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Optimization techniques in landslides modelling

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Abstract. The stationary anti-plane problem for a two dimensional Bingham fluid is considered. We take into account the inhomogeneous yield limit of the fluid, which is well adjusted to the description of landslides. The blocking property is analyzed and we introduce the safety factor which is connected to two optimization problems in terms of velocities and stresses. Concerning the velocity analysis the minimum problem is equivalent to a shape optimization problem. We describe a numerical method to compute the safety factor through this equivalence. For the stress optimization problem we give a stream function formulation in order to deduce a minimum problem in $W^{1,\infty}(\Omega)$ and we prove the existence of an minimizer. The $L^p(\Omega)$ approximation technique is used to get a sequence of minimum problems for smooth functionals. The finite element approach and a Newton method is used to obtain a numerical scheme for the safety factor. Some numerical results are given in order to compare the two methods. The shape optimization method is sharp in detecting the sliding zones but the convergence is very sensitive to the choice of the parameters. The stress optimization method is more robust, gives precise safety factors but the results cannot be easily compiled to obtain the sliding zone.

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1. Introduction

A lot of efforts have been devoted in analyzing and modelling landslides in order to evaluate the risk and to obtain a prediction. A stability analysis may provide information on the safety factor of stable mass of soil.

Recently the inhomogeneous (or density-dependent) Bingham fluid was considered in landslides modelling [8, 4, 9, 18]. This rigid visco-plastic model is very simple exhibiting only two constitutive constants: viscosity and yield stress. Another important advantage of using this model is the fact that the initial distribution of the stress in the soil is not required.

Although the Bingham model deals with fluids, it was also seen as a solid, called the "Bingham solid" (see for instance [25]) and investigated to describe the deformation and displacement of many solid bodies. The inhomogeneous yield limit is a key point in describing landslides phenomenon. Indeed, due to their own weight, the geomaterials are compacted so that the mechanical properties also vary with depth. Therefore the yield limit g and the viscosity coefficient η cannot be supposed homogeneous. In opposition to the previous works dealing only with homogeneous Bingham fluids [14, 19, 20, 24], we are interested here in a fluid whose yield limit is inhomogeneous.

A particularity of the Bingham model lies in the presence of rigid zones located in the interior of the flow of the Bingham solid/fluid. As the yield limit g increases,

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these rigid zones become larger and may completely block the flow. When modelling landslides, the solid is blocked in its natural configuration and the beginning of a flow can be seen as a "disaster". The "safety factor" was introduced in [17] to obtain a qualitative and quantitative evaluation of the blocking phenomenon. More precisely, through the safety factor we study the link between the yield limit distribution and the external forces distribution (or the mass density distribution) for which the flow of the Bingham fluid is blocked.

Let us give the outline of the paper. The stationary anti-plane problem (two dimensional) is considered in section 2. The variational formulations in terms of velocities and stresses are recalled from [18]. In section 3 the blocking property is analyzed and we introduce the safety factor which is connected to two optimization problems (velocity and stress formulations). The analysis in terms of velocities is given in section 4. Here we recall from [21] the equivalence between the minimum problem in $BV(\Omega)$ and a shape optimization problem. In section 5 we study the stress optimization problem. We give a stream function formulation in order to deduce a minimum problem in $W^{1,\infty}(\Omega)$ and we prove the existence of a minimizer. The $L^p(\Omega)$ approximation technique is used to get a sequence of minimum problems for smooth functionals. Finally, we propose a numerical approach following the analysis presented before. First, we describe a numerical method to compute the safety factor through the equivalence with the shape optimization problem. After that we use a finite element discretization and a Newton method to obtain a numerical scheme for the safety factor through the stress analysis. We compare and we analyze the two approaches through two numerical examples. The shape optimization method is sharp in detecting the sliding zones but the convergence is very sensitive to the choice of the parameters. The finite element method is more robust, giving more precise safety factor but the results cannot be easily compiled to obtain the sliding zone.

2. The anti-plane flow

We consider here the equations describing the stationary anti-plane flow of an inhomogeneous Bingham fluid in a domain $\mathcal{D} = \Omega \times \mathbb{R} \subset \mathbb{R}^3$, where Ω is a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$. We are looking for a flow in the Ox_3 direction, *i.e.* the velocity field $\boldsymbol{u}$ is given by $\boldsymbol{u} = (0, 0, u)$, where u does not depend on x_3 and t so that $u = u(x_1, x_2)$ (see Figure 1). The non-vanishing stress components are $\sigma_{13}(x_1, x_2), \sigma_{23}(x_1, x_2)$ denoted by $\boldsymbol{\sigma} = (\sigma_{13}, \sigma_{23})$. The momentum balance law in the Eulerian coordinates reads

$$\mathbf{div}\,\boldsymbol{\sigma} + f = 0 \quad \text{in } \Omega, \tag{1}$$

where f denotes the body forces in the x_3 direction. The rate deformation is described by ∇u and the constitutive equation of the Bingham fluid [3] can be written as follows:

$$\boldsymbol{\sigma} = \eta \nabla u + g \frac{\nabla u}{|\nabla u|} \qquad \text{if } |\nabla u| \neq 0, \tag{2}$$

$$|\boldsymbol{\sigma}| \leq g \qquad \text{if } |\nabla u| = 0, \tag{3}$$

where $\eta = \eta(x_1, x_2)$ is the viscosity distribution and $g = g(x_1, x_2)$ is a positive continuous function which stands for the yield limit distribution in $\mathcal{D}$. The behavior described by equations (2–3) can be observed in the case of some oils or sediments used in the process of oil drilling [14, 16]. The Bingham model, also denominated "Bingham solid" (see for instance [25]) was considered in order to describe the deformation of many solid bodies. For instance, in metal-forming processes, it was introduced in the

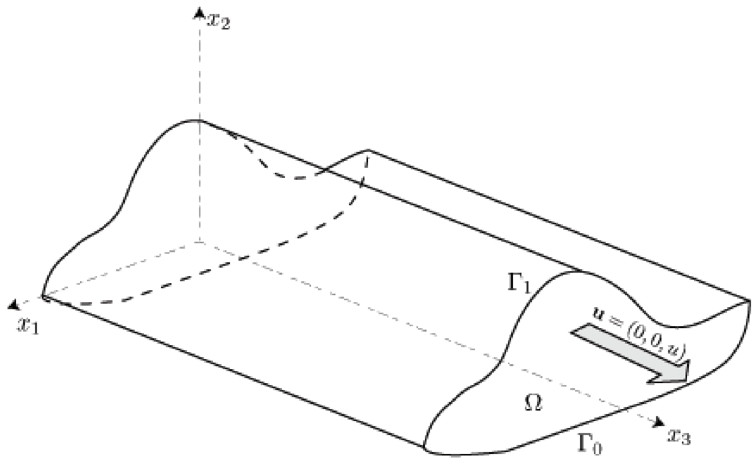

Figure 1: The anti-plane flow geometry.

study of wire drawing [7, 20]. Recently the inhomogeneous (or density-dependent) Bingham fluid was chosen in landslides modelling [4, 9, 18]. Note that in this latter case the fluid/solid cannot be supposed to be homogeneous as for oil drilling or metal forming modelling.

In order to complete equations (1–3) with some boundary conditions we assume that the boundary of Ω is divided into two parts $\Gamma = \Gamma_0 \cup \Gamma_1$. On Γ_0 we suppose an adherence condition and Γ_1 will be considered as a (stress) free surface (called also "rigid roof"). More precisely we have

$$u = 0 \quad \text{on} \quad \Gamma_0, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_1, \tag{4}$$

where $\boldsymbol{n}$ stands for the outward unit normal on $\partial\Omega$.

We suppose in the following that

$$f, g, \eta \in L^\infty(\Omega), \quad g(x) \geq g_1 > 0, \quad \eta(x) \geq \eta_0 > 0, \text{ a.e. } x \in \Omega.$$

If we define

$$V = \{v \in H^1(\Omega); \quad v = 0 \quad \text{on} \quad \Gamma_0\}$$

then the variational formulation for the anti-plane flow is

$$u \in V, \qquad \int_\Omega \eta(x) \nabla u(x) \cdot \nabla(v(x) - u(x))\, dx + \int_\Omega g(x)|\nabla v(x)|\, dx$$
$$- \int_\Omega g(x)|\nabla u(x)|\, dx \geq \int_\Omega f(x)(v(x) - u(x))\, dx, \qquad \forall v \in V. \tag{5}$$

The above problem is a standard variational inequality. If $\text{meas}(\Gamma_0) > 0$ then it has a unique solution u. If $\Gamma_0 = \emptyset$ and $\int_\Omega f(x)\, dx = 0$ then a solution exists and it is unique up to an additive constant. In the following we will always assume that the former holds; the other one can be deduced with obvious minor changes.

In order to give a variational formulation in terms of stresses for (5) we define

$$A_f = \{\boldsymbol{\tau} \in (L^2(\Omega))^2; \quad \text{div}\, \boldsymbol{\tau} = -f \quad \text{in } \ \Omega, \quad \boldsymbol{\tau} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_1\}, \tag{6}$$

where $\boldsymbol{\tau}\cdot\boldsymbol{n}$ is considered in $H^{-\frac{1}{2}}(\Gamma)$. Let $T : L^2(\Omega)^2 \to \mathbb{R}$ be defined by

$$T(\boldsymbol{\tau}) = \int_\Omega \frac{1}{2\eta(x)}[|\boldsymbol{\tau}(x)| - g(x)]_+^2\, dx, \tag{7}$$

where $[\quad]_+$ is the positive part. We recall from [18] the following result

Theorem 1. *i) There exists at least a* $\boldsymbol{\sigma} \in A_f$ *minimizing* T *on* A_f, i.e. $T(\boldsymbol{\sigma}) \le T(\boldsymbol{\tau})$, *for all* $\tau \in A_f$*, which is characterized by*

$$\boldsymbol{\sigma} \in A_f \quad \textit{and} \quad \int_\Omega \frac{[|\boldsymbol{\sigma}(x)| - g(x)]_+}{\eta(x)|\boldsymbol{\sigma}(x)|}\boldsymbol{\sigma}(x)\cdot\boldsymbol{\tau}(x)\, dx = 0, \quad \forall\, \boldsymbol{\tau} \in A_0, \tag{8}$$

(where A_0 *is* A_f *with* $f = 0$*).*

ii) Let u *be the solution of (5). Then we have*

$$\nabla u(x) = \frac{[|\boldsymbol{\sigma}(x)| - g(x)]_+}{\eta(x)|\boldsymbol{\sigma}(x)|}\boldsymbol{\sigma}(x), \quad \textit{a.e.} \quad x \in \Omega. \tag{9}$$

When considering a viscoplastic model of Bingham type, one can observe rigid zones (*i.e.* zones where $\nabla u = 0$) in the interior of the flow of the solid/fluid. The above theorem gives the opportunity to describe the rigid zones Ω_r and the shearing zones Ω_s defined by

$$\Omega_r = \{x \in \Omega;\quad |\nabla u(x)| = 0\}, \quad \Omega_s = \{x \in \Omega;\quad |\nabla u(x)| > 0\}.$$

From (9) we deduce $|\boldsymbol{\sigma}(x)| = g(x) + \eta(x)|\nabla u(x)|$ in Ω_s and that the solution $\boldsymbol{\sigma}$ of (8) is unique in Ω_s, (*i.e.* if $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$ are two solutions of (8) then $\boldsymbol{\sigma}_1(x) = \boldsymbol{\sigma}_2(x)$ a.e. $x \in \Omega_s$). For any $\boldsymbol{\sigma}$ solution of (8) we have

$$\Omega_r = \{x \in \Omega;\quad |\boldsymbol{\sigma}(x)| \le g(x)\}, \quad \Omega_s = \{x \in \Omega;\quad |\boldsymbol{\sigma}(x)| > g(x)\}. \tag{10}$$

3. The blocking property and the safety factor

The previous description of the rigid zones can be used to study the blocking property, *i.e.* when the whole Ω is a rigid zone ($\Omega = \Omega_r$). When g increases, the rigid zones are growing and if g becomes sufficiently large, the fluid stops flowing [16]. Commonly called the *blocking property*, such a behavior can lead to unfortunate consequences in oil transport in pipelines, in the process of oil drilling or in the case of metal forming. On the contrary, in landslides modelling, it is precisely the blocking phenomenon which ensures stability of the soil.

Proposition 1. *The following three statements are equivalent.*

i) *The Bingham fluid is blocked* i.e. $u \equiv 0$ *is the solution of (5).*

ii) *The blocking inequality holds:*

$$\int_\Omega g(x)|\nabla v(x)|\, dx \ge \int_\Omega f(x)v(x)\, dx, \qquad \forall v \in V. \tag{11}$$

iii) *There exists* $\boldsymbol{\sigma} \in A_f$ *such that* $|\boldsymbol{\sigma}(x)| \le g(x)$ *a.e.* $x \in \Omega$.

In order to give another characterization of the blocking property let us define:

$$B(v) := \frac{\displaystyle\int_\Omega g(x)|\nabla v(x)|\, dx}{\left|\displaystyle\int_\Omega f(x)v(x)\, dx\right|}, \qquad s := \inf_{v\in V} B(v) \tag{12}$$

$$S(\boldsymbol{\tau}) := \operatorname*{ess\,sup}_{x\in\Omega} \frac{|\boldsymbol{\tau}(x)|}{g(x)}, \qquad \mu := \inf_{\boldsymbol{\tau}\in A_f} S(\boldsymbol{\tau}). \tag{13}$$

Then we have the following result of [17].

Proposition 2. *The following equality holds*

$$s = \frac{1}{\mu}. \tag{14}$$

Moreover the Bingham fluid is blocked if and only if $s = 1/\mu \geq 1$.

In the case of landslides modelling $s = 1/\mu$ appears here as a *safety factor*. To see this one can consider a loading parameter t, *i.e.* we put tf in (1) instead of f and a family of variational inequalities where f is replaced by tf and $u_t \in V$ is the family of solutions:

$$u_t \in V, \qquad \int_\Omega \eta(x)\nabla u_t(x) \cdot \nabla(v(x) - u_t(x))\, dx + \int_\Omega g(x)|\nabla v(x)|\, dx -$$
$$\int_\Omega g(x)|\nabla u_t(x)|\, dx \geq t \int_\Omega f(x)(v(x) - u_t(x))\, dx, \qquad \forall v \in V. \tag{15}$$

We obtain the existence of a critical loading $t_{cr} = s$ which characterize the blocking phenomenon: the blocking occurs (i.e. $u_t \equiv 0$) if and only if $t \leq t_{cr}$.

4. Velocity analysis

Let us suppose, all over this section, that g and f are continuous functions on $\overline{\Omega}$. Since the trace map is not lower semi-continuous with respect to the weak* topology of $BV(\Omega)$ we have to relax the boundary condition $v = 0$ on Γ_0. Indeed, if we denote by

$$\mathcal{W} := \{v \in BV(\mathbb{R}^2) : v = 0 \text{ a.e. in } \mathbb{R}^N \setminus \overline{\Omega}\},$$

then for all $v \in \mathcal{W}$ there exist $\phi_n \in C_c^\infty(\mathbb{R}^2) \cap V$ such that

$$\int_\Omega f(x)\phi_n(x)\, dx \quad \to \quad \int_\Omega f(x)v(x)\, dx, \tag{16}$$

$$\int_\Omega g(x)\, |\nabla \phi_n(x)|\, dx \quad \to \quad \int_\Omega g(x)\, d\, |\nabla v|\, (x) + \int_{\Gamma_0} g(x)|v(x)|\, dS, \tag{17}$$

where $|\nabla v|$ is the variation measure of v and $|v|$ on Γ_0 have to be understood in the sense of the trace map on $BV(\Omega)$. That means that we have to introduce the boundary condition into the functional B, i.e. to extend B for all $v \in \mathcal{W}$ as follows

$$B(v) = \frac{\displaystyle\int_{\Omega\cup\Gamma_0} g(x)\, d\, |\nabla v|\, (x)}{\displaystyle\int_\Omega f(x)v(x)\, dx}. \tag{18}$$

Note that Γ_0 may be non-negligible with respect to the variation measure $|\nabla v|$. Let us also remark that if $v \in BV(\Omega)$ then

$$\int_{\Omega\cup\Gamma_0} g(x)\, d\, |\nabla \bar{v}|\, (x) = \int_\Omega g(x)\, d\, |\nabla v|\, (x) + \int_{\Gamma_0} g(x)|v(x)|\, dS, \tag{19}$$

where $\bar{v} : \mathbb{R}^2 \to \mathbb{R}$ is the function v extended by 0 outside Ω (*i.e.* $\bar{v}(x) = v(x)$ for $x \in \Omega$ and $\bar{v}(x) = 0$, for $x \in \mathbb{R}^2 \setminus \Omega$).

From the above equality it is clear now that B has an extension on $\mathcal{W}$, i.e. $B(\bar{v})$ given by (18) coincides with $B(v)$ given by (12) for all $v \in V$. Then we have [21]:

Theorem 2. *There exists* $v^* \in \mathcal{W}$ *such that*

$$s = B(v^*) = \min_{v \in \mathcal{W}} B(v). \tag{20}$$

Remark 4.1. *In the special case* $f \equiv 1 \equiv g$ *and* $\Gamma_1 = \emptyset$ *the minimum in (20) is the first eigenvalue of the so-called* 1*-laplacian operator* [12, 13, 22]*. The latter is the limit, as* $p \to 1$*, of the p-laplacian operator whose first eigenvalue* $\lambda_p(\Omega)$ *can be estimated from below by the Cheeger constant* $h(\Omega) = \inf_{\omega \subset \Omega} |\partial\omega|/|\omega|$ *with* $\lambda_p(\Omega) \geq (h(\Omega)/p)^p$*, see* [23]*. For* $p = 2$*, this is the well-known Cheeger's inequality* [6]*, which was the initial motivation for the study of the Cheeger's problem.*

We consider now the functional $\mathcal{J}$ defined for open sets $\omega \subset \Omega$ with regular boundary by:

$$\mathcal{J}(\omega) \;=\; \frac{\int_{\partial\omega \setminus \Gamma_1} g(x)\, dS}{\int_\omega f(x)\, dx}. \tag{21}$$

Let $\omega \subset \Omega$ be given and 1_ω be its characteristic function. Then one can check that $1_\omega \in \mathcal{W}$ and since $\partial\omega \cap (\Omega \cup \Gamma_0) = \partial\omega \setminus \Gamma_1$ we have $\mathcal{J}(\omega) = B(1_\omega)$. The integrals in (21) can be considered for any set ω with finite perimeter (that is, such that its characteristic function 1_ω belongs to $BV(\mathbb{R}^N)$). Hence we may extend the definition of $\mathcal{J}(\omega)$ for these sets. In this case $\partial\omega$ has to be replaced by the reduced boundary $\partial^*\omega$ of ω, (see [15, section 5.7]) and we can write $\int_{\partial^*\omega \setminus \Gamma_1} g(x)\, dS$, instead of $\int_{\Omega \cup \Gamma_0} g\, d\,|\nabla 1_\omega|$. Even if the set ω does not have finite perimeter, then the integral on the boundary $\partial^*\omega$ can be considered infinite and we will define $\mathcal{J}(\omega) = +\infty$ regardless of the value of $\int_\omega f(x)\, dx$. Since we shall investigate a minimization problem for $\mathcal{J}$, such a set ω is not relevant, because it is not a minimizer. Finally we have

$$\mathcal{J}(\omega) = B(1_\omega), \quad \forall \omega \in \mathcal{O},$$

where $\mathcal{O}$ is the set of open subsets of Ω with finite perimeter. We denote in the following by $\mathcal{O}_1 \subset \mathcal{O}$, the set of simply connected open subsets of Ω.

We recall from [21] the link between the blocking inequality (11) and a shape optimization problem (*i.e.* a minimum problem for $\mathcal{J}$), and some existence and regularity results:

Theorem 3. *We have*

$$s = \inf_{\omega \in \mathcal{O}} \mathcal{J}(\omega). \tag{22}$$

Moreover if Ω *is simply connected then the infimum in (22) is attained by some simply connected open set* X*,* i.e.

$$s = \mathcal{J}(X) = \min_{\omega \in \mathcal{O}_1} \mathcal{J}(\omega). \tag{23}$$

Additionally, if $g \in C^1(\overline{\Omega})$*, then any minimizer* X *of* $\mathcal{J}$ *has a boundary of class* C^2 *in* Ω *and* C^1 *in any point* $x_0 \in \Gamma_0$ *where the tangent cone* $K(x_0) = \cup_{\lambda>0} \lambda(\Omega - x_0)$ *is a convex set. If* ∂X *crosses* Γ_1 *at some point* $x_0 \in \Gamma_1$ *where* Γ_1 *is* C^1*, then* ∂X *has a tangent line orthogonal to* Γ_1 *at* x_0*.*

We shall investigate now the physical interpretation of the optimal subset X. For this we consider u_t the family of solutions of (15) and let

$$\Omega_t^0 := \{x \in \Omega; u_t(x) = 0\}, \quad \Omega_t^{sl} := \{x \in \Omega; u_t(x) \neq 0\} = \Omega \setminus \Omega_t^0 \tag{24}$$

be the family of subsets of Ω where the fluid is at rest or sliding, respectively. As it follows from the previous section we have $\Omega_t^0 = \Omega, \Omega_t^{sl} = \emptyset$ for all $t \in (0, s]$. We conjecture the following result:

Conjecture. *The optimal subset X is the part of the land which slides whenever the loading parameter t becomes greater than s. More precisely, there exists* $\lim_{t\to s+}\Omega_t^{sl}$ *and*

$$X = \lim_{t\to s+}\Omega_t^{sl} = \Omega \setminus \lim_{t\to s+}\Omega_t^{0}$$

is a solution of the shape optimization problem (23).

As it's proved in [17], the conjecture is true for the one dimensional flow ($\Omega \subset \mathbb{R}$) between an infinite plane ($x = 0$) and a rigid roof ($x = \ell$) which models landslides on a natural slope (see [8]). In this case the body forces f are positive and are given by $f(x) = \gamma\rho(x)\sin\theta > 0$, where θ is the angle of the slope, $\rho(x) > 0$ is the mass density distribution and γ is the vertical gravitational acceleration.

5. Stress analysis

We shall suppose in this section that Γ_1 is simply connected and $|\Gamma_1| > 0$. From Proposition 2 we have that the problem of the safety factor $s = 1/\mu$ in terms of stresses reduces to

$$\mu \;=\; \inf_{\boldsymbol{\tau}\in A_f} S(\boldsymbol{\tau}). \tag{25}$$

We shall suppose in the following that $A_f^\infty := A_f \cap (L^\infty(\Omega))^2$ is not empty and let $\boldsymbol{\tau}_f = (\tau_1^f, \tau_2^f) \in A_f^\infty$. Then we have $A_f^\infty = A_0^\infty + \boldsymbol{\tau}_f$, where A_0^∞ is A_f^∞ with $f = 0$. For all $\boldsymbol{\tau} = (\tau_1, \tau_2) \in A_0^\infty$ the condition $\partial_{x_1}\tau_1 + \partial_{x_2}\tau_2 = 0$ implies that there exists a function ϕ (the stream function) such that $\partial_{x_1}\phi = -\tau_2$, $\partial_{x_2}\phi = \tau_1$ in Ω. The condition $\boldsymbol{\tau}\cdot\boldsymbol{n} = 0$ on Γ_1 means that the tangential derivative of ϕ on Γ_1 is vanishing. This means that ϕ is constant on Γ_1 and since ϕ is only defined up to an additive constant, we can assume $\phi = 0$ on Γ_1.

In order to give a formulation of (25) in terms of stream functions we define

$$W_\infty := \{\phi \in W^{1,\infty}(\Omega) \mid \phi = 0 \text{ on } \Gamma_1\},$$

$$P(\phi) := \operatorname*{ess\,sup}_{x\in\Omega} \frac{|\nabla\phi(x) + \boldsymbol{\sigma}_f(x)|}{g(x)},$$

where $\boldsymbol{\sigma}_f = (-\tau_2^f, \tau_1^f) \in L^\infty$. Since for all $\phi \in W_\infty$ we have $\boldsymbol{\tau} = (\partial_{x_2}\phi, -\partial_{x_1}\phi) \in A_0^\infty$ and $S(\boldsymbol{\tau} + \boldsymbol{\tau}_f) = P(\phi)$ we can replace (25) with a minimization problem for P on the space of the stream functions.

Since the integrand $x \to |A + \boldsymbol{\sigma}_f(x)|/g(x)$ is not regular enough we cannot directly apply a result of [1] to deduce the lower semicontinuity of P with respect to the weak-$*$ topology in $W^{1,\infty}(\Omega)$. This property follows from the uniform coercivity property of the integrand (see [5]). More precisely we have the following result of [17]:

Proposition 3. *There exists $\psi \in W_\infty$ solution of*

$$\mu = P(\psi) = \min_{\phi\in W_\infty} P(\phi). \tag{26}$$

In order to use the L^p approximation method we introduce the following sequence of spaces and functionals for $p > 1$

$$W_p = \{\phi \in W^{1,p}(\Omega) \mid \phi = 0 \text{ on } \Gamma_1\}$$

$$Q_p(\phi) = \frac{1}{|\Omega|} \int_\Omega \frac{|\nabla\phi(x) + \boldsymbol{\sigma}_f(x)|^p}{g(x)^p}\, dx, \quad P_p(\phi) = Q_p(\phi)^{1/p}.$$

Theorem 4. *For all $p > 1$ there exists an unique solution ψ_p of the optimization problem for P_p*

$$\mu_p := P_p(\psi_p) = \min_{\phi\in W_p} P_p(\phi). \tag{27}$$

The minimum μ_p converges to μ as p goes to ∞. Moreover there exist a subsequence of $(\psi_p)_{p>1}$, again denoted by $(\psi_p)_{p>1}$ and $\psi \in \mathcal{W}_\infty$ a solution of (26) such that $\psi_p \rightharpoonup \psi$ in W_q for all $q > 1$.

The proof given in [17] does not make use of the Γ-convergence method introduced in [11] (see [10] for more details). Since the integrand does not satisfy the linear growth condition given in [5, Proposition 3.6] we need to give here a direct proof.

6. Numerical approach

In this section we present some numerical results obtained for the two approaches described in sections 4 and 5 in order to compare them. It is not our intention to give here a numerical analysis (convergence, optimal parameters, ...) for the numerical methods described below. We just want to compare velocity and stress analysis through some numerical experiments in order to get some general features of these two approaches.

6.1. Shape optimization approach. We give here a numerical approach of the blocking property following the velocity analysis (described in section 4). For this, let us consider $r : \Omega \to \mathbb{R}$ such that

$$\frac{\partial r}{\partial x_2}(x) = f(x), \quad \forall x = (x_1, x_2) \in \Omega.$$

Then the integral on ω can be reduced on its boundary

$$\int_\omega f(x)\, dx = -\int_{\partial\omega} r(x)\, dx_1.$$

In order to discretize the shape optimization problem (22) let $n \in \mathbb{N}$ and let us consider γ_n a piecewise linear Jordan curve with n edges $[x^i, x^{i+1}]$, $1 \le i \le n$ (where $x^{n+1} = x^1$). We denote by ω_n the interior of γ_n, *i.e.* $\gamma_n = \partial\omega_n$. Let us consider $F_n, G_n : \Omega^n \to \mathbb{R}$:

$$F_n(x^1, ..., x^n) = -\sum_{i=1}^n (x_1^{i+1} - x_1^i) \int_0^1 r(x^i + t(x^{i+1} - x^i))\, dt$$

$$G_n(x^1, ..., x^n) = \sum_{i=1}^n \eta_i |x^{i+1} - x^i| \int_0^1 g(x^i + t(x^{i+1} - x^i))\, dt$$

where $\eta_i = 0$ if $[x^i, x^{i+1}] \subset \Gamma_1$ and $\eta_i = 1$ otherwise. Then we can define $J_n : \Omega^n \to \mathbb{R}$ as

$$J_n(x^1, ..., x^n) := G_n(x^1, ..., x^n)/F_n(x^1, ..., x^n) = \mathcal{J}(\omega_n).$$

The discretization of (22) consists of constructing a sequence X^n of domains with the boundary $\partial X^n = \bigcup_{i=1}^n [z^i, z^{i+1}]$ a piecewise linear Jordan curve with n edges such that

$$s_n = J_n(z^1, ..., z^n) = \min_{(x^1,...,x^n)\in\Omega^n} J_n(x^1, ..., x^n). \tag{28}$$

We have considered a shape discretization of 200 points. Since it is not the goal of this paper to discuss the more appropriate method for the non-convex optimization problem (28) we have chosen a basic one: the gradient method. As expected, we have remarked that a large number of iterations are needed but the method is quite precise. On the other hand, in some cases the method is converging to local minima different from the global minimum. Even with a choice of an initial shape close to the optimal shape, the step has to be very small to ensure the convergence to the global minimum.

6.2. Supremal functional approach. As it follows from section 5 the sequence of optimization problems for the functional Q_p

$$\mu_p^p = Q_p(\psi_p) = \min_{\phi \in W_p} Q_p(\phi), \tag{29}$$

approximates the stream minimization problem (26).

Our aim now is to obtain a finite element approximation of the problem (29). Let be given a family of finite dimensional subspaces $W^h \subset W_\infty$ where h denotes the discretization parameter. The finite dimensional problem consists then of finding $\psi_{ph} \in W^h$ such that:

$$\psi_{ph} \in W_h, \quad Q_p(\psi_{ph}) = \min_{\phi_h \in W^h} Q_p(\phi_h). \tag{30}$$

In order to solve the optimization problem (30) we shall use a classical Newton iterative method. Hence we shall consider a sequence of linear algebraic systems with the solutions $(\psi_{ph}^n)_n$ recursively defined

$$\psi_{ph}^n \in W^h, \quad Q_p''(\psi_{ph}^n)(\psi_{ph}^{n+1} - \psi_{ph}^n; \phi_h) = -Q_p'(\psi_h^n)(\phi_h), \quad \forall \phi_h \in W^h, \tag{31}$$

where Q_p' and Q_p'' denotes the gradient and the Hessian matrix given by:

$$Q_p'(\psi)(\phi) = \frac{p}{|\Omega|} \int_\Omega g^{-p} |\nabla\psi + \boldsymbol{\sigma}_f|^{p-2} (\nabla\psi(x) + \boldsymbol{\sigma}_f) \cdot \nabla\phi \, dx,$$

$$Q_p''(\psi)(\phi;\varphi) = \frac{p}{|\Omega|} \int_\Omega g^{-p} |\nabla\psi + \boldsymbol{\sigma}_f|^{p-4} \Big\{ (p-2)[(\nabla\psi + \boldsymbol{\sigma}_f) \cdot \nabla\phi]$$
$$[(\nabla\psi + \boldsymbol{\sigma}_f) \cdot \nabla\varphi] + |\nabla\psi + \boldsymbol{\sigma}_f|^2 \nabla\phi \cdot \nabla\varphi \Big\} \, dx.$$

For the stress analysis we have used an uniform mesh of 100×100 finite elements. The parameter p was chosen up to 200 and a rapid convergence of the Newton method was remarked. As a matter of fact we have changed p at each iteration to ensure the convergence with respect to these two parameters in the same time. Due to the L^p approximation, in some cases, the distribution of $|\nabla\psi_{ph}^n + \boldsymbol{\sigma}_f|/g$ in Ω is not sharp enough to distinguish the sliding and rigid zones.

6.3. Numerical results and comparison. A rectangular domain $\Omega = (-1, 1) \times (-1, 1)$ with the free surface (rigid roof) $\Gamma_1 = (-1, 1) \times \{1\}$ was the geometry we have considered. Firstly we have considered $f = g = 1$, *i.e.* the homogeneous case. The computed safety factors are very close $\mu = 1/s = 0.71037$ for the shape optimization and $\mu = 1/s = 0.70954$ for the other method. The computed optimal set X (or the sliding domain) has as interior boundary two quarters of a circle (see Figure 2). This fact, which was proved in [21], is not so evident in the stress analysis.

In the second example we have considered the nonhomogeneous case with functions which depend on the vertical variable y. The distribution is linear with respect to the depth for the body forces $f(y) = 3 + 2(1 - y)$ and quadratic for the yield stress

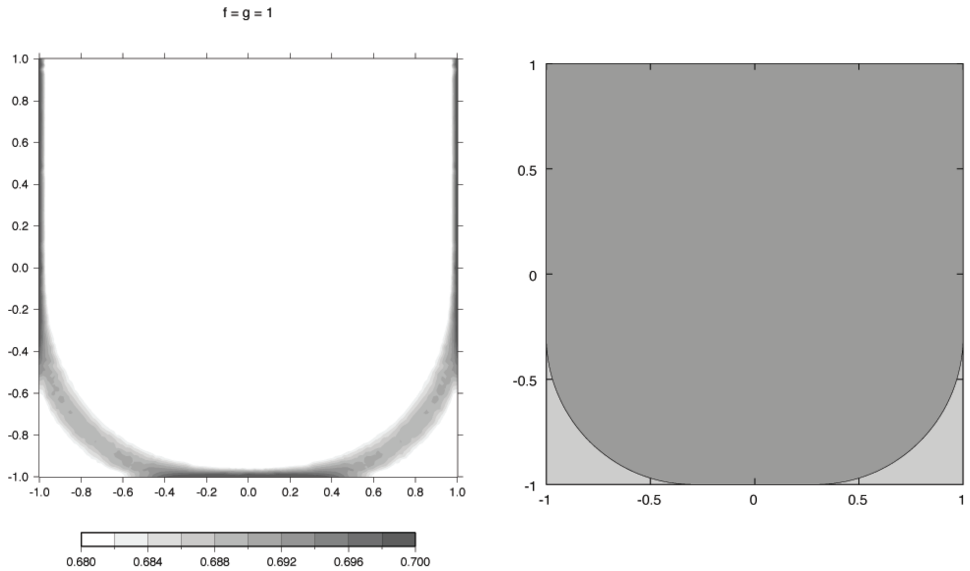

Figure 2: The homogeneous case $f = g = 1$ on $\Omega = (-1, 1) \times (-1, 1)$ (light grey) with the (stress) free surface $\Gamma_1 = (-1, 1) \times \{1\}$. Left: the computed distribution of $|\nabla\psi_p + \boldsymbol{\sigma}_f|/g$ solution of the stream optimization problem (27) with $\mu = 1/s = 0.70954$. Right: the computed domain X (dark grey) solution of the shape optimization problem (23) with $\mu = 1/s = 0.71037$.

$g(y) = 2 + 2(1 - y)^2$ as proposed in [4]. The shape optimization method and the stream function method gives close results for the safety factors $\mu = 1/s = 0.66461$ and $\mu = 1/s = 0.66917$ respectively. We remark that in this case the sliding domain (optimal set X) is smaller and does not touch the bottom (see Figure 3).

If we compare the two approaches the conclusions are:

(1) The shape optimization method is sharp in detecting the zones in sliding but the convergence to a global minimum, which is very sensitive to the choice of the parameters, is not ensured.

(2) The stress optimization method is more robust, gives precise safety factors but the results cannot be easily compiled to obtain the sliding zone.

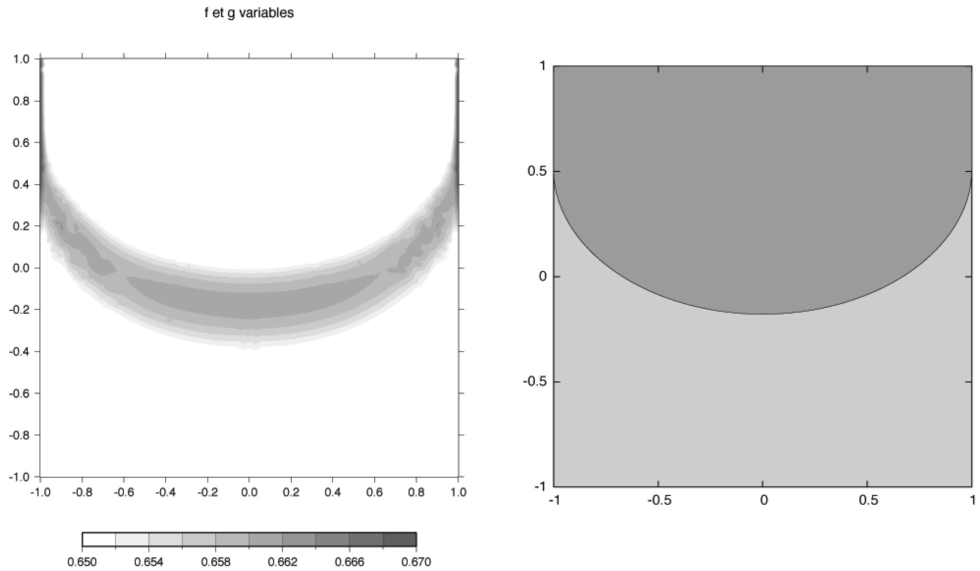

Figure 3: The domain $\Omega = (-1,1) \times (-1,1)$ (light grey) with the (stress) free surface $\Gamma_1 = (-1,1) \times \{1\}$ with non-homogeneous body forces and yield limit $f = f(y) = 3 + 2(1-y)$ and $g = 2 + 2(1-y)^2$. Left : the computed distribution of $|\nabla \psi_p + \boldsymbol{\sigma}_f|/g$ solution of the stream optimization problem (27) with $\mu = 1/s = 0.66917$. Right : the computed domain X (dark grey) solution of the shape optimization problem (23) with $\mu = 1/s = 0.66461$.

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