \mathcal{K}_\infty$*, then for all* $a, b \geq 0$*, we have*

$$\alpha(a+b) \leq \alpha(2a) + \alpha(2b).$$

3. Main results

In this section, we assume that $h \in \mathcal{HCB}_+$ is a differentiable function on $\mathbf{R}_+$.

3.1. Sufficient conditions for generalized practical $h-$stability. In this section, we give some sufficient conditions to ensure the practical $h-$stability of fuzzy system (2) by using Lyapunov's second method. First, we start by investigating the generalized practical $h-$stability of the system (2) with the input control $u \equiv 0$.

Theorem 3.1. *Suppose there are a Lyapunov-Like function* $V \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$*,* $\delta \in \mathbf{R}_+$ *and two class* $\mathcal{K}_\infty$ *functions* α_1, α_2 *satisfying the following properties:*
- **i):** $\alpha_1(D_0[x, \hat{0}]) \leq V(t,x) \leq \alpha_2(D_0[x,\hat{0}]) + \delta,\ (t,x) \in \mathbf{R}_+ \times E^n$,
- **ii):** $D^+V_{(2)}(t,x(t))\Big|_{u\equiv 0} \leq \frac{h'(t)}{h(t)}\Big(V(t,x(t)) - \delta\Big),\ t \in \mathbf{R}_+$.

Then, the fuzzy system (2) with the control $u = 0$*, is generalized practically uniformly* $h-$*stable. Moreover, if* $\delta = 0$ *the fuzzy system (2) with the control* $u = 0$*, is generalized uniformly* $h-$*stable.*

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of system (2) through $(t_0, x_0) \in \mathbf{R}_+ \times E^n$. By condition ii), we have

$$D^+V_{(2)}(t,x(t))\Big|_{u\equiv 0} \leq \varphi(t, x(t)), t \in \mathbf{R}_+$$

where

$$\varphi(t,z) = \frac{h'(t)}{h(t)}\Big(z - \delta\Big) := p(t)z + q(t) \text{ with } p(t) = \frac{h'(t)}{h(t)}, q(t) = -\delta\frac{h'(t)}{h(t)}.$$

Then, by Lemma 2.3, we obtain

$$\begin{aligned} V(t,x(t)) \leq & V(t_0, x(t_0))e^{\int_0^t \frac{h'(\tau)}{h(\tau)}d\tau} - \delta h(t)h^{-1}(t_0) + \delta \\ =& (V(t_0,x(t_0)) - \delta)h(t)h^{-1}(t_0) + \delta \\ \leq & \alpha_2(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0) + \delta. \end{aligned}$$

By condition i), we get, for all $t \geq t_0$

$$\begin{aligned} \alpha_1(D_0[x(t), \hat{0}]) &\leq V(t, x(t)) \\ &\leq \alpha_2(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0) + \delta. \end{aligned}$$

Consequently, by using Lemma 2.5, the solution $x(t)$ of system (2) satisfies

$$D_0[x(t),\hat{0}] \leq \alpha_1^{-1}\Big(\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)+\delta\Big)$$
$$\leq \alpha_1^{-1}\Big(2\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big)+\alpha_1^{-1}(2\delta).$$

Then, the fuzzy system (2) with the control $u=0$ is generalized uniformly practically $h-$stable. If $\delta=0$, the solution $x(t)$ satisfies

$$D_0[x(t),\hat{0}] \leq \alpha_1^{-1}\Big(2\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big).$$

Then, the fuzzy system (2) with the control $u=0$, is generalized uniformly $h-$stable. □

Remark 3.1. In Theorem 3.1, if

(1) $\alpha_1(r)=c_1r^p$, $\alpha_2(r)=c_2r^p$ where $c_1,c_2,p\in\mathbf{R}_+^*$, then the system (2) is practically $h^{\frac{1}{p}}-$stable (see [9]).
(2) $h(t)=c$, for a positive constant c, then the system (2) is practically stable (see [9], [2]).
(3) $h(t)=e^{-\lambda t}$, for positive constant λ and $\alpha_1(r)=c_1r^p$, $\alpha_2(r)=c_2r^p$ where $c_1,c_2,p\in\mathbf{R}_+^*$, then the system (2) is practically exponentially stable (see [9], [2]).
(4) $h(t)$ is a strictly decreasing function such that $\lim\limits_{t\longrightarrow+\infty}h(t)=0$, then the ball $\mathcal{B}_{\alpha_1^{-1}(2\delta)}$ is uniformly asymptotically stable on $\mathcal{B}_r$. More precisely, the solutions of system (2) converge to the ball $\mathcal{B}_{\alpha_1^{-1}(2\delta)}$ of radius $\alpha_1^{-1}(2\delta)$ (i.e., $\limsup\limits_{t\longrightarrow+\infty}D_0[x(t,t_0,x_0),\hat{0}]\leq\alpha_1^{-1}(2\delta)$, for $x_0\in\mathcal{B}_r$) (see [9]).

With the help of the notion of $h-$stable functions, we obtain the stability result given by the following corollary.

Corollary 3.2. *Suppose there are a Lyapunov-Like function $V\in\mathcal{C}(\mathbf{R}_+\times E^n,\mathbf{R}_+)$, $\delta\in\mathbf{R}_+$, two class $\mathcal{K}_\infty$ functions α_1,α_2 and a scalar function $\mu\in\mathcal{PC}[\mathbf{R}_+,\mathbf{R}]$ satisfying the following properties:*

i): $\alpha_1(D_0[x,\hat{0}])\leq V(t,x)\leq\alpha_2(D_0[x,\hat{0}])+\delta$, $(t,x)\in\mathbf{R}_+\times E^n$,

ii): $D^+V_{(2)}(t,x(t))\Big|_{u\equiv0}\leq\mu(t)\Big(V(t,x(t))-\delta\Big)$, $t\in\mathbf{R}_+$.

(1) *If the scalar function μ is $h-$stable, then the fuzzy system (2) with the control $u=0$, is generalized practically $h-$stable.*
(2) *If the scalar function μ is uniformly $h-$stable, then the fuzzy system (2) with the control $u=0$, is generalized uniformly practically $h-$stable.*

Proof. The proof is the same as that of Theorem 3.1. If we take, $\hbar(t)=e^{\int_0^t\mu(s)ds}$, then condition *ii)* of Corollary 3.2 becomes

$$D^+V_{(2)}(t,x(t))\Big|_{u\equiv0}\leq\frac{\hbar'(t)}{\hbar(t)}\Big(V(t,x(t))-\delta\Big),\ t\in\mathbf{R}_+. \tag{15}$$

Equation (15) is similar to that *ii)* of Theorem 3.1. Then, we deduce directly that we have

$$D_0[x(t),\hat{0}]\leq\alpha_1^{-1}\Big(2\alpha_2(D_0[x(t_0),\hat{0}])e^{\int_{t_0}^t\mu(s)ds}\Big)+\alpha_1^{-1}(2\delta). \tag{16}$$

- If the scalar function μ is $h-$stable, then by (13), we have

$$\hbar(t)\hbar^{-1}(t_0) = e^{\int_{t_0}^{t}\mu(s)ds} \leq h(t)h^{-1}(t_0)e^{\zeta(t_0)}$$

  and by equation (16), we obtain

$$D_0[x(t),\hat{0}] \leq \alpha_1^{-1}\Big(2K(t_0)\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big) + \alpha_1^{-1}(2\delta)$$

  where $K(t_0) = e^{\xi(t_0)}$. Consequently, the fuzzy system (2) with the control $u = 0$, is generalized practically $h-$stable.
- In the same way, if the scalar function μ is uniformly $h-$stable, then through (13) and (16), we obtain

$$D_0[x(t),\hat{0}] \leq \alpha_1^{-1}\Big(2e^{\xi}\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big) + \alpha_1^{-1}(2\delta)$$

  which means that system (2) with the control $u = 0$, is generalized uniformly practically $h-$stable.

□

3.2. Input-to-state practical $h-$stability analysis. In this part, we consider the system (2) with a control $u \in \mathcal{L}_\infty^n(\mathbf{R}_+)$ not necessarily zero. We start by giving sufficient conditions to guarantee the generalized input-to-state uniform practical $h-$stability of the system (2).

Theorem 3.3. *Suppose there are a Lyapunov-Like function $V(t,x) \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$, $\delta \in \mathbf{R}_+$, a class-$\mathcal{K}$ function ρ and two class $\mathcal{K}_\infty$ functions α_1, α_2 satisfying the following properties:*

i): $\alpha_1(D_0[x,\hat{0}]) \leq V(t,x) \leq \alpha_2(D_0[x,\hat{0}]) + \delta$, $(t,x) \in \mathbf{R}_+ \times E^n$,

ii): $D^+V_{(2)}(t,x(t)) \leq \frac{h'(t)}{h(t)}\Big(V(t,x(t)) - \delta\Big)$ *if* $V(t,x(t)) \geq h(t)\rho(D[u(t),\hat{0}]) + \delta$, $t \in \mathbf{R}_+$.

Then, the non-linear time-varying fuzzy system (2) is generalized input-to-state uniformly practically $h-$stable.

Proof. Let consider the Lyapunov-Like function

$$W(t,x) = V(t,x)e^{-\int_0^t \frac{h'(s)}{h(s)}ds} = h(0)h^{-1}(t)V(t,x).$$

The upper right-hand derivative of $W(t,x)$ along the trajectories of system (2) satisfies

$$\begin{aligned} D^+W_{(2)}(t,x(t)) \leq& \Big[D^+V_{(2)}(t,x(t)) - \frac{h'(t)}{h(t)}V(t,x(t))\Big]h(0)h^{-1}(t), \\ \leq& -\delta h(0)\frac{h'(t)}{h^2(t)}. \end{aligned}$$

It follows that

$$D^+W_{(2)}(t,x(t)) \leq \varphi(t, W(t,x(t))) \text{ with } \varphi(t,z) = -\delta h(0)\frac{h'(t)}{h^2(t)}. \tag{17}$$

Now, let consider the following inequality

$$V(\tau,x(\tau)) \geq h(\tau)\rho(D[u(\tau),\hat{0}]) + \delta. \tag{18}$$

Two cases arise :

i): If the inequality (18) is true for all $\tau \in [t_0, t]$, then it follows from (17) and Lemma 2.3, that

$$W(t, x(t)) \leq W(t_0, x(t_0)) - \delta h(0) \int_{t_0}^{t} \frac{h'(s)}{h^2(s)} ds = W(t_0, x(t_0)) + \delta h(0)\Big(\frac{1}{h(t)} - \frac{1}{h(t_0)}\Big).$$

Then,

$$h(0)h^{-1}(t)V(t, x(t)) \leq h(0)h^{-1}(t_0)V(t_0, x(t_0)) + \delta h(0)\Big(\frac{1}{h(t)} - \frac{1}{h(t_0)}\Big).$$

Consequently, we obtain

$$V(t, x(t)) \leq \alpha_2(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0) + \delta,\ t \geq t_0,\ x(t_0) \in E^n.$$

ii): If (18) does not hold true for almost all $\tau \in [t_0, t]$, then the set

$$\Gamma := \Big\{\tau \in [t_0, t], V(\tau, x(\tau)) \leq h(\tau)\rho(D[u(\tau), \hat{0}]) + \delta\Big\}$$

is non-empty. Denote $\tau^\star = \sup \Gamma$.

- If $\tau^\star = t$, it follows that

$$\begin{aligned} V(t, x(t)) &= V(\tau^\star, x(\tau^\star)), \\ &\leq h(\tau^*)\rho(D[u(\tau^\star), \hat{0}]) + \delta, \\ &\leq \sup_{s\in[t_0,t]} \{h(s)\}\rho(\sup_{s\in[t_0,t]} \{D[u(s), \hat{0}]\}) + \delta, \\ &\leq \|h\|_\infty \rho(\|u\|_{[t_0,t]}) + \delta. \end{aligned}$$

- If $\tau^\star < t$, then $V(\tau, x(\tau)) \geq h(\tau)\rho(D[u(\tau), \hat{0}]) + \delta$ for all $\tau \in [\tau^\star, t]$. It follows from condition $ii)$, that

$$D^+V_{(2)}(\tau, x(\tau)) \leq \frac{h'(\tau)}{h(\tau)}\big(V(\tau, x(\tau)) - \delta\big),\ \forall \tau \in [\tau^\star, t],$$

from which it follows that for all $\tau \in [\tau^\star, t]$

$$V(\tau, x(\tau)) \leq V(\tau^*, x(\tau^*))h(\tau)h^{-1}(\tau^*) + \delta - \delta h(\tau)h^{-1}(\tau^*).$$

In particular, if $\tau = t$ we have

$$\begin{aligned} V(t, x(t)) &\leq V(\tau^*, x(\tau^*))h(t)h^{-1}(\tau^*) + \delta - \delta h(t)h^{-1}(\tau^*), \\ &= \Big(h(\tau^*)\rho(D[u(\tau^\star), \hat{0}]) + \delta\Big)h(t)h^{-1}(\tau^*) + \delta - \delta h(t)h^{-1}(\tau^*), \\ &= h(\tau^*)\rho(D[u(\tau^\star), \hat{0}])h(t)h^{-1}(\tau^\star) + \delta, \\ &\leq \|h\|_\infty \rho(\|u\|_{[t_0,t]}) + \delta. \end{aligned}$$

By combining the two cases thus discussed above, we obtain

$$V(t, x(t)) \leq \alpha_2(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0) + \|h\|_\infty \rho(\|u\|_{[t_0,t]}) + \delta,\ t \geq t_0,\ x(t_0) \in E^n.$$

By Lemma 2.5, we get, for all $t \geq t_0$ and $x(t_0) \in E^n$

$$\begin{aligned} D_0[x(t), \hat{0}] \leq& \alpha_1^{-1}(V(t, x(t))) \\ \leq& \alpha_1^{-1}\Big(2\alpha_2(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0)\Big) + \alpha_1^{-1}(4\rho(\|u\|_{[t_0,t]})) + \alpha_1^{-1}(4\delta) \end{aligned}$$

which shows that the non-linear time-varying fuzzy system (2) is generalized input-to-state uniformly practically $h-$stable. □

Remark 3.2. In Theorem 3.3, if $\delta = 0$, we obtain sufficient conditions to ensure the generalized input-to-state uniform $h-$stability of the non-linear time-varying fuzzy system (2).

The generalized input-to-state practical $h-$stability can be obtained with the help of $h-$ stable function.

Corollary 3.4. *Suppose there are a Lyapunov-Like function $V(t,x) \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$, $\delta \in \mathbf{R}_+$, a class $\mathcal{K}$ function ρ, two class $\mathcal{K}_\infty$ functions α_1, α_2 and a scalar function $\mu \in \mathcal{PC}[\mathbf{R}_+, \mathbf{R}]$ satisfying the following properties:*

i): $\alpha_1(D_0[x,\hat{0}]) \leq V(t,x) \leq \alpha_2(D_0[x,\hat{0}]) + \delta,\ (t,x) \in \mathbf{R}_+ \times E^n,$

ii): $D^+V_{(2)}(t,x(t)) \leq \mu(t)\Big(V(t,x(t)) - \delta\Big)$ *if* $V(t,x(t)) \geq e^{\int_0^t \mu(s)ds}\rho(D[u(t),\hat{0}]) + \delta,$ $t \in \mathbf{R}_+$.

(1) *If the scalar function μ is $h-$stable, then the non-linear time-varying fuzzy system (2) is generalized input-to-state practically $h-$stable.*

(2) *If the scalar function μ is uniformly $h-$stable then, the non-linear time-varying fuzzy system (2) is generalized input-to-state uniformly practically $h-$stable.*

Proof. Let us consider the function $\hbar(t) = e^{\int_0^t \mu(s)ds}$, $t \geq 0$. Then, we get

$$D^+V_{(2)}(t,x(t)) \leq \frac{\hbar'(t)}{\hbar(t)}\Big(V(t,x(t)) - \delta\Big) \text{ if } V(t,x(t)) \geq \hbar(t)\rho(D[u(t),\hat{0}]) + \delta, t \in \mathbf{R}_+.$$

It is clear that the function $V(t,x)$ satisfies conditions i) and ii) of Theorem 3.3. Then, by adopting the same calculation techniques used in the proof of Theorem 3.3, we end up showing the following inequality

$$V(t,x(t)) \leq \alpha_2(D_0[x(t_0),\hat{0}])e^{\int_{t_0}^t \mu(s)ds} + e^{\int_0^t \mu(s)ds}\rho(\|u\|_{[t_0,t]}) + \delta,\ t \geq t_0,\ x(t_0) \in E^n.$$

- If the scalar function μ is $h-$stable, then by Proposition 2.2, we get

$$\begin{aligned} V(t,x(t)) \leq & \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)e^{\xi(t_0)} \\ & + \|h\|_\infty h^{-1}(0)e^{\xi(0)}\rho(\|u\|_{[t_0,t]}) + \delta,\ t \geq t_0,\ x(t_0) \in E^n \end{aligned}$$

  Therefore, we get

$$\begin{aligned} D_0[x(t),\hat{0}] \leq & \alpha_1^{-1}\Big(2K(t_0)\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big) \\ & + \alpha_1^{-1}\Big(4\|h\|_\infty h^{-1}(0)e^{\zeta(0)}\rho(\|u\|_{[t_0,t]})\Big) + \alpha_1^{-1}\Big(4\delta\Big),\ t \geq t_0,\ x(t_0) \in E^n \end{aligned}$$

  with $K(t_0) = e^{\zeta(t_0)}$. Consequently, the system (2) is generalized input-to-state practically $h-$stable.
- If the function μ is uniformly $h-$stable, then by Proposition 2.2, we get

$$\begin{aligned} D_0[x(t),\hat{0}] \leq & \alpha_1^{-1}\Big(2e^{\zeta}\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big) + \alpha_1^{-1}\Big(4\|h\|_\infty h^{-1}(0)e^{\zeta}\rho(\|u\|_{[t_0,t]})\Big) \\ & + \alpha_1^{-1}\Big(4\delta\Big),\ t \geq t_0,\ x(t_0) \in E^n. \end{aligned}$$

  Then, the system (2) is generalized input-to-state uniformly practically $h-$stable. □

In what remains, sufficient conditions will be established to ensure the generalized integral input-to-state practical $h-$stability of system (2).

Theorem 3.5. *Suppose there are a Lyapunov-Like function $V(t,x) \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$, $\delta \in \mathbf{R}_+$, two class $\mathcal{K}$ functions π_1, π_2 and two class $\mathcal{K}_\infty$ functions α_1, α_2 satisfying the following properties:*

i): $\alpha_1(D_0[x,\hat{0}]) \leq V(t,x) \leq \alpha_2(D_0[x,\hat{0}]) + \delta$, $(t,x) \in \mathbf{R}_+ \times E^n$,

ii): $D^+V_{(2)}(t,x(t)) \leq \Big(\frac{h'(t)}{h(t)} + \pi_1(D_0[u(t),\hat{0}])\Big)(V(t,x(t)) - \delta) + h(t)\pi_2(D_0[u(t),\hat{0}])$, $t \in \mathbf{R}_+$.

Then, the non-linear time-varying fuzzy system (2) is generalized integral input-to-state practically $h-$stable.

Proof. By condition ii), we have $D^+V_{(2)}(t,x(t)) \leq \varphi(t, V(t,x(t)))$ where

$$\varphi(t,z) = \Big(\frac{h'(t)}{h(t)} + \pi_1(D_0[u(t),\hat{0}])\Big)(z-\delta) + h(t)\pi_2(D_0[u(t),\hat{0}]).$$

Consequently, by Lemma 2.3, we obtain

$$\begin{aligned}
V(t,x(t)) \leq & V(t_0,x(t_0))e^{\int_{t_0}^t \pi_1(D_0[u(s),\hat{0}]) + \frac{h'(s)}{h(s)})ds} \\
& + \int_{t_0}^t e^{\int_s^t \pi_1(D_0[u(\tau),\hat{0}]) + \frac{h'(\tau)}{h(\tau)})d\tau} h(s)\pi_2(D_0[u(s),\hat{0}])ds \\
& - \delta h(t)\int_{t_0}^t e^{\int_s^t \pi_1(D_0[u(\tau),\hat{0}]))d\tau} \frac{h'(s) + h(s)\pi_1(D_0[u(s),\hat{0}])}{h^2(s)} ds \\
\leq & V(t_0,x(t_0))h(t)h^{-1}(t_0)e^{\int_{t_0}^t \pi_1(D_0[u(s),\hat{0}])} \\
& + e^{\int_{t_0}^t \pi_1(D_0[u(\tau),\hat{0}])d\tau} \int_{t_0}^t \frac{h(t)}{h(s)} h(s)\pi_2(D_0[u(s),\hat{0}])ds \\
& - \delta h(t)\int_{t_0}^t e^{-\int_t^s \pi_1(D_0[u(\tau),\hat{0}]))d\tau} \frac{h'(s) + h(s)\pi_1(D_0[u(s),\hat{0}])}{h^2(s)} ds \\
\leq & (\alpha_2(D_0[x(t_0),\hat{0}]) + \delta)h(t)h^{-1}(t_0)e^{\int_{t_0}^t \pi(D_0[u(s),\hat{0}])} \\
& + \|h\|_\infty e^{\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau} \int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds \\
& + \delta h(t)\int_{t_0}^t \Big[-\frac{e^{-\int_t^s \pi_1(D_0[u(\tau),\hat{0}]))d\tau}}{h(s)}\Big]' ds \\
\leq & \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)e^{\int_{t_0}^t \pi(D_0[u(s),\hat{0}])} \\
& + \|h\|_\infty e^{\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau} \int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds + \delta
\end{aligned}$$

where $\pi = \pi_1 \vee \pi_2$. Letting $\theta_1 = \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)$ and $\theta_2 = \int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau$.

By using the inequality $\forall\ z_1, z_2 \in \mathbf{R}_+,\ \ z_1 e^{z_2} = z_1 + z_1(e^{z_2} - 1) \leq z_1 + \frac{1}{2}z_1^2 + \frac{1}{2}(e^{z_2} - 1)^2$ we obtain $V(t,x(t)) \leq \xi_1(\theta_1) + \xi_2(\theta_2) + \delta$, $\forall\, t \geq t_0$ with $\xi_1(r) = r + \frac{r^2}{2} \in \mathcal{K}_\infty$ and

$\xi_2(r) = \frac{1}{2}(e^r - 1)^2 + \|h\|_\infty r e^r \in \mathcal{K}_\infty$. By condition i), we have

$$\begin{aligned}\alpha_1(D_0[x(t), \hat{0}]) \leq & V(t, x(t)), \\ \leq & \xi_1(\theta_1) + \xi_2(\theta_2)) + \delta, \\ \leq & \alpha_2(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0) + \frac{1}{2}\alpha_2^2(D_0[x(t_0), \hat{0}])h^2(t)h^{-2}(t_0) \\ & + \xi_2(\int_{t_0}^{t} \pi(D_0[u(\tau), \hat{0}])d\tau)) + \delta, \\ \leq & \Big[\alpha_2(D_0[x(t_0), \hat{0}]) + \frac{\|h\|_\infty}{2h(t_0)}\alpha_2^2(D_0[x(t_0), \hat{0}])\Big]h(t)h^{-1}(t_0) \\ & + \xi_2(\int_{t_0}^{t} \pi(D_0[u(\tau), \hat{0}])d\tau)) + \delta, \\ \leq & \max\{1, \frac{\|h\|_\infty}{h(t_0)}\}\Big[\alpha_2(D_0[x(t_0), \hat{0}]) + \frac{1}{2}\alpha_2^2(D_0[x(t_0), \hat{0}])\Big]h(t)h^{-1}(t_0) \\ & + \xi_2(\int_{t_0}^{t} \pi(D_0[u(\tau), \hat{0}])d\tau)) + \delta, \\ := & K(t_0)\xi_1(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0) + \xi_2(\int_{t_0}^{t} \pi(D_0[u(\tau), \hat{0}])d\tau)) + \delta,\end{aligned}$$

where $K(t_0) = \frac{\|h\|_\infty}{h(t_0)} = \max\{1, \frac{\|h\|_\infty}{h(t_0)}\} \geq 1$. By Lemma 2.5, we get

$$\begin{aligned}D_0[x(t), \hat{0}] \leq & \alpha_1^{-1}\Big(2K(t_0)\xi_1(D_0[x(t_0), \hat{0}])h(t)h^{-1}(t_0)\Big) \\ & + \alpha_1^{-1}\Big(4\xi_2(\int_{t_0}^{t} \pi(D_0[u(\tau), \hat{0}])d\tau)\Big) + \alpha_1^{-1}(4\delta).\end{aligned}$$

Then, the non-linear time-varying fuzzy system (2) is generalized integral input-to-state practical $h-$stable. □

Remark 3.3. In Theorem 3.5, if $\delta = 0$, we obtain sufficient conditions to ensure the generalized integral input-to-state $h-$stability of the non-linear time-varying fuzzy system (2).

A similar result of generalized integral input-to-state practical $h-$stability is obtained using the notion of $h-$stable functions.

Corollary 3.6. *Suppose there exists a Lyapunov-Like function $V(t, x) \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$, $\delta \in \mathbf{R}_+$, two class $\mathcal{K}_\infty$ functions α_1, α_2 and a scalar function $\mu \in \mathcal{PC}[\mathbf{R}_+, \mathbf{R}]$ satisfying the following properties :*

i): $\alpha_1(D_0[x, \hat{0}]) \leq V(t, x) \leq \alpha_2(D_0[x, \hat{0}]) + \delta$, $(t, x) \in \mathbf{R}_+ \times E^n$,

ii): $D^+V_{(2)}(t, x(t)) \leq \Big(\mu(t) + \pi_1(D_0[u(t), \hat{0}])\Big)(V(t, x(t)) - \delta) + e^{\int_0^t \mu(s)ds}\pi_2(D_0[u(t), \hat{0}])$, $t \in \mathbf{R}_+$.

If the scalar function μ is $h-$stable, then the non-linear time-varying fuzzy system (2) is generalized integral input-to-state practically $h-$stable.

Proof. Let consider the function $\hbar(t) = e^{\int_0^t \mu(s)ds}$. It's easy too see that condition $ii)$ of Corollary 3.6 becomes

$$D^+V_{(2)}(t,x(t)) \leq \Big(\frac{\hbar'(t)}{\hbar(t)} + \pi_1(D_0[u(t),\hat{0}])\Big)(V(t,x(t)) - \delta) + \hbar(t)\pi_2(D_0[u(t),\hat{0}]),\ t \in \mathbf{R}_+.$$

Based on the proof of Theorem 3.5, we get

$$\begin{aligned}
V(t,x(t)) \leq & V(t_0,x(t_0))e^{\int_{t_0}^t \pi_1(D_0[u(s),\hat{0}]) + \frac{\hbar'(s)}{\hbar(s)})ds} \\
& + \int_{t_0}^t e^{\int_s^t \pi_1(D_0[u(\tau),\hat{0}]) + \frac{\hbar'(\tau)}{\hbar(\tau)})d\tau}\hbar(s)\pi_2(D_0[u(s),\hat{0}])ds \\
& - \delta\hbar(t)\int_{t_0}^t e^{\int_s^t \pi_1(D_0[u(\tau),\hat{0}]))d\tau}\frac{\hbar'(s) + \hbar(s)\pi_1(D_0[u(s),\hat{0}])}{\hbar^2(s)}ds \\
\leq & \alpha_2(D_0[x(t_0),\hat{0}])\hbar(t)\hbar^{-1}(t_0)e^{\int_{t_0}^t \pi(D_0[u(s),\hat{0}])} \\
& + \hbar(t)e^{\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau}\int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds + \delta
\end{aligned}$$

where $\pi = \pi_1 \vee \pi_2$. If the function μ is $h-$stable, then

$$\hbar(t)\hbar^{-1}(t_0) = e^{\int_{t_0}^t \mu(s)ds} \leq h(t)h^{-1}(t_0)e^{\zeta(t_0)} \text{ and } \hbar(t) = e^{\int_0^t \mu(s)ds} \leq h(t)h^{-1}(0)e^{\zeta(0)}.$$

Consequently, we obtain

$$\begin{aligned}
V(t,x(t)) \leq & \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)e^{\zeta(t_0)}e^{\int_{t_0}^t \pi(D_0[u(s),\hat{0}])} \\
& + h(t)h^{-1}(0)e^{\zeta(0)}e^{\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau}\int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds + \delta \\
\leq & \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)e^{\zeta(t_0)}e^{\int_{t_0}^t \pi(D_0[u(s),\hat{0}])} \\
& + \|h\|_\infty h^{-1}(0)e^{\zeta(0)}e^{\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau}\int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds + \delta
\end{aligned}$$

If we take, $\theta_1 = \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)e^{\zeta(t_0)}$ and $\theta_2 = \int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau$ then, as in the proof of Theorem 3.5, we get

$$V(t,x(t)) \leq \psi_1(\theta_1) + \psi_2(\theta_2) + \delta,\ \forall t \geq t_0$$

with

$$\psi_1(r) = r + \frac{r^2}{2} \in \mathcal{K}_\infty \text{ and } \psi_2(r) = \frac{1}{2}(e^r - 1)^2 + \frac{\|h\|_\infty}{h(0)}e^{\zeta(0)}re^r \in \mathcal{K}_\infty.$$

Finally, the solution of system (2) satisfies

$$\begin{aligned}
D_0[x(t),\hat{0}] \leq & \alpha_1^{-1}\Big(2\psi_1(\theta_1)\Big) + \alpha_1^{-1}\Big(4\psi_2(\theta_2)\Big) + \alpha_1^{-1}(4\delta) \\
\leq & \alpha_1^{-1}\Big(2K(t_0)\psi_1(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big) \\
& + \alpha_1^{-1}\Big(4\psi_2(\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau)\Big) + \alpha_1^{-1}(4\delta)
\end{aligned}$$

where $K(t_0) = \max\{e^{\zeta(t_0)}, \frac{\|h\|_\infty e^{2\zeta(t_0)}}{h(t_0)}\}$. Then the non-linear time-varying fuzzy system (2) is generalized integral input-to-state practically $h-$stable. $\square$

As for generalized integral input-to-state uniform practical $h-$stability, we have this result given by the following Theorem.

Theorem 3.7. *Suppose there are a Lyapunov-Like function $V(t,x) \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$, $\delta \in \mathbf{R}_+$, two class $\mathcal{K}_\infty$ functions α_1, α_2 and a class $\mathcal{K}$ function π satisfying the following properties :*

i): $\alpha_1(D_0[x,\hat{0}]) \leq V(t,x) \leq \alpha_2(D_0[x,\hat{0}]) + \delta$, $(t,x) \in \mathbf{R}_+ \times E^n$,

ii): $D^+V_{(2)}(t,x(t)) \leq \Big(\frac{h'(t)}{h(t)} - \pi(D_0[u(t),\hat{0}])\Big)\Big(V(t,x(t)) - \delta\Big) + h(t)\pi(D_0[u(t),\hat{0}])$, $t \in \mathbf{R}_+$.

Then, the non-linear time-varying fuzzy system (2) is generalized integral input-to-state uniformly practically $h-$stable.

Proof. Let $x(t) = x(t,t_0,x_0)$ be the solution of system (2) through $(t_0,x_0) \in \mathbf{R}_+ \times E^n$. By condition ii), we have

$$D^+V_{(2)}(t,x(t)) \leq \varphi(t,x(t)), t \in \mathbf{R}_+$$

where

$$\begin{aligned}\varphi(t,z) =& \Big(\frac{h'(t)}{h(t)} - \pi(D_0[u(t),\hat{0}])\Big)\Big(z - \delta\Big) + h(t)\pi(D_0[u(t),\hat{0}])\\ :=& p(t)z + q(t)\end{aligned}$$

with

$$p(t) = \frac{h'(t)}{h(t)} - \pi(D_0[u(t),\hat{0}]), q(t) = -\delta\Big(\frac{h'(t)}{h(t)} - \pi(D_0[u(t),\hat{0}])\Big) + h(t)\pi(D_0[u(t),\hat{0}]).$$

Using the same techniques of the proof of Theorem 3.5, we obtain

$$\begin{aligned}V(t,x(t)) \leq& \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)e^{-\int_{t_0}^t \pi(D_0[u(s),\hat{0}])}\\ &+ \|h\|_\infty \int_{t_0}^t e^{-\int_s^t \pi(D_0[u(\tau),\hat{0}])d\tau}\pi(D_0[u(s),\hat{0}])ds + \delta\\ \leq& \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0) + \|h\|_\infty \int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds + \delta.\end{aligned}$$

Then, by using condition i) and Lemma 2.5, we obtain, for all $t \geq t_0$ and $x(t_0) \in E^n$,

$$\begin{aligned}D_0[x(t),\hat{0}] \leq& \alpha_1^{-1}\Big(2\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big)\\ &+ \alpha_1^{-1}\Big(4\|h\|_\infty \int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds\Big) + \alpha_1^{-1}(4\delta).\end{aligned}$$

Consequently, the system (2) is generalized integral input-to-state uniformly practically $h-$stable. □

Using the notion of $h-$stable functions, we have the following result.

Corollary 3.8. *Suppose there exists a Lyapunov-Like function $V(t,x) \in \mathcal{C}(\mathbf{R}_+ \times E^n, \mathbf{R}_+)$, $\delta \in \mathbf{R}_+$, two class $\mathcal{K}_\infty$ functions α_1, α_2, a class $\mathcal{K}$ function π and a scalar function $\mu \in \mathcal{PC}[\mathbf{R}_+, \mathbf{R}]$ satisfying the following properties :*

i): $\alpha_1(D_0[x,\hat{0}]) \leq V(t,x) \leq \alpha_2(D_0[x,\hat{0}]) + \delta$, $(t,x) \in \mathbf{R}_+ \times E^n$,

ii): $D^+V_{(2)}(t,x(t)) \leq \Big(\mu(t) - \pi(D_0[u(t),\hat{0}])\Big)(V(t,x(t)) - \delta) + e^{\int_0^t \mu(s)ds}\pi(D_0[u(t),\hat{0}])$, $t \in \mathbf{R}_+$.

If the scalar function μ is uniformly $h-$stable, then the non-linear time-varying fuzzy system (2) is generalized integral input-to-state uniformly practically $h-$stable.

Proof. Let us consider the function $\hbar(t) = e^{\int_0^t \mu(s)ds}$. The proof is similar to that of Theorem 3.7. More precisely, after having followed the same steps of calculating of the proof of theorem 3.7, we obtain

$$\begin{aligned} V(t,x(t)) \leq& \alpha_2(D_0[x(t_0),\hat{0}])\hbar(t)\hbar^{-1}(t_0)e^{-\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau} \\ &+ \hbar(t)\int_{t_0}^t e^{-\int_s^t \pi(D_0[u(\tau),\hat{0}])d\tau}\pi(D_0[u(s),\hat{0}])ds + \delta \\ =& \alpha_2(D_0[x(t_0),\hat{0}])e^{\int_{t_0}^t \mu(s)ds}e^{-\int_{t_0}^t \pi(D_0[u(\tau),\hat{0}])d\tau} \\ &+ e^{\int_0^t \mu(s)ds}\int_{t_0}^t e^{-\int_s^t \pi(D_0[u(\tau),\hat{0}])d\tau}\pi(D_0[u(s),\hat{0}])ds + \delta \\ \leq& \alpha_2(D_0[x(t_0),\hat{0}])e^{\int_{t_0}^t \mu(s)ds} + e^{\int_0^t \mu(s)ds}\int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds + \delta. \end{aligned}$$

If the function μ is uniformly $h-$stable, then by Proposition 2.2, we get

$$V(t,x(t)) \leq \alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)e^{\xi} + h(t)h^{-1}(0)e^{\xi}\int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds + \delta.$$

By condition $i)$ and Lemma 2.5, we obtain

$$\begin{aligned} D_0[x(t),\hat{0}] \leq& \alpha_1^{-1}\Big(2e^{\xi}\alpha_2(D_0[x(t_0),\hat{0}])h(t)h^{-1}(t_0)\Big) \\ &+ \alpha_1^{-1}\Big(4\frac{\|h\|_\infty}{h(0)}e^{\xi}\int_{t_0}^t \pi(D_0[u(s),\hat{0}])ds\Big) + \alpha_1^{-1}\Big(4\delta\Big). \end{aligned}$$

Then, the non-linear time-varying fuzzy system (2) is generalized integral input-to-state uniformly practically $h-$stable. □

4. Conclusion

A new concept of practical $h-$stability of fuzzy differential equations have been investigated in this work. Lyapunov stability of fuzzy differential equations have been obtained by using lyapunov-like functions. This concept include generalized practical $h-$stability, practical input-to-state practical $h-$stability and integral input-to-state practical $h-$stability. The stability analysis is also achieved with the help of the comparison principle and the concept of scalar $h-$stable functions.

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