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Bifurcation Analysis of a Class of Polyharmonic Semilinear Equations with Perturbed Potential

Imed Abid and Sami Baraket

Abstract. This paper focuses on the investigation of bifurcation phenomena in a polyharmonic semilinear problem, considering both the Dirichlet and Navier boundary conditions. We explore the existence and uniqueness of positive solutions, as well as the presence of critical values and the uniqueness of extremal solutions. Additionally, we address various bifurcation scenarios that arise in a class of elliptic problems, and we establish the asymptotic behavior of the solution in the vicinity of the bifurcation point.

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1. Introduction and main results

Many problems of mechanics were described by polyharmonic equations [10]. Moreover, the areas of physics and geometry where such operators occur, include the study of the Kirchhoff plate equation in the theory of elasticity, and the study of the Paneitz-Branson operator in conformal geometry, see [10]. Inverse spectral problems for a potential perturbation of the polyharmonic operator were studied in [13].

The polyharmonic operator $(-\Delta)^m$ is the prototype of an elliptic operator of order $2m$. A general theory for boundary value problems for linear elliptic operators of order $2m$ was developed by Agmon-Douglis-Nirenberg in [3, 4] and Berchio-Gazzola in [6].

Although the material is quite technical, it turns out that the L^p-theory can be developed to a large extent analogously to second order equations. As long as existence, regularity and stability results are concerned, the theory of semi-linear higher order problems is already quite well developed. This is no longer true as soon as qualitative properties of the solution related to the bifurcation problems are investigated.

Shang and Wang have studied in [17] the following polyharmonic problem

$$\begin{cases} (-\Delta)^m u = \lambda\alpha(x)|u|^{q-2}u + \beta(x)|u|^{m^*-2}u, & x \in \Omega, \\ u \in H_0^m(\Omega), \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2m+1$, $1 < q < 2$, $\lambda > 0$ and $m^* = \dfrac{2n}{n-2m}$ is the critical Sobolev exponent. The functions α, β are continuous on Ω which are somewhere positive but which may change sign on Ω. By extracting the Palais-Smale sequence in the Nehari manifold, the authors showed the existence of multiple nontrivial solutions to problem (1). In the case $\alpha, \beta \equiv 1$ and $q = 2$,

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some authors have showed the existence and nonexistence of nontrivial solutions for the polyharmonic problem (1). In particular, when $m = 1$ and $n \geq 4$, Brezis and Nirenberg have discovered [8] a remarkable phenomenon: There exists a positive solution to (1) if and only if $\lambda \in (0, \lambda_1^{(1)})$ and there is no positive solution if $\lambda \geq \lambda_1^{(1)}$, where $\lambda_1^{(m)}$ denote the first eigenvalue of $(-\Delta)^m$ in $H_0^m(\Omega)$. In [9, 10, 16], the authors showed, for $m = 2$ and $n \geq 8$, that the biharmonic problem (1) possesses at least one nontrivial solution if $\lambda \in (0, \lambda_1^{(2)})$. For existence and nonexistence of nontrivial solutions for the general polyharmonic problems with critical growth and linear or superlinear perturbation, peoples can refer the papers [9, 10, 11, 16].

Various authors have studied the existence of weak solutions for the bifurcation problem

$$\begin{cases} -\Delta u &= \lambda\, f(u) \quad \text{in}\, \Omega, \\ \quad u &> 0 \quad\quad\quad \text{in}\, \Omega, \\ \quad u &= 0 \quad\quad\quad \text{on}\, \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Mironescu and Rădulescu in [15], Martel in [14] have proved that there exists $0 < \lambda^* < \infty$, a critical value of the parameter λ, such that (2) has a minimal, positive, classical solution u_λ for $0 < \lambda < \lambda^*$ and does not have a weak solution for $\lambda > \lambda^*$. Abid et al. generalized in [2] the same result for the bi-Laplace operator, and Abid in [1] discussed the existence, uniqueness, and stability of a positive solution. He also proved the existence of a critical value and the uniqueness of extremal solutions for a class of parametric fractional Schrödinger equations.

In this paper, we are interested in the following perturbed polyharmonic equation for $m \geq 2$

$$(P_\lambda): \ (-\Delta)^m u - \nabla^{m-1}(\theta(x)\nabla^{m-1}u) = \lambda f(u), \ u > 0 \ \text{ in } \ \Omega,$$

with the Dirichlet boundary conditions

$$u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} \ = \ 0 \ \text{ on } \ \partial\Omega, \tag{3}$$

or with the Navier boundary conditions

$$u = \Delta u = \cdots = \Delta^{m-1} u \ = \ 0 \ \text{ on } \ \partial\Omega, \tag{4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $\lambda > 0$ is a parameter. Let $f \in C^{0,\nu}(\Omega)$ and $\theta \in C^{0,\nu}(\overline{\Omega})$, $0 < \nu < 1$ fulfill the hypotheses

(A_1) f is positive, nondecreasing and convex on $(0, +\infty)$.

(A_2) $\lim\limits_{t \to +\infty} \dfrac{f(t)}{t} = a \in (0, \infty)$.

(A_3) There exist two positive constant $c_1, c_2 > 0$ such that $c_1 \leq \theta(x) \leq c_2$.

The objective of this study is to explore the critical polyharmonic equation's behavior under either Dirichlet or Navier boundary conditions. To accomplish this, we employ the Maximum Principle, which guarantees the desired outcome under the condition: There exists $\epsilon = \epsilon(n, \Omega)$ such that

$$\|\theta\|_\infty < \epsilon,$$

for more detail, see [10, Theorem 5.19].

The function spaces $\Sigma_m(\Omega)$ are defined for polyharmonic problems, which involve higher-order elliptic equations. These spaces are used to specify the boundary conditions for the problems. There are two types of boundary conditions considered: Dirichlet and Navier. In the Dirichlet boundary condition, the function space is defined as:

$$\Sigma_m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega)$$

and in the Navier boundary condition, the function space is defined as:

$$\Sigma_m(\Omega) = \{\varphi \in H^m(\Omega) \,/\, \varphi = \Delta\varphi = \cdots = \Delta^{m-1}\varphi \ = \ 0 \ \text{ on } \ \partial\Omega\}$$

These spaces are used to define the problem and ensure that the solutions satisfy the required boundary conditions.

Given the involvement of multiple orders of differentiation, various equivalent norms are applicable within these function spaces. The selection of the norm relies heavily on $\|\cdot\|_2$, which represents the $L^2(\Omega)$-norm. Additionally, we denote $\|\cdot\|$ as the Σ_m-norm, which is defined by

$$\|u\| = \Big(\int_\Omega |D^m u|^2\Big)^{\frac{1}{2}},$$

where

$$D^m = \begin{cases} \nabla\Delta^{\frac{m-1}{2}} & \text{if } m \text{ is odd,} \\ \Delta^{\frac{m}{2}} & \text{if } m \text{ is even.} \end{cases}$$

Let us begin by defining weak solutions for problem (P_λ).

Definition 1.1. A function $u \in \Sigma_m(\Omega)$, is a weak solution of problem (P_λ) if

$$\int_\Omega D^m u \cdot D^m\varphi + \int_\Omega \theta(x)\nabla^{m-1}u\nabla^{m-1}\varphi = \lambda\int_\Omega f(u)\varphi \tag{5}$$

for all $\varphi \in C^m(\Omega) \cap \Sigma_m(\Omega)$.

These solutions are commonly referred to as weak energy solutions. For brevity, we will simply refer to them as solutions, assured by the following lemma.

Lemma 1.1. *Since $f(t) \le at + f(0)$, if $u \in \Sigma_m(\Omega)$ is a weak solution of (P_λ), it is easily seen by a standard bootstrap argument that u is always a classical solution.*

For more details, see [10, Proposition 7.15]. In the rest of this article, we denote by a solution of (P_λ) any weak or classical solution.

A weak super-solution (resp. sub-solution) is a function that verifies (5) with equality replaced by $\ge$ (resp. $\le$) for every nonnegative test function.

Let φ_1 be a positive eigenfunction (see [10, section 3.1.3]) associated with the first eigenvalue λ_1 of the operator $(-\Delta)^m - \nabla^{m-1}(\theta(x)\nabla^{m-1})$, namely

$$\begin{cases} (-\Delta)^m\varphi_1 - \nabla^{m-1}(\theta(x)\nabla^{m-1}\varphi_1) &= \lambda_1\varphi_1 \quad \text{in } \Omega, \\ \varphi_1 &> 0 \quad \text{in } \Omega, \\ \|\varphi_1\|_2 &= 1, \end{cases} \tag{6}$$

with the Dirichlet boundary conditions

$$\varphi_1 = \frac{\partial\varphi_1}{\partial\nu} = \cdots = \frac{\partial^{m-1}\varphi_1}{\partial\nu^{m-1}} \ = \ 0 \ \text{ on } \ \partial\Omega,$$

or with the Navier boundary conditions

$$\varphi_1 = \Delta\varphi_1 = \cdots = \Delta^{m-1}\varphi_1 \ = \ 0 \ \text{ on } \ \partial\Omega.$$

A solution u of problem (P_λ) is stable if and only if the first eigenvalue $\mu_1(\lambda, u)$ of the linearized operator

$$v \mapsto L_\lambda(v) := (-\Delta)^m v - \nabla^{m-1}(\theta(x)\nabla^{m-1}u)v - \lambda f'(u)v,$$

given by

$$\mu_1(\lambda, u) := \inf_{\varphi\in\Sigma_m(\Omega)-\{0\}} \frac{\displaystyle\int_\Omega |D^m\varphi|^2 + \int_\Omega \theta(x)|\nabla^{m-1}\varphi|^2 - \lambda\int_\Omega f'(u)\varphi^2}{\|\varphi\|_2^2},$$

is nonnegative. In other words,

$$\lambda\int_\Omega f'(u)\varphi^2 \le \int_\Omega |D^m\varphi|^2 + \int_\Omega \theta(x)|\nabla^{m-1}\varphi|^2, \quad \text{for any } \varphi\in\Sigma_m(\Omega). \tag{7}$$

If $\mu_1(\lambda, u) < 0$, the solution u is said to be *unstable.*

Next, we define:

- $\Lambda := \{\lambda > 0 : \ (P_\lambda) \text{ admits a solution } \}$.
- $\lambda^* := \sup\Lambda \le +\infty$.
- r as the infimum of the function $f(t)/t$ for $t > 0$, denoted as

$$r := \inf_{t>0} \frac{f(t)}{t}.$$

The values of a and r defined earlier play a crucial role in understanding the bifurcation phenomena. Specifically, there exists a finite positive number λ^*, $(\lambda_1/a \le \lambda^* \le \lambda_1/r)$, referred to as the extremal value. For $0 < \lambda < \lambda^*$, problem $\lambda > \lambda^*$ possesses at least one positive solution. However, for $\lambda > \lambda^*$, no solution exists, even in the weak sense.

Our first main result establishes the existence of the critical value λ^*.

Theorem 1.2. *Under the assumptions* $(A_1), (A_2)$ *and* (A_3) *there exists a critical value* $\lambda^* \in (0, \infty)$ *such that the following properties hold:*

(i) *For any* $\lambda \in (0, \lambda^*)$*, problem* (P_λ) *has a minimal solution* u_λ*, which is the unique stable solution of* (P_λ)*.*

(ii) $\lambda_1/a \le \lambda^*$ *and for any* $\lambda \in (0, \lambda_1/a)$*,* u_λ *is the unique solution of problem* (P_λ)*.*

(iii) *For evry* $x \in \Omega$*, the function* $\lambda \longmapsto u_\lambda(x)$ *is a increasing.*

(iv) *If* $\lambda = \lambda^*$*, the problem* (P_λ) *has a solution, then* $u^* := \lim_{\lambda\to\lambda^*} u_\lambda$ *is a stable solution. In particular,* $\mu_1(s, \lambda^*, u^*) = 0$*.*

The next natural focus of our study provides us with more detailed information regarding λ^*.

An essential role in our arguments will be played by

$$l := \lim_{t\to+\infty}\Big(f(t) - at\Big).$$

We classify two distinct situations that heavily depend on the sign of the parameter l.

Theorem 1.3. *Under the assumptions* $(A_1), (A_2)$ *and* (A_3)*, let us consider the case where* $l \ge 0$*. Then*

(i) $\lambda^* = \lambda_1/a$*.*

(ii) *The problem* (P_λ) *has no solution.*

(iii) $\lim\limits_{\lambda\to\lambda^*} u_\lambda = \infty$ *uniformly on compact subsets of* Ω*.*

Theorem 1.4. *Under the hypotheses* $(A_1), (A_2)$, *and* (A_3), *consider the case where* $l < 0$. *Then*

(i) *The critical value* λ^* *belongs to the interval* $(\lambda_1/a, \lambda_1/r)$.
(ii) *For* $\lambda = \lambda^*$, *the problem* (P_λ) *has a unique solution* u^*.
(iii) *The problem* (P_λ) *has an unstable solution* v_λ *for any* $\lambda \in (\lambda_1/a, \lambda^*)$ *and the sequence* $(v_\lambda)_\lambda$ *satisfies:*
  (a) $\lim_{\lambda \to \lambda_1/a} v_\lambda = \infty$ *uniformly on compact subsets of* Ω,
  (b) $\lim_{\lambda \to \lambda^*} v_\lambda = u^*$ *uniformly in* Ω.

This paper is organized as follows: In the next section, we present the proof of Theorem 1.2 which focuses on the existence of "minimal" solutions. We demonstrate there exists a limiting parameter λ^* such that one has the existence of stable and regular minimal solutions to (P_λ) for λ in the interval $(0, \lambda^*)$. Furthermore, we establish that for $\lambda > \lambda^*$, not even singular solutions exist. In Sections 3 and 4, we delve into the proofs of Theorems 1.3 and 1.4, respectively. In these sections, we address bifurcation problems for a class of elliptic problems and examine the asymptotic behavior of the solution around the bifurcation point.

2. Proof of Theorem 1.2

We begin by establishing the following preliminary lemma.

Lemma 2.1. *Problem* (P_λ) *has no solution for any* $\lambda > \lambda_1/r$, *but has at least one solution provided* λ *is positive and small enough.*

Proof. We begin by proving that (P_λ) has a solution for small λ using the barrier method. Since $f(0) > 0$, $\underline{w} \equiv 0$ is a strict sub-solution of (P_λ) for every $\lambda > 0$. Let $\overline{w} \in \Sigma_m(\Omega)$ be the solution of

$$(-\Delta)^m \overline{w} - \nabla^{m-1}(\theta(x)\nabla^{m-1}\overline{w}) = 1 \quad \text{in } \Omega,$$

is a bounded super-solution of (P_λ) for small λ, precisely when $\lambda < 1/f(\|\overline{w}\|_\infty)$.

Next, we define a sequence $w_n \in \Sigma_m(\Omega)$ by

$$(-\Delta)^m w_{n+1} - \nabla^{m-1}(\theta(x)\nabla^{m-1} w_{n+1}) = \lambda f(w_n) \quad \text{in } \Omega. \tag{8}$$

By the maximum principle (see [10]) we have

$$\underline{w} \le w_n \le w_{n+1} \le \overline{w} \quad \text{for all } n \in \mathbb{N},$$

implying that the sequence $(w_n)_{n\ge 0}$ is increasing and bounded. Therefore, it converges, and we conclude that problem $(P\lambda)$ has a solution for small λ.

Next, we prove that there is no weak solution for large $\lambda > 0$.

Assume that u is a solution of (P_λ) for some $\lambda > 0$. Using φ_1 given in (6) as a test function and integrating by parts, we obtain

$$\begin{aligned}
\lambda_1 \int_\Omega \varphi_1 u &= \int_\Omega (-\Delta)^m \varphi_1 u - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1}\varphi_1)u \\
&= \int_\Omega (-\Delta)^m u\varphi_1 - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1}u)\varphi_1 \\
&= \lambda \int_\Omega f(u)\varphi_1 \ \ge \lambda r \int_\Omega u\varphi_1,
\end{aligned}$$

From this, we can deduce the inequality

$$(\lambda_1 - \lambda r)\int_\Omega \varphi_1 u \geq 0.$$

Since $\varphi_1 > 0$ and $u > 0$, it follows that the parameter λ must belong to the interval $(0, \lambda_1/r)$. This completes our proof. □

It follows from the previous analysis that λ^* is a real number. We now state another useful result as follows

Lemma 2.2. *Assume that (P_λ) has a solution for some $\lambda \in (0, \lambda^*)$. then there exists a minimal solution denoted by u_λ. Moreover, for any $\lambda' \in (0, \lambda)$ the problem $(P_{\lambda'})$ is resolvable.*

Proof. Let $\lambda \in (0, \lambda^*)$ and consider u be a solution of (P_λ). Using the barrier method, we can construct a sequence $(w_n)_{n\geq 0}$ defined in (8),which is increasing and bounded by u. Thus, the sequence converges to a solution $u\lambda$, which is independent of the choice of u and therefore represents a minimal solution.

Furthermore, if u is a solution of (P_λ), it also serves as a super-solution for the problem $(P_{\lambda'})$ for any $\lambda' \in (0, \lambda)$. Similarly, 0 can be used as a sub-solution. This completes the proof. □

Remark 2.1. Thanks to Lemmas 2.1 and 2.2, the set Λ is a non-empty bounded interval.

By combining Lemmas 2.1 and 2.2, we are able to establish the Theorem 1.2, which provides important insights into the minimal solution u_λ and its stability.

2.1. Proof of (i). First, we claim that u_λ is stable. Let's assume the contrary, i.e., suppose that the first eigenvalue $\mu_1(\lambda, u_\lambda)$ is negative. This implies the existence of a positive eigenfunction $\psi \in \Sigma_m(\Omega)$ such that

$$(-\Delta)^m\psi - \nabla^{m-1}(\theta(x)\nabla^{m-1}\psi) - \lambda f'(u_\lambda)\psi = \mu_1\psi \quad \text{in } \Omega.$$

Now consider the function $u^\varepsilon := u_\lambda - \varepsilon\psi$. By linearity, we have

$$\begin{aligned}
&(-\Delta)^m u^\varepsilon - \nabla^{m-1}(\theta(x)\nabla^{m-1}u^\varepsilon) - \lambda f(u^\varepsilon)\\
&= (-\Delta)^m u_\lambda - \varepsilon(-\Delta)^m\psi - \nabla^{m-1}(\theta(x)\nabla^{m-1}u_\lambda) + \varepsilon\nabla^{m-1}(\theta(x)\nabla^{m-1}\psi) - \lambda f(u^\varepsilon)\\
&= \lambda f(u_\lambda) - \varepsilon(\lambda f'(u_\lambda)\psi + \mu_1\psi) - \lambda f(u_\lambda - \varepsilon\psi)\\
&= \lambda\Big(- f(u_\lambda - \varepsilon\psi) + f(u_\lambda) - \varepsilon f'(u_\lambda)\psi\Big) - \varepsilon\mu_1\psi\\
&= \lambda o_\varepsilon(\varepsilon\psi) - \varepsilon\mu_1\psi\\
&= \varepsilon\psi(\lambda o_\varepsilon(1) - \mu_1).
\end{aligned}$$

Since $\mu_1(\lambda, u_\lambda) < 0$, for $\varepsilon > 0$ sufficiently small, we have

$$(-\Delta)^m u^\varepsilon - \nabla^{m-1}(\theta(x)\nabla^{m-1}u^\varepsilon) - \lambda f(u^\varepsilon) \geq 0 \quad \text{in } \Omega.$$

Applying the strong maximum principle, we conclude that $u^\varepsilon \geq 0$ is a super-solution of (P_λ). However, since $u^\varepsilon < u_\lambda$, this contradicts the minimality of u_λ. Therefore, u_λ must be stable.

We now aim to show that (P_λ) has at most one stable solution. Let's assume the existence of another stable solution $v \neq u_\lambda$ of problem (P_λ). Consider the fonction

$w := v - u_\lambda$. By maximum principle, we have $w > 0$ in Ω. Taking w as a test function in (7), we obtain

$$\begin{aligned}\lambda \int_\Omega f'(v)w^2 &\le \int_\Omega |D^m w|^2 + \int_\Omega \theta(x)|\nabla^{m-1}w|^2 \\ &\le \int_\Omega D^m w.D^m w - \int_\Omega \theta(x)\nabla^{m-1}w.\nabla^{m-1}w \\ &\le \int_\Omega (-\Delta)^m w\, w - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1}w)\, w \\ &\le \int_\Omega \Big[(-\Delta)^m v - \nabla^{m-1}(\theta(x)\nabla^{m-1}v)\Big] w \\ &- \int_\Omega \Big[(-\Delta)^m u_\lambda + \nabla^{m-1}(\theta(x)\nabla^{m-1}u_\lambda)\Big] w \\ &\le \lambda \int_\Omega \Big[f(v) - f(u_\lambda)\Big] w.\end{aligned}$$

Therefore

$$\int_\Omega \Big[f(v) - f(u_\lambda) - f'(v)(v - u_\lambda)\Big] w \ge 0.$$

Since f is convex, the term in the brackets is nonpositive, yielding

$$f(v) - f(u_\lambda) - f'(v)(v - u_\lambda) = 0 \quad \text{in } \Omega.$$

This implies that f is affine over $[u_\lambda(x), v(x)]$ for any $x \in \Omega$, and consequently, f is affine in the interval $[0, \max_\Omega v]$. Thus, there exists two real numbers α and $\beta > 0$ such that

$$f(x) = \alpha x + \beta \qquad \text{in } [0, \max_\Omega v].$$

Finally, since u_λ and v are two solutions to $(-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1}w) = \lambda\alpha w + \lambda\beta$, we obtain

$$\begin{aligned}0 &= \int_\Omega \Big(u_\lambda(-\Delta)^m\, v - v\,(-\Delta)^m u_\lambda\Big) \\ &- \int_\Omega \Big(u_\lambda\, \nabla^{m-1}(\theta(x)\nabla^{m-1}v) - \nabla^{m-1}(\theta(x)\nabla^{m-1}u_\lambda)\, v\Big) \\ &= \lambda\beta \int_\Omega (u_\lambda - v).\end{aligned}$$

This is impossible since $\lambda > 0$, $\beta = f(0) > 0$ and $w = v - u_\lambda$ is positive in Ω. Hence, we conclude that (P_λ) has at most one stable solution. □

2.2. Proof of (ii). Recall that λ_1 is defined in (6). Using the convexity of f, we deduce that $a = \sup_{\mathbb{R}_+} f'(t)$. Let u be a solution to (P_λ) for $\lambda \in (0, \lambda_1/a)$. Suppose that u is unstable. Then, we can find $\varphi = \varphi_1 \in \Sigma_m(\Omega)$ satisfying

$$\begin{aligned}\lambda a \int_\Omega v_1^2 \ge \lambda \int_\Omega f'(u)v_1^2 &> \int_\Omega |D^m\varphi_1|^2 + \int_\Omega \theta(x)|\nabla^{m-1}\varphi_1|^2 \\ &= \int_\Omega (-\Delta)^m\varphi_1\, \varphi_1 - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1}\varphi_1)\, \varphi_1 = \lambda_1 \int_\Omega \varphi_1^2,\end{aligned}$$

which implies that

$$(\lambda a - \lambda_1)\int_\Omega \varphi_1^2 > 0.$$

However, this is impossible for $\lambda \in (0, \lambda_1/a)$. Thus, we conclude that $\mu_1(\lambda, u) \geq 0$. By part (i), this implies the uniqueness of u.

To establish the existence of solutions, we observe that problem (P_λ) can be formulated as the Euler-Lagrange equation of the functional $\mathcal{J}_\lambda : \Sigma_m(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{J}(u) := \frac{1}{2}\int_\Omega |D^m u|^2 + \frac{1}{2}\int_\Omega \theta(x)|\nabla^{m-1}u|^2 - \lambda\int_\Omega F(u),$$

for all $u \in \Sigma_m(\Omega)$ with

$$F(u) := \int_0^u f(s^+)ds \quad \text{and} \quad s^+ := \max(s, 0),$$

The functional $\mathcal{J}_\lambda$ is well-defined and Fréchet differentiable at $u \in \Sigma_m(\Omega)$, and its derivative $\mathcal{J}'_\lambda(u)(\varphi)$ for $\varphi \in \Sigma_m(\Omega)$ is given by

$$\mathcal{J}'_\lambda(u)(\varphi) = \int_\Omega D^m u \cdot D^m \varphi + \int_\Omega \theta(x)\nabla^{m-1}u \cdot \nabla^{m-1}\varphi - \lambda\int_\Omega f(u)\varphi.$$

Therefore, critical points of $\mathcal{J}_\lambda$ correspond to weak solutions of (P_λ). In order to find these critical points, we need to show that $\mathcal{J}_\lambda$ is well-defined, sequentially weakly lower semicontinuous, coercive, and belongs to $\mathcal{C}^1(\Sigma_m(\Omega))$.

If $\lambda \in (0, \lambda_1/a)$, there exist $\varepsilon > 0$ and $A > 0$ depending on λ such that

$$2\lambda F(t) \leq (\lambda_1 - \varepsilon)t^2 + A, \qquad \forall t \in \mathbb{R}.$$

Using standard arguments, we can conclude that $\mathcal{J}_\lambda(u)$ is coercive, bounded from below and weakly lower semicontinuous in $\Sigma_m(\Omega)$. Hence, the minimum of $\mathcal{J}_\lambda$ is attained by some function $u \in \Sigma_m(\Omega)$. Therefore, the critical point u of $\mathcal{J}_\lambda$ gives a solution of (P_λ).

2.3. Proof of (iii). Using sub- and super-solution method, as described in Lemma 2.2 we can conclude that the mapping $\lambda \mapsto u_\lambda$ is increasing. This establishes the result stated in (iii). □

2.4. Proof of (iv). Suppose that (P_{λ^*}) has a solution u. Then, for every $\lambda \in (0, \lambda^*)$ we have $u_\lambda \leq u$ in Ω. Utilizing the monotonicity of u_λ, we can define the function

$$u^* = \lim_{\lambda \to \lambda^*} u_\lambda$$

which is well-defined in Ω and serves as a stable solution for problem (P_{λ^*}). Consequently,we have $\mu_1(\lambda^*, u^*) \geq 0$.

Now, let us consider the nonlinear operator

$$\begin{array}{rcl} G : (0, +\infty) \times C^{2m,\alpha}(\overline{\Omega}) \cap E & \longrightarrow & C^{0,\alpha}(\overline{\Omega}) \\ (\lambda, u) & \longmapsto & (-\Delta)^m u - \nabla^{m-1}(\theta(x)\nabla^{m-1}u) - \lambda f(u), \end{array}$$

where $\alpha \in (0,1)$ and E is the function space defined as follows:

$$E = \begin{cases} \{u \in W^{2m,2}(\Omega) \ / \ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1} u}{\partial \nu^{m-1}} = 0 \text{ on } \partial\Omega\} & \text{if we use (3)} \\ \{u \in W^{2m,2}(\Omega) \ / \ u = \Delta u = \cdots = \Delta^{m-1} u = 0 \text{ on } \partial\Omega\} & \text{if we use (4)} \end{cases} \tag{9}$$

Assuming that the first eigenvalue $\mu_1(\lambda^*, u^*)$ is positive, we can apply the implicit function theorem to the operator G. For any λ in a neighborhood of λ^* and u in a neighborhood of u^*, we have $G(\lambda, u) = 0$, which proves that the problem (P_λ) has a solution for λ in a neighborhood of λ^*. However, this contradicts the definition of λ^*. Hence, we conclude that $\mu_1(\lambda^*, u^*) = 0$, completing the proof of Theorem 1.2. □

3. Proof of Theorem 1.3

Remark 3.1. Thanks to Lemma 2.1 and Theorem 1.2, the critical value λ^* satisfies

$$\lambda_1/a \leq \lambda^* \leq \lambda_1/r.$$

In order to prove this theorem, we will demonstrate the equivalence of the three assertions and establish the validity of one of them. To aid us in this proof, we will utilize the following auxiliary result, which is a reformulation of a theorem by Hörmander [12].

Lemma 3.1. *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$ with smooth boundary. Let (u_n) be a sequence of super-harmonic nonnegative functions defined on Ω. Then the following alternatives hold:*

(i) *Either* $\lim_{n\to\infty} u_n = \infty$ *uniformly on compact subsets of* Ω,
or
(ii) (u_n) *contains a subsequence which converges in* $L^1_{\text{loc}}(\Omega)$ *to some function* u.

Remark 3.2. The result by Hörmander is also true if (u_n) is a sequence of a super-biharmonic nonnegative functions.

3.1. Proof (i)⇒(ii). First, let us assume by contradiction that $\lambda^* = \frac{\lambda_1}{a}$. If (P_{λ^*}) has a solution u^*, then according to observation (iv) in Theorem 1.2 we have $\mu_1(\lambda^*, u^*) = 0$. This implies the existence of $\psi \in H^{2m}(\Omega)$ satisfying:

$$\begin{aligned} (-\Delta)^m \psi - \nabla^{m-1}(\theta(x)\nabla^{m-1}\psi) - \lambda^* f'(u^*)\psi = 0 \quad & \text{in } \Omega \\ \psi > 0 \quad & \text{in } \Omega, \end{aligned}$$

subject Dirichlet boundary conditions (3) or Navier bounday conditions (4). By using φ_1, given in (6), as a test function and integrating by parts, we obtain

$$\int_\Omega \Big((-\Delta)^m \varphi_1 - \nabla^{m-1}(\theta(x)\nabla^{m-1}\varphi_1)\Big)\psi - \lambda^* \int_\Omega f'(u^*)\psi\varphi_1 = 0,$$

which leads to

$$\int_\Omega \big(\lambda_1 - \lambda^* f'(u^*)\big)\psi\varphi_1 = 0.$$

Since $\varphi_1 > 0$, $\psi > 0$, $\lambda^* = \frac{\lambda_1}{a}$ and $a = \sup_{t>0} f'(t)$, we have $\lambda_1 - \lambda^* f'(u^*) \geq 0$. This equation implies $\lambda_1 - \lambda^* f'(u^*) = 0$. Consequently,

$$f'(u^*) \equiv a \quad \text{in} \quad \Omega.$$

This implies that $f(t) = at + b$ in $[0, \max_{\Omega} u^*]$ for some scalar $b > 0$. However, there is no positive function in Ω satisfying

$$(-\Delta)^m u - \nabla^{m-1}(\theta(x)\nabla^{m-1}u) = \lambda^* au + \lambda^* b \quad \text{in } \Omega.$$

To see this, suppose such a function exists. Using φ_1 as a test function and integrating by parts, we have

$$\int_\Omega (-\Delta)^m u \ \varphi_1 - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1}u) \ \varphi_1 = \lambda^* a \int_\Omega u \ \varphi_1 + \lambda^* b \int_\Omega \varphi_1$$

which simplifies to

$$\int_\Omega \Big((-\Delta)^m \varphi_1 - \nabla^{m-1}(\theta(x)\nabla^{m-1}\varphi_1)\Big) u = \lambda_1 \int_\Omega u \ \varphi_1 + \lambda^* b \int_\Omega \varphi_1$$

This implies

$$0 = \lambda^* b \int_\Omega \varphi_1,$$

which is impossible. Therefore, problem (P_{λ^*}) has no solution and (i) implies (ii). □

3.2. Proof. (ii)⇒(iii). Let us assume that (ii) occurs and we aim to prove that $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$ uniformly on compact subsets of Ω. By contradiction, suppose that (iii) does not hold. According to Lemma 3.1 and considering a subsequence if necessary, we have (u_λ) converging locally in $L^1(\Omega)$ to u^* as $\lambda \to \lambda^*$.

Now, we utilize the following result:

Lemma 3.2. *The minimal solution u_λ of the problem (P_λ) is bounded in $L^2(\Omega)$.*

Proof. Assume that u_λ is not bounded in $L^2(\Omega)$. We can define

$$u_\lambda := l_\lambda w_\lambda,$$

where

$$\|w_\lambda\|_2 = 1 \quad \text{and} \quad l_\lambda \to +\infty \quad \text{as} \quad \lambda \to \lambda^*.$$

Since $f(t) \le at + f(0)$, we have

$$\begin{aligned}
\int_\Omega |D^m w_\lambda|^2 &\le \int_\Omega |D^m w_\lambda|^2 + \int_\Omega \theta(x)|\nabla^{m-1}w_\lambda|^2 \\
&= \int_\Omega (-\Delta)^m w_\lambda \ w_\lambda - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1}w_\lambda) \ w_\lambda = \int_\Omega \frac{\lambda f(u_\lambda)}{l_\lambda} w_\lambda \\
&\le \lambda^* \int_\Omega \Big(a\, w_\lambda^2 + \frac{f(0)}{l_\lambda} w_\lambda\Big) \le \lambda^* a + c_\lambda \int_\Omega w_\lambda \\
&\le \lambda^* a + c_\lambda \sqrt{|\Omega|},
\end{aligned}$$

where c_λ is a positive constant independent of λ.

Recalling that w_λ satisfies $(-\Delta)^m w_\lambda - \nabla^{m-1}(\theta(x)\nabla^{m-1}w_\lambda) = \dfrac{\lambda f(l_\lambda w_\lambda)}{l_\lambda}$ and f is quasilinear, we can deduce that (w_λ) is bounded in $H^{2m}(\Omega)$. Hence, by taking a subsequence, we have

$$w_\lambda \to w \ \text{ weakly in } \ H^{2m}(\Omega) \quad \text{and} \quad w_\lambda \to w \ \text{ in strongly in } \ H^{2m-1}(\Omega) \ \text{ as } \ \lambda \to \lambda^*.$$

Moreover, by the trace Theorem, we obtain

$$\begin{aligned} w = \frac{\partial w}{\partial \nu} = \cdots = \frac{\partial^{m-1} w}{\partial \nu^{m-1}} &= 0 \text{ on } \partial\Omega, \\ w = \Delta w = \cdots = \Delta^{m-1} w &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Consequently,

$$(-\Delta)^m w_\lambda - \nabla^{m-1}(\theta(x)\nabla^{m-1} w_\lambda) = \frac{\lambda f(u_\lambda)}{l_\lambda} \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega) \quad \text{as } \lambda \to \lambda^*.$$

This implies that $(-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1} w) = 0$ in $\mathcal{D}'(\Omega)$. By applying the boundary conditions, we deduce that $w \equiv 0$ in Ω. However, this contradicts the fact that $\|w\|_2 = \lim_{\lambda\to\lambda^*} \|w_\lambda\|_2 = 1$. Hence, the proof of the lemma is complete. □

Consequently, (u_λ) is bounded in $L^2(\Omega)$ and by employing similar arguments as above, it is also bounded in $H^{2m}(\Omega)$. Thus, we can conclude that, up to a subsequence,

$$u_\lambda \to u \text{ weakly in } H^{2m}(\Omega) \quad \text{and} \quad u_\lambda \to u \text{ in strongly in } H^{2m-1}(\Omega) \text{ as } \lambda \to \lambda^*.$$

Moreover, we have

$$(-\Delta)^m u - \nabla^{m-1}(\theta(x)\nabla^{m-1} u) = \lambda^* f(u) \text{ in } \Omega.$$

However, this is impossible based on the hypothesis (*ii*). Hence, we can conclude that (*ii*) implies (*iii*). Moreover, this demonstrates that (*ii*) and (*iii*) are equivalent.

However, this is impossible based on the hypothesis (*ii*). Hence, we can conclude that (*ii*) implies (*iii*). Moreover, this demonstrates that (*ii*) and (*iii*) are equivalent.

3.3. Proof. (iii)⇒(i). Assume that (P_{λ^*}) has a solution u^*, and consider the sequence (u_λ) converges to u^* as λ. If $\lim_{\lambda\to\lambda^*} u_\lambda = \infty$, then the sequence cannot converge to u^*. Hence, we can conclude that (*iii*) implies (*i*).

Suppose both (*ii*) and (*iii*) occur. We have $\lim_{\lambda\to\lambda^*} \|u_\lambda\|_2 = \infty$. Let us write

$$u_\lambda = l_\lambda w_\lambda \quad \text{with } \|w_\lambda\|_2 = 1,$$

and consider a subsequence such that

$$w_\lambda \rightharpoonup w \text{ weakly in } H^{2m}(\Omega) \text{ and } w_\lambda \to w \text{ strongly in } H^{2m-1}(\Omega) \text{ as } \lambda \to \lambda^*.$$

We also have,

$$\frac{\lambda}{l_\lambda} f(l_\lambda w_\lambda) \to \lambda^* a w \quad \text{in } L^2(\Omega) \quad \text{as } \lambda \to \lambda^*.$$

$$(-\Delta)^m w_\lambda - \nabla^{m-1}(\theta(x)\nabla^{m-1} w_\lambda) \to (-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1} w) \quad \text{in } \mathcal{D}'(\Omega)$$

as $\lambda \to \lambda^*$, and then

$$\begin{gathered} (-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1} w) = \lambda^* a w \quad \text{in } \Omega, \\ w = \frac{\partial w}{\partial \nu} = \cdots = \frac{\partial^{m-1} w}{\partial \nu^{m-1}} = 0 \text{ on } \partial\Omega, \\ \text{or} \\ w = \Delta w = \cdots = \Delta^{m-1} w = 0 \text{ on } \partial\Omega. \end{gathered}$$

By multiplying by φ_1, as defined in (6), we obtain

$$\begin{aligned}\int_\Omega \lambda^* a w \varphi_1 &= \int_\Omega (-\Delta)^m w \varphi_1 - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1} w)\varphi_1 \\ &= \int_\Omega (-\Delta)^m \varphi_1 w - \int_\Omega \nabla^{m-1}(\theta(x)\nabla^{m-1}\varphi_1) w \\ &= \int_\Omega \lambda_1 \varphi_1 w.\end{aligned}$$

Since $\varphi_1 > 0$ and $w > 0$ in Ω, we have $\lambda^* = \frac{\lambda_1}{a}$ which proves (i).

To complete the proof of Theorem 1.3, it remains to show that $(P_{\lambda_1/a})$ has no solution.

Assume that u is a solution of $(P_{\lambda_1/a})$. Since

$$l := \lim_{t\to\infty} \big(f(t) - at\big) \geq 0,$$

we have $f(t) - at \geq 0$. Therefore,

$$(-\Delta)^m u - \nabla^{m-1}(\theta(x)\nabla^{m-1} u) = \frac{\lambda_1}{a} f(u) \geq \lambda_1 u \quad \text{in } \Omega.$$

Multiplying the previous equation by φ_1 and integrating by parts, we obtain $f(u) = au$ in Ω, which contradicts $f(0) > 0$. This establishes that $(P_{\lambda_1/a})$ has no solution.

The proof of Theorem 1.3 is now complete. □

Remark 3.3. Note that the equivalence of the statements in Theorem 1.3 holds regardless of the sign of l.

4. Proof of Theorem 1.4

4.1. Proof (i). For the first part of Theorem 1.4, we already established in Remark 3.1 that $\lambda_1/a \leq \lambda^* \leq \lambda_1/r$. Therefore, it suffices to show that $\lambda^* \neq \lambda_1/a$ and $\lambda^* \neq \lambda_1/r$.

First, assume that $\lambda^* = \lambda_1/a$. By Remark 3.3, we know that

$$\lim_{\lambda\to\lambda^*} u_\lambda = \infty \text{ uniformly on compact subsets of } \Omega.$$

Let u_λ be the minimal solution to (P_λ). Multiplying (P_λ) by φ_1 and integrating, we obtain:

$$0 = \int_\Omega \Big(\lambda_1 u_\lambda - \lambda f(u_\lambda)\Big)\varphi_1 dx = \int_\Omega \Big((\lambda_1 - a\lambda)u_\lambda - \lambda(f(u_\lambda) - au_\lambda)\Big)\varphi_1 dx.$$

This leads to:

$$\lambda \int_\Omega \varphi_1 \Big(f(u_\lambda) - au_\lambda\Big) dx \geq 0.$$

Taking the limit as λ approaches λ^*, in the last inequality, we find:

$$0 \leq l\lambda^* \int_\Omega \varphi_1 dx < 0,$$

which is a contradiction. Therefore, $\lambda^* \neq \frac{\lambda_1}{a}$.

Next, assume that $\lambda^* = \lambda_1/r$ and let u be a solution of problem (P_{λ^*}). Multiplying (P_{λ^*}) by φ_1 and integrating by parts, we have

$$\lambda_1 \int_\Omega u\varphi_1 dx = \frac{\lambda_1}{r} \int_\Omega f(u)\varphi_1 dx.$$

This implies:

$$\int_\Omega (f(u) - ru)\varphi_1 dx = 0.$$

Hence, we obtain $f(u) = r\,u$ in Ω, which implies $f(t) = rt$ for $t \in [0, \max_\Omega u]$. However, this contradicts the fact that $f(0) > 0$. Therefore, $\lambda^* \neq \frac{\lambda_1}{r}$.

This completes the proof of (i) in Theorem 1.4. □

4.2. Proof (ii). Since $\lambda^* > \lambda_1/a$, the existence of a solution to (P_{λ^*}) with λ^* is assured by Remark 3.3. It remains to prove the uniqueness. Assume that u is another solution to (P_{λ^*}) and let $w := u - u^*$. Since $u_\lambda < u$ and $\lim_{\lambda\to\lambda^*} u_\lambda = u^*$, we have $w \geq 0$.

By convexity of f, we have :

$$(-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1}w) = \lambda^*(f(u) - f(u^*)) \geq \lambda^* f'(u^*)w \quad \text{in } \Omega.$$

Recall that $\mu_1(\lambda^*, u^*) = 0$, so let ψ be the corresponding eigenfunction. Multiplying the last inequality by ψ and integrating by parts, we find:

$$0 = \int_\Omega \lambda^* \Big(f(u) - f(u^*) - f'(u^*)w \Big) \psi \geq 0.$$

Therefore, we must have equality $f(u) - f(u^*) = f'(u^*)w$ in Ω, which implies that f is linear in $[0, \max_\Omega u]$. However, this leads to a contradiction as in the proof of Theorem 1.2.

Hence, the solution u^* of (P_{λ^*}) is unique, and this completes the proof of (ii) in Theorem 1.4. □

4.3. Proof (iii). To establish the existence of a non-stable solution v_λ for (P_λ), we will make use of the mountain pass theorem introduced by Ambrosetti and Rabinowitz [5]. The theorem is stated as follows:

Theorem 4.1. *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. Assume that J satisfies the Palais-Smale condition and the following geometric assumptions:*

(*) *there exist positive constants R and ρ such that*

$$J(u) \geq J(u_0) + \rho, \text{ for all } u \in E \text{ with } \|u - u_0\| = R.$$

(**) *there exists $v_0 \in E$ such that $\|v_0 - u_0\| > R$ and $J(v_0) \leq J(u_0)$.*

Then the functional J possesses at least a critical point. The critical value is characterized by

$$c := \inf_{g\in\Gamma} \max_{u\in g([0,1])} J(u),$$

where

$$\Gamma := \Big\{ g \in C([0,1], E) : g(0) = u_0,\, g(1) = v_0 \Big\}$$

and satisfies $c \geq J(u_0) + \rho$.

In our case, we define the functional $\mathcal{J}_\lambda$ as follows:

$$\begin{array}{llll}\mathcal{J}_\lambda: & E & \longrightarrow & \mathbb{R}\\ & u & \longmapsto & \dfrac{1}{2}\displaystyle\int_\Omega |D^m u|^2 + \dfrac{1}{2}\int_\Omega \theta(x)|\nabla^{m-1}u|^2 - \int_\Omega F(u),\end{array}$$

where

$$F(t) = \lambda \int_0^t f(s)ds, \quad \text{for all } t \geq 0,$$

and E is the function space defined in (9). In this context, we choose u_0 to be the stable solution u_λ for each $\lambda \in (\lambda_{1/a}, \lambda^*)$. It is important to note that the operator D^m is given by:

$$D^m = \begin{cases} \nabla\Delta^{\frac{m-1}{2}} & \text{for } m \text{ odd} \\ \Delta^{\frac{m}{2}} & \text{for } m \text{ even} \end{cases},$$

Remark 4.1. The energy functional $\mathcal{J}_\lambda$ is continuously differentiable and its derivative is given by:

$$\langle \mathcal{J}'_\lambda(u), v\rangle = \int_\Omega D^m u \cdot D^m v + \int_\Omega \theta(x)\nabla^{m-1}u \cdot \nabla^{m-1}v - \lambda\int_\Omega f(u)v,$$

for all $u, v \in E$.

Since $\mu_1(\lambda, u_\lambda) > 0$, the function u_λ is a strict local minimum for $\mathcal{J}_\lambda$.Therefore, we can apply the mountain pass theorem to $\mathcal{J}_\lambda$.

Next, we will prove the compactness condition of $\mathcal{J}_\lambda$, which is known as the Palais-Smale condition.

Lemma 4.2. *Let $(u_n) \subset E$ be a Palais-Smale sequence, which means that it satisfies the following conditions:*

$$\sup_{n\in\mathbb{N}} |\mathcal{J}_\lambda(u_n)| < +\infty, \tag{10}$$

$$\|\mathcal{J}'_\lambda(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{11}$$

Then (u_n) is relatively compact in E.

Proof. Since any subsequence of (u_n) satisfies (10) and (11) it is enough to prove that (u_n) contains a convergent subsequence in E. Specifically, we aim to show that (u_n) contains a bounded subsequence in E.

Suppose that $\|u_n\| \to \infty$. Let $u_n = k_n w_n$ with $k_n \to \infty$ and $\|w_n\|_2 = 1$. Then we have

$$0 = \lim_{n\to\infty} \frac{\mathcal{J}_\lambda(u_n)}{k_n^2} = \lim_{n\to\infty}\left[\frac{1}{2}\int_\Omega \left(|D^m w_n|^2 + \theta(x)|\nabla^{m-1}w_n|^2\right) - \frac{1}{k_n^2}\int_\Omega F(u_n)dx\right].$$

However, since $|f(t)| \leq a|t| + b$, we have

$$|F(u_n)| = |F(k_n w_n)| \leq \frac{a\lambda}{2}k_n^2 w_n^2 + b\lambda|k_n w_n|.$$

This shows that

$$\frac{1}{k_n^2}\int_\Omega F(u_n)dx \leq \frac{a\lambda}{2}\int_\Omega w_n^2 dx + \frac{b\lambda}{k_n}\int_\Omega w_n dx < \infty.$$

We claim that

$$(-\Delta)^m u - \nabla^{m-1}(\theta(x)\nabla^{m-1}u) = a\lambda w^+ \quad \text{where } w^+ := \max\{0, w\}. \tag{12}$$

To prove this claim, we consider (11) divided by k_n, which gives

$$\int_\Omega \Big(D^m w_n \cdot D^m v + \theta(x)\nabla^{m-1} w_n \cdot \nabla^{m-1} v\Big) - \lambda \int_\Omega \frac{f(u_n)}{k_n} v dx \to 0, \tag{13}$$

for each $v \in E$. Now, we have

$$\int_\Omega \Big(D^m w_n \cdot D^m v + \theta(x)\nabla^{m-1} w_n \cdot \nabla^{m-1} v\Big) \to \int_\Omega \Big(D^m w \cdot D^m v + \theta(x)\nabla^{m-1} w \cdot \nabla^{m-1} v\Big).$$

Hence (12) can be concluded from (13) if we show that $1/k_n f(u_n)$ converges (up to a subsequence) to aw^+ in $L^2(\Omega)$.

Now, we have $1/k_n f(u_n) = 1/k_n f(k_n w_n)$ and it is easy to see that the required limit is equal to aw in the set $\{x \in \Omega : w_n(x) \to w(x) \neq 0\}$.

If $w(x) = 0$ and $w_n(x) \to w(x)$, let $\varepsilon > 0$ and n_0 be such that $|w_n(x)| < \varepsilon$ for $n \geq n_0$. Then

$$\frac{f(k_n w_n)}{k_n} \leq a\varepsilon + \frac{b}{k_n} \quad \text{for such } n,$$

which implies that the required limit is 0. Thus, $f(u_n)/k_n \to aw^+$ almost everywhere.

Here $b = f(0)$. Now $w_n \to w$ in $L^2(\Omega)$ and, thus, up to a subsequence, w_n is dominated in $L^2(\Omega)$ see ([7], Theorem IV.9).

Since $1/k_n f(u_n) \leq a|w_n| + 1/k_n b$, it follows that $1/k_n f(u_n)$ is also dominated. Hence (12) is obtained. Now, (12) and the maximum principle imply that $w \geq 0$ and (12) becomes

$$(-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1} w) = a\lambda\, w \quad \text{in } \Omega, \quad w \geq 0 \text{ in } \Omega, \quad \|w\|_2 = 1 \text{ in } \Omega.$$

Thus from (6), we have $\lambda a = \lambda_1$ and $w = \varphi_1$, which contradicts the fact that $\lambda \neq \lambda_1/a$.

The second step is to show that (u_n) admits a strongly convergent subsequence in E. Then, up to a subsequence, $u_n \to u$ weakly in E, strongly in $L^2(\Omega)$. Now (11) gives

$$(-\Delta)^m u_n - \nabla^{m-1}(\theta(x)\nabla^{m-1} u_n) - \lambda f(u_n) \to 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Note that $f(u_n) \to f(u)$ in $L^2(\Omega)$ because $|f(u_n) - f(u)| \leq a|u_n - u|$. This shows that

$$(-\Delta)^m u_n - \nabla^{m-1}(\theta(x)\nabla^{m-1} u_n) \to \lambda f(u) \quad \text{in } \mathcal{D}'(\Omega).$$

That is

$$(-\Delta)^m u - \nabla^{m-1}(\theta(x)\nabla^{m-1} u) - \lambda f(u) = 0.$$

The above equality multiplied by u gives

$$\int_\Omega |D^m u|^2 + \int_\Omega \theta(x)|\nabla^{m-1} u|^2 - \lambda \int_\Omega f(u)u = 0. \tag{14}$$

Now (11) multiplied by (u_n) gives

$$\int_\Omega |D^m u_n|^2 + \int_\Omega \theta(x)|\nabla^{m-1} u_n|^2 - \lambda \int_\Omega f(u_n)u_n = 0. \to 0 \tag{15}$$

in view of the boundedness of (u_n) and the $L^2(\Omega)$-convergence of u_n and $f(u_n)$. We have

$$\lambda \int_\Omega f(u_n)u_n \to \lambda \int_\Omega f(u)u.$$

Hence, (14) and (15) give

$$\int_\Omega \Big(|D^m u_n|^2 \theta(x) |\nabla^{m-1} u_n|^2 \Big) \to \int_\Omega \Big| D^m u_n|^2 + \theta(x)|\nabla^{m-1} u_n|^2 \Big),$$

which ensures us that $u_n \to u$ in E. Actually, it is enough to prove that (u_n) is (up to a subsequence) bounded in $L^2(\Omega)$. The $L^2(\Omega)$-boundedness of (u_n) implies that E-boundedness of (u_n) as can be seen by examining (10).

Therefore, (u_n) contains a convergent subsequence in E, and the proof is complete. □

To establish the validity of the two geometric assumptions of Theorem 4.1, we proceed as follows:

Firstly, consider u_λ which is a local minimum of $\mathcal{J}_\lambda$. This implies the existence of $R > 0$ such that for any $u \in E$ satisfying $\|u - u_\lambda\| = R$, we have $\mathcal{J}_\lambda(u) \geq \mathcal{J}_\lambda(u_\lambda)$. Consequently, we obtain the expression

$$\mathcal{J}_\lambda(u) - \mathcal{J}_\lambda(u_\lambda) = \mathcal{J}_\lambda''(u_\lambda)(u - u_\lambda, u - u_\lambda) + \rho,$$

where $\rho > 0$. Thus, u_λ becomes a strict local minimum for $\mathcal{J}$, establishing the validity of assumption $(*)$.

Next, using the definition of φ_1 given in (6), we have, $\forall\, t \in \mathbb{R}$

$$\mathcal{J}_\lambda(t\varphi_1) = \frac{\lambda_1}{2} t^2 - \int_\Omega F(t\varphi_1)$$

Considering that $\lim\limits_{t\to+\infty} (f(t) - a\,t)$ is finite, there exists $\beta \in \mathbb{R}$ such that

$$f(t) \geq a\,t + \beta, \quad \forall\, t > 0.$$

Hence, we deduce

$$F(t) \geq \frac{a\,\lambda}{2} t^2 + \beta\lambda t, \quad \forall\, t > 0.$$

This yields

$$\frac{\mathcal{J}_\lambda(t\varphi_1)}{t^2} \leq \Big(\frac{\lambda_1}{2} - \frac{a\lambda}{2} \Big) - \frac{\beta\lambda}{t} \int_\Omega \varphi_1,$$

which implies

$$\limsup_{t\to+\infty} \frac{1}{t^2}\, \mathcal{J}_\lambda(t\varphi_1) \leq \frac{\lambda_1 - a\lambda}{2} < 0, \quad \forall\, \lambda > \lambda_1/a.$$

Therefore,

$$\lim_{t\to+\infty} \mathcal{J}_\lambda(t\varphi_1) = -\infty.$$

Thus, there exists $v_0 \in E$ such that $\mathcal{J}_\lambda(v_0) \leq \mathcal{J}_\lambda(u_\lambda)$, which establishes the validity of assumption $(**)$.

Finally, let $\tilde{v}$ (respectively $\tilde{c}$) denote the critical point (respectively critical value) of $\mathcal{J}_\lambda$ Recall that the function $\tilde{v}$ belongs to E and satisfies

$$(-\Delta)^m \tilde{v} - \nabla^{m-1}(\theta(x)\nabla^{m-1}\tilde{v}) = \lambda f(\tilde{v}) \quad \text{in } \Omega \quad \text{and} \quad \mathcal{J}(\tilde{v}) = \tilde{c}.$$

The next Lemma states that the limit of a sequence of unstable solutions is also unstable

Lemma 4.3. *Let $u_n \rightharpoonup u$ in E and $\eta_n \to \eta$ such that $\mu_1(\eta_n, u_n) < 0$. Then, $\mu_1(\eta, u) < 0$.*

Proof. We begin by assuming that $\mu_1(\eta_n, u_n) < 0$, which implies the existence of a sequence (φ_n) in $\Sigma_m(\Omega)$ satisfying

$$\int_\Omega |D^m\varphi_n|^2 + \int_\Omega \theta(x)|\nabla^{m-1}\varphi_n|^2 \leq \eta_n \int_\Omega f'(u_n)\varphi_n^2 \quad \text{with} \int_\Omega \varphi_n^2 = 1. \tag{16}$$

Since $f' \leq a$, inequality (16) implies that the sequence (φ_n) is bounded in $\Sigma_m(\Omega)$. Therefore, there exists a subsequence, still denoted by (φ_n) for simplicity, such that $\varphi_n \rightharpoonup \varphi$ in $\Sigma_m(\Omega)$.

Taking the weak lower semicontinuity property of norms into account, we have

$$\int_\Omega |D^m\varphi|^2 \leq \liminf \int_\Omega |D^m\varphi_n|^2, \ \int_\Omega \theta(x)|\nabla^{m-1}\varphi|^2 \leq \liminf \int_\Omega \theta(x)|\nabla^{m-1}\varphi_n|^2.$$

Since $\|\varphi\|_2 = 1$, we obtain

$$\int \Omega|D^m\varphi|^2 + \int_\Omega \theta(x)|\nabla^{m-1}\varphi|^2 \leq \liminf \left(\int_\Omega |D^m\varphi_n|^2 + \int_\Omega \theta(x)|\nabla^{m-1}\varphi_n|^2\right).$$

Moreover, as $\eta_n \to \eta$, we have

$$\eta_n \int_\Omega f'(u_n)\varphi_n^2 \to \eta \int_\Omega f'(u)\varphi^2.$$

Combining the above inequalities and limits, we conclude that

$$\int_\Omega |D^m\varphi|^2 + \int_\Omega \theta(x)|\nabla^{m-1}\varphi|^2 \leq \eta \int_\Omega f'(u)\varphi^2.$$

Thus, $\mu_1(\eta, u) < 0$, which completes the proof. □

It is evident that the function v belongs to $C^{2m}(\overline{\Omega}) \cap E$ due to a bootstrap argument. In fact, the subsequent paragraph provides substantial additional information regarding the behavior of the unstable solution v_λ.

4.3.1. Proof (iii) (a). By contradiction, thanks to Lemma 3.1, we can assume the existence of a sequence of positive scalars (η_n) and a sequence (v_n) of unstable solutions to P_{η_n} such that $v_n \to v$ in $L^1_{\text{loc}}(\Omega)$ as $\eta_n \to \lambda_1/a$ for some function v.

Firstly, we claim that the sequence (v_n) cannot be bounded in E. If it were bounded, there would exist $w \in E$ such that, up to a subsequence,

$$v_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad v_n \to w \text{ strongly in } L^2(\Omega)$$

Consequently, we would have

$$(-\Delta)^m v_n - \nabla^{m-1}(\theta(x)\nabla^{m-1}v_n) \to (-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1}w) \text{ in } \mathcal{D}'(\Omega),$$
$$f(v_n) \to f(w) \text{ in } L^2(\Omega),$$

which implies that

$$(-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1}w) = \frac{\lambda_1}{a} f(w) \text{ in } \Omega.$$

Hence, $w \in E$ and solves $P_{\lambda_1/a}$. However, this contradicts the uniqueness of the solution to $P_{\lambda_1/a}$.

Now, since

$$(-\Delta)^m v_n - \nabla^{m-1}(\theta(x)\nabla^{m-1}v_n) = \eta_n f(v_n),$$

the unboundedness of (v_n) in E implies that this sequence is also unbounded in $L^2(\Omega)$. To see this, let us consider the decomposition $v_n = k_n w_n$, where $k_n > 0$, $\|w_n\|_2 = 1$, and $k_n \to \infty$. It follows that

$$(-\Delta)^m w_n - \nabla^{m-1}(\theta(x)\nabla^{m-1} w_n) = \frac{\eta_n}{k_n} f(v_n) \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega).$$

Hence, the sequence (w_n) is bounded in E due to standard arguments. Consequently, we obtain

$$(-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1} w) = 0 \quad \text{and} \quad \|w\|_2 = 1.$$

which leads to the desired contradiction since $w \in E$.

4.3.2. Proof (iii) (b). We end the proof by showing that v_λ tends to u^* uniformly in Ω when λ tends to λ^*.

As before, it is sufficient to prove the $L^2(\Omega)$ boundedness of v_λ near λ^* and utilize the uniqueness property of u^*. Suppose that $\|v_n\|_2 \to \infty$ as $\eta_n \to \lambda^*$, where v_n is a solution to (P_{η_n}). Once again, we express $v_n = l_n w_n$. Then, we have

$$(-\Delta)^m w_n - \nabla^{m-1}(\theta(x)\nabla^{m-1} w_n) = \frac{\eta_n}{l_n} f(v_n). \tag{17}$$

The boundedness of the right-hand side of (17) in $L^2(\Omega)$ implies that the sequence (w_n) is bounded in E. Let (w_n) be such that (up to a subsequence)

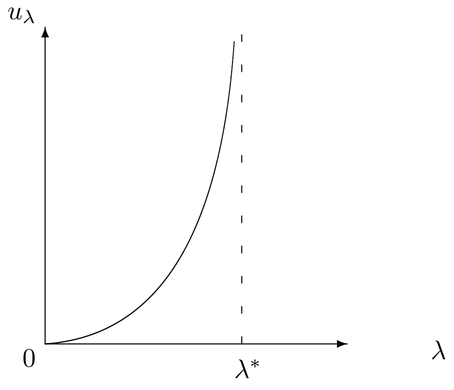
$$w_n \rightharpoonup w \text{ weakly in } E \quad \text{and} \quad w_n \to w \text{ strongly in } L^2(\Omega).$$

A previously established computation shows that

$$(-\Delta)^m w - \nabla^{m-1}(\theta(x)\nabla^{m-1} w) = \lambda^* a w, \quad w \geq 0 \text{ and } \|w\|_2 = 1,$$

which implies that $\lambda^* = \lambda_1/a$. This contradiction concludes the proof. □

In conclusion: The results obtained provide a comprehensive understanding of the solution behavior in the quasilinear case with $a \in (0, +\infty)$. An essential element in our arguments is the quantity $l := \lim_{t\to\infty}\Big(f(t) - at\Big)$, which plays a crucial role. Based on the sign of l, we can distinguish two distinct situations that have a significant impact on the solutions.



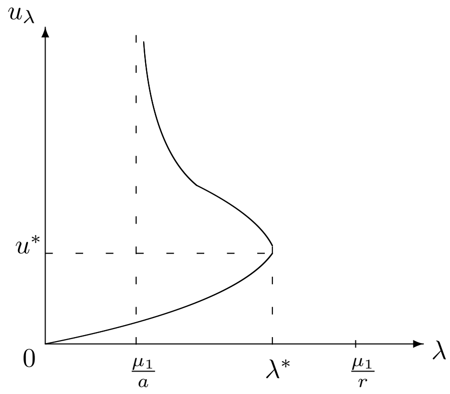


Fig 1 : Behavior of the minimal solution, $l > 0$. Fig 2 : Bifurcation branches, $l < 0$.

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(Imed Abid) University of Tunis El Manar, Higher Institut of Medicals Technologies of Tunis, Tunisia
E-mail address: imed.abid@istmt.utm.tn

(Sami Baraket) Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, 11623, Saudi Arabia
E-mail address: SMBaraket@imamu.edu.sa