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Lower and Upper Bounds of Integral Mean Estimate for Polar Derivative of a Polynomial

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Abstract. If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for $r \ge 1$, Aziz [J. Approx. Theory, 55 (1988), 232–239] proved

$$\left\{\int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta\right\}^{\frac{1}{r}} \max_{|z|=1} |p'(z)| \ge n \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}},$$

whereas, Devi et al. [Note Mat., 41 (2021), 19–29] proved that if $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \le 1$, then for $r > 0$,

$$k^n n \left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{\frac{1}{r}} \le \left\{\int_0^{2\pi} |e^{i\theta} + k^n|^r d\theta\right\}^{\frac{1}{r}} \left\{n \max_{|z|=1} |p(z)| - \max_{|z|=1} |p'(z)|\right\},$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, where $q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.
In this paper, we not only obtain improved extensions of the above inequalities into polar derivatives by involving the leading coefficient and the constant term of the polynomial but also give generalized integral extensions of inequalities on polar derivatives recently proved by Singh et al. [Complex Anal. Synerg. 9 (2023), Art. 3].

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1. Introduction

The study of extremal problems of functions of complex variables and the results where some approaches to obtaining the polynomial inequalities are developed using various methods of the geometric function theory is a fertile area in analysis for researchers.

Historically, the question relating to polynomials, for example, the solution of polynomial equations and the approximation by polynomials give rise to some of the most important problems of the day. The well-known Russian mathematician Chebyshev (1821-1894) studied some properties of polynomials with the least deviation from a given continuous function and introduced the concept of best approximation in mathematical analysis. Various interesting inequalities concerning the estimate of the sup-norm of the derivative as an upper bound in terms of the sup-norm of the polynomial itself known as Bernstein-type inequality plays a key role in the literature

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for proving the inverse theorems in approximation theory (see Borwein and Erdélyi [6], Ivanov [17], Lorentz [19], Telyakovskii [36]), and of course, have their intrinsic interests. The first result in this area was connected with some investigation of the well-known Russian chemist Mendeleev [24]. In fact, Mendeleev's problem was to determine $\max_{-1\le x\le 1}|p'(x)|$, where $p(x)$ is a quadratic polynomial of real variable x with real coefficients and satisfying $-1\le p(x)\le 1$ for $-1\le x\le 1$. He was able to prove that if $p(x)$ is a quadratic polynomial and $|p(x)|\le 1$ on $[-1,1]$, then $|p'(x)|\le 4$ on the same interval. A. A. Markov [23] generalized this result for a polynomial of degree n in the real line. In fact, he proved that if $p(x)$ is an algebraic polynomial of degree at most n with real coefficients, then

$$\max_{-1\le x\le 1}|p'(x)|\le n^2\max_{-1\le x\le 1}|p(x)|.$$

After about twenty years, Bernstein [5] needed the analog of Markov's Theorem for the unit disk in the complex plane instead of the interval $[-1,1]$ to prove an inverse theorem of approximation (see Borwein and Erdélyi [6, p. 241]) to estimate how well a polynomial of a certain degree approximates a given continuous function in terms of its derivatives and Lipschitz constants. This leads to the famous well-known result known as Bernstein's inequality which states that if $t\in\tau_n$ (the set of all real trigonometric polynomials of degree at most n), then for $K:=[0,2\pi)$,

$$\max_{\theta\in K}|t^{(m)}(\theta)|\le n^m\max_{\theta\in K}|t(\theta)|. \tag{1}$$

The above inequality remains true for all $t\in\tau_n^c$ (the set of all complex trigonometric polynomials of degree at most n), which implies, as a particular case, the following algebraic polynomial version of Bernstein's inequality on the unit disk.

Theorem 1.1. *If $p(z)$ is a polynomial of degree n, then*

$$\max_{|z|=1}|p'(z)|\le n\max_{|z|=1}|p(z)|. \tag{2}$$

Equality holds in (2) if and only if $p(z)$ has all its zeros at the origin.

It is really of interest both in theoretical and practical aspects that continuous functions are approximated by polynomials. One such approach of approximation is made through the applications of Bernstein's inequality, particularly the trigonometric version and in this regard, we have the following interesting result (Theorem 1.2) [6, p. 241, Part (a) of E.18] which approximates m times differentiable real-valued function on a half-closed interval $[0,2\pi)$ by trigonometric polynomials. For the sake of convenience of the readers, we state the above result more precisely.

Let Lip_α, $\alpha\in(0,1]$, denote the family of all real-valued functions g defined on K satisfying

$$\sup\left\{\frac{|g(x)-g(y)|}{|x-y|^\alpha}:x\ne y\in K\right\}<\infty.$$

If $C(K)$ denotes the set of all continuous functions on K, then for $f\in C(K)$, let

$$E_n(f):=\inf\left\{\sup_{\theta\in K}|t-f|:t\in\tau_n\right\}.$$

Theorem 1.2. *(Direct theorem) Suppose f is m times differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0,1]$. Then there is a constant C depending only on f so that*

$$E_n(f) \leq Cn^{-(m+\alpha)}, \qquad n = 1, 2, \dots .$$

On the other hand, the converse (inverse) of Theorem 1.2 is essentially of interest and is stated below.

Theorem 1.3. *(Inverse theorem) Suppose m is a non-negative integer, $\alpha \in (0,1)$, and $f \in C(K)$. Suppose there is a constant $C > 0$ depending only on f such that*

$$E_n(f) \leq Cn^{-(m+\alpha)}, \qquad n = 1, 2, \dots .$$

Then f is m times continuously differentiable on K and $f^{(m)} \in \mathrm{Lip}_\alpha$.

The proof of Theorem 1.3 is done by the application of the well-known result due to Bernstein (inequality (1)) given in [6].

From the above discussion, it is worth to note that Bernstein and Markov-type inequalities play a significant role in approximation theory. Direct and inverse theorems of approximation and related matters may be found in many books on approximation theory, including Cheney [8], Lorentz [19], and DeVore and Lorentz [11].

Moreover, inequality (2) shows how fast a polynomial of degree at most n can change, and is of interest both in mathematics, especially in approximation theory, and in the application areas such as physical systems. Various analogs of this inequality are known in which the underlying intervals, the sup-norms, and the families of polynomials, are replaced by more general sets, norms, and families of functions, respectively. One such generalization is replacing the sup-norm by a factor involving integral mean.

Let $p(z)$ be a polynomial of degree n over the set of complex numbers and $q(z)$ represent the polynomial $z^n\overline{p\left(\frac{1}{\bar{z}}\right)}$. For each real number $r > 0$, we define the integral mean of $p(z)$ on the unit circle $|z| = 1$ by

$$\|p\|_r = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}} .$$

If we take limit as $r \to \infty$ in the above equality and make use of the well-known fact from analysis [31, 35] that

$$\lim_{r\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|p(e^{i\theta})\right|^r d\theta \right\}^{\frac{1}{r}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Similarly, we can define

$$\|p\|_0 = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |p(e^{i\theta})| d\theta \right\},$$

and it follows easily that $\lim_{r\to 0^+} \|p\|_r = \|p\|_0$. It would be of further interest that by taking limit as $r \to 0^+$ the stated results on integral mean inequalities holding for $r > 0$, hold for $r = 0$ as well.

Inequality (2) can be obtained by letting $r \to \infty$ in the inequality

$$\|p'\|_r \leq n\|p\|_r, \quad r > 0. \tag{3}$$

Inequality (3) was proved by Zygmund [38] for $r \geq 1$, and by Arestov [1] for $0 < r < 1$.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then inequalities (2) and (3) can be respectively improved as

$$\|p'\|_\infty \leq \frac{n}{2}\|p\|_\infty \tag{4}$$

and

$$\|p'\|_r \leq \frac{n}{\|1+z\|_r}\|p\|_r, \ r > 0. \tag{5}$$

Inequality (4) was conjectured by Erdös and later verified by Lax [18], whereas inequality (5) was proved by de-Bruijn [9] for $r \geq 1$, and by Rahman and Schmeisser [29] for $0 < r < 1$.

On the other hand, in 1939 (see [37]), Turán obtained a lower bound for the maximum of $|p'(z)|$ on $|z| = 1$, by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then

$$\|p'\|_\infty \geq \frac{n}{2}\|p\|_\infty. \tag{6}$$

As a generalization of (6), Govil [14] proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq k$, $k \geq 1$, then

$$\|p'\|_\infty \geq \frac{n}{1+k^n}\|p\|_\infty. \tag{7}$$

Whereas for the class of polynomials not vanishing in $|z| < k$, $k \leq 1$, the precise upper bound estimate for the maximum of $|p'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable. For quite some time, it was believed that if $p(z)$ has no zero in $|z| < k$, $k \leq 1$, then the inequality that generalizes (4) should be

$$\|p'\|_\infty \leq \frac{n}{1+k^n}\|p\|_\infty,$$

until E. B. Saff gave the example $p(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)$ to counter this belief.

Thus, the approximation does not seem to be known in general, and this problem is still open. However, some special cases in this direction have been considered by many people where some partial extensions of (4) are established. In 1980, it was again Govil [13], who generalized (4) with an extra condition by proving that if $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \leq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1+k^n}\|p\|_\infty, \tag{8}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

For the first time in 1984, Malik [20] extended inequality (6) proved by Turán [37] into integral mean version and proved that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then for $r > 0$,

$$\|1+z\|_r\|p'\|_\infty \geq n\|p\|_r.$$

The result is sharp and equality holds for $p(z) = (z+1)^n$.

In 1988, Aziz [2] obtained the integral mean extension of inequality (7) and proved

Theorem 1.4. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k, k \ge 1$, then for $r \ge 1$,*

$$\|1 + k^n z\|_r \|p'\|_\infty \ge n\|p\|_r. \tag{9}$$

The result is sharp and equality holds for $p(z) = \alpha z^n + \beta k^n, |\alpha| = |\beta|$.

On the other hand, Devi et al. [10] established the integral analog of inequality (8) and proved

Theorem 1.5. *If $p(z)$ is a polynomial of degree n having no zero in $|z| < k, k \le 1$, then for $r > 0$,*

$$k^n n\|p\|_r \le \|z + k^n\|_r\{n\|p\|_\infty - \|p'\|_\infty\}, \tag{10}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Before proceeding to some other results, let us introduce the concept of the polar derivative involved. For a polynomial $p(z)$ of degree n, we define

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z),$$

the polar derivative of $p(z)$ with respect to the point α (see [22] and [12, Chap. 6]). The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative $p'(z)$ in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p^{'}(z),$$

uniformly with respect to z for $|z| \le R$, $R > 0$.

Various results of majorization on the polar derivative of a polynomial can be found in the comprehensive books of Milovanović et al. [25], Marden [22], and Rahman and Schmeisser [28], where some approaches to obtaining polynomial inequalities are developed by applying the methods and results of the geometric function theory.

In 1998, Aziz and Rather [3] established the polar derivative generalization of (7) by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every complex number α with $|\alpha| \ge k$,

$$\|D_\alpha p\|_\infty \ge n\left(\frac{|\alpha| - k}{1 + k^n}\right)\|p\|_\infty, \tag{11}$$

whereas the corresponding polar derivative analog of (8) was recently given by Mir and Breaz [26]. They proved that if $p(z)$ is a polynomial of degree n which does not vanish in $|z| < k$, $k \le 1$, then for every complex number α with $|\alpha| \ge 1$,

$$\|D_\alpha p\|_\infty \le n\left(\frac{|\alpha| + k^n}{1 + k^n}\right)\|p\|_\infty, \tag{12}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Aziz and Rather [4] further generalized (11) by using a parameter β and established that if $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$ and $m = \min_{|z|=k} |p(z)|$, then for every complex number α, β with $|\alpha| \ge k$, $|\beta| < 1$,

$$\|D_\alpha p(z) + \beta mn\|_\infty \ge n\left(\frac{|\alpha| - k}{1 + k^n}\right)\{\|p\|_\infty + |\beta|m\}, \tag{13}$$

whereas, if $p(z)$ has no zero in $|z| < k$, $k \le 1$, then Mir and Breaz [26] established the following refinement of (12) as

$$\|D_\alpha p\|_\infty \le \frac{n}{1+k^n}\{(|\alpha| + k^n)\|p\|_\infty - (|\alpha| - 1)m\}, \tag{14}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

It is easy to see that the inequalities (13) and (14) sharpen the inequalities (11) and (12) respectively, but both have a drawback that if there is a zero of $p(z)$ on $|z| = k$, then $\min\limits_{|z|=k} |p(z)| = 0$, and so the inequalities (13) and (14) fail to give any improvement over (11) and (12) respectively. Hence, it is entirely reasonable to inquire whether it is feasible to achieve improved bounds for the polynomial $p(z)$ under various restrictions on its zeros and is more informative than the ones given in (11) and (12). Inspired by this, recently, Singh et al. [33] improved the bound of inequality (11) by incorporating the leading coefficient and the constant term of the polynomial and proved the following result.

Theorem 1.6. *If $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every complex number α with $|\alpha| \ge k$,*

$$\begin{aligned}\max_{|z|=1} |D_\alpha p(z)| \ &\ge \left(\frac{|\alpha| - k}{1+k^n}\right)\left(n + \frac{\sqrt{|a_n|k^n} - \sqrt{|a_0|}}{\sqrt{|a_n|k^n}}\right)\left(1 + \frac{(|a_n|k^n - |a_0|)(k-1)}{2(|a_n|k^n + k|a_0|)}\right) \\ &\times \max_{|z|=1} |p(z)|. \end{aligned} \tag{15}$$

In the same paper [33], they also established the following result which gives the improved bound of inequality (8).

Theorem 1.7. *If $p(z) = \sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k$, $k \le 1$, then*

$$\begin{aligned}\max_{|z|=1} |p'(z)| \ &\le \frac{1}{1+k^n}\left[n - k^n\left\{\frac{\sqrt{|a_0|} - \sqrt{|a_n|k^n}}{\sqrt{|a_0|}} + \frac{(|a_0| - |a_n|k^n)(1-k)}{2(|a_0|k + k^n|a_n|)}\right.\right. \\ &\times \left.\left.\left(n + \frac{\sqrt{|a_0|} - \sqrt{|a_n|k^n}}{\sqrt{|a_0|}}\right)\right\}\right] \max_{|z|=1} |p(z)|, \end{aligned} \tag{16}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

For about 20 years, there has been no generalization or extension of inequality (11) due to Aziz and Rather [3] appeared in 1998 into integral analog. In an attempt to obtain the integral versions of Turán-type inequalities of the class of polynomials with zeros lying in $|z| \le k$, $k \ge 1$, it was only in 2017 that Rather and Bhat [30] gave the extension of inequality (11) in an integral mean setting by applying Gauss-Lucas theorem and a well-known property of subordination [16]. In fact, they proved

Theorem 1.8. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every complex number α with $|\alpha| \ge k$ and for each $r > 0$,*

$$n(|\alpha| - k)\|p\|_r \le C_r \|D_\alpha p\|_\infty, \tag{17}$$

where

$$C_r = \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^r d\theta \right\}^{\frac{1}{r}}.$$

It is worth mentioning that the polynomial inequalities in sup-norm on the unit circle in the complex plane are a special case of polynomial inequalities in an integral setting. For example, if we let $r \to \infty$ in (17), and noting that $C_r \to 1 + k^n$, we get inequality (11) due to Aziz and Rather [3] in polar derivative. Moreover, if we divide both sides of inequality (11) by $|\alpha|$ and let $|\alpha| \to \infty$, it reduces to the ordinary derivative inequality (7) proved by Govil [14]. Thus the direction of extending inequalities concerning ordinary or polar derivatives into integral versions, has better higher orders or meanings in the sense of the above discussion. But in the current paper, we have proved the generalized integral version of inequality (15) of Theorem 1.6, which further provides the improved integral extension of inequality (11) in a simpler approach than done by Rather and Bhat [30] entirely based on some existing inequalities on polynomials.

On the other hand, for the last more than 40 years, it has been of interest to obtain the integral setting of inequality (8) due to Govil [13], until 2021, Devi et al. [10] gave the integral extension of inequality (8). In seeking improvement and extension into polar derivative of inequality (10) due to Devi et al. [10], we also prove an improved integral extension of it in polar derivative which is the improved and generalized integral extension of inequality (16) of Theorem 1.7 in the polar derivative and it further provides an improved integral version of inequality (12) due to Mir and Breaz [26]. As a consequence, we additionally derive a polar derivative generalization of Theorem 1.7, enhancing the bound stated in inequality (12), which inequality (14) fails to improve on certain cases due to its limitations.

The paper is organized as follows. In Section 2, we present the main results in integral norms along with remarks and corollaries. In Section 3, we present some auxiliary results necessary to prove the main results. Then the proofs of our main results are given in Section 4. Finally, Section 5 contains the conclusion.

2. Main results

We begin by proving the result for the class of polynomials having all its zeros in $|z| \le k$, $k \ge 1$ which is the generalized integral extension of inequality (15) of Theorem 1.6. More precisely, we prove

Theorem 2.1. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z| \le k$, $k \ge 1$, then for every complex number α, β with $|\alpha| \ge k$, $|\beta| < 1$, for each γ, $0 \le \gamma < 2\pi$, and $r > 0$,*

$$\left\| D_\alpha p(z) + \beta mn \right\|_\infty \ge \frac{|\alpha| - k}{2} A E_r \left\| p(e^{i\theta}) + \beta m \right\|_r, \tag{18}$$

where

$$A = \left\{ n + \frac{\sqrt{|a_n|k^n} - \sqrt{|a_0 + \beta m|}}{\sqrt{|a_n|k^n}} \right\}, \qquad E_r = \frac{\left\{ \int\limits_0^{2\pi} \left| \frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk} + e^{i\gamma} \right|^r d\gamma \right\}^{\frac{1}{r}}}{\left\{ \int\limits_0^{2\pi} |1 + e^{i\gamma}k^n|^r \, d\gamma \right\}^{\frac{1}{r}}}$$

and $m = \min_{|z|=k} |p(z)|$.

Remark 2.1. Letting $r \to \infty$ on both sides of (18), we obtain

$$\max_{|z|=1} |D_\alpha p(z) + \beta mn| \geq \frac{|\alpha| - k}{2} A \frac{\left\{ \frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk} + 1 \right\}}{(1 + k^n)} \max_{|z|=1} |p(z) + \beta m|. \tag{19}$$

Suppose z_0 on $|z| = 1$ be such that $\max_{|z|=1} |p(z)| = |p(z_0)|$. Then, in particular,

$$\max_{|z|=1} |p(z) + \beta m| \geq |p(z_0) + \beta m|. \tag{20}$$

Now to appropriate choice of the argument of β, we can get

$$|p(z_0) + \beta m| = |p(z_0)| + |\beta| m. \tag{21}$$

Using (21) in (20), we have

$$\max_{|z|=1} |p(z) + \beta m| \geq |p(z_0)| + |\beta| m. \tag{22}$$

On combining (19) and (22), we have

$$\begin{aligned} \max_{|z|=1} |D_\alpha p(z) + \beta mn| \quad &\geq \quad \frac{1}{2} \left(\frac{|\alpha| - k}{1 + k^n} \right) A \left\{ \frac{|a_n|k^{n+1} + |a_0| + |\beta|m}{|a_n|k^n + k|a_0| + |\beta|mk} + 1 \right\} \\ &\times \quad \left(\max_{|z|=1} |p(z)| + |\beta| m \right), \end{aligned}$$

which on simplification, gives the following interesting result and it is an improvement of inequality (13) due to Aziz and Rather [4], and a generalization of Theorem 1.6.

Corollary 2.2. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k \geq 1$, *then for every complex number* α *with* $|\alpha| \geq k$, *and a fixed complex number* β *with* $|\beta| < 1$,

$$\begin{aligned} \max_{|z|=1} |D_\alpha p(z) + \beta mn| &\geq \left(\frac{|\alpha| - k}{1 + k^n} \right) \left(n + \frac{\sqrt{|a_n|k^n} - \sqrt{|a_0 + \beta m|}}{\sqrt{|a_n|k^n}} \right) \\ &\times \left(1 + \frac{(|a_n|k^n - |a_0| - |\beta|m)(k-1)}{2(|a_n|k^n + k|a_0| + |\beta|mk)} \right) \left(\max_{|z|=1} |p(z)| + |\beta| m \right), \end{aligned} \tag{23}$$

where $m = \min_{|z|=k} |p(z)|$.

If we divide both sides of (18) of Theorem 2.1 by $|\alpha|$ and let $|\alpha| \to \infty$, we get an interesting result, which gives the generalized integral extension of an inequality in ordinary derivative due to Singh et al. [33, Corollary 3] which further improves inequality (9).

Corollary 2.3. *If $p(z)=\sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\ge 1$, then for every complex number β with $|\beta|<1$, for each γ, $0\le\gamma<2\pi$, and $r>0$,*

$$\|p'\|_\infty \ge \frac{AE_r}{2}\left\|p(e^{i\theta})+\beta m\right\|_r, \tag{24}$$

where A, E_r and m are as defined in Theorem 2.1.

Further, taking limit as $r\to\infty$ on both sides of (24) and following similar arguments of Remark 2.1 for routing inequality (22), namely, $\max\limits_{|z|=1}|p(z)+\beta m|\ge\max\limits_{|z|=1}|p(z)|+|\beta|m$ for some suitable β, we get the following generalized result of Singh et al. [33, Corollary 3], and it sharpens inequality (7).

Corollary 2.4. *If $p(z)=\sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\ge 1$, then for a fixed complex number β with $|\beta|<1$,*

$$\begin{aligned}\max_{|z|=1}|p'(z)| \;&\ge\; \frac{1}{1+k^n}\left(n+\frac{\sqrt{|a_n|k^n}-\sqrt{|a_0+\beta m|}}{\sqrt{|a_n|k^n}}\right)\\ &\times\;\left(1+\frac{(|a_n|k^n-|a_0|-|\beta|m)(k-1)}{2(|a_n|k^n+k|a_0|+|\beta|mk)}\right)\left(\max_{|z|=1}|p(z)|+|\beta|m\right),\end{aligned}$$

where $m=\min\limits_{|z|=k}|p(z)|$.

Putting $\beta=0$ in (18) of Theorem 2.1, we further get the following integral extension of Theorem 1.6.

Corollary 2.5. *If $p(z)=\sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\ge 1$, then for every complex number α with $|\alpha|\ge k$, for each γ, $0\le\gamma<2\pi$, and $r>0$,*

$$\|D_\alpha p\|_\infty \ge \frac{|\alpha|-k}{2}\left\{n+\frac{\sqrt{|a_n|k^n}-\sqrt{|a_0|}}{\sqrt{|a_n|k^n}}\right\}\frac{\left\{\int\limits_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int\limits_0^{2\pi}\left|1+e^{i\gamma}k^n\right|^r d\gamma\right\}^{\frac{1}{r}}}\|p\|_r. \tag{25}$$

If we divide both sides of (25) by $|\alpha|$ and let $|\alpha|\to\infty$, we get the following interesting result, which gives the integral mean analog of the result due to Singh et al. [33, Corollary 3] which also improves inequality (9).

Corollary 2.6. *If $p(z)=\sum\limits_{j=0}^{n} a_j z^j$ is a polynomial of degree n having all its zeros in $|z|\le k$, $k\ge 1$, then for each γ, $0\le\gamma<2\pi$, and $r>0$,*

$$\|p'\|_\infty \ge \frac{1}{2}\left\{n+\frac{\sqrt{|a_n|k^n}-\sqrt{|a_0|}}{\sqrt{|a_n|k^n}}\right\}\frac{\left\{\int\limits_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int\limits_0^{2\pi}\left|1+e^{i\gamma}k^n\right|^r d\gamma\right\}^{\frac{1}{r}}}\|p\|_r. \tag{26}$$

Remark 2.2. Further, taking limit as $r \to \infty$ on both sides of (26), we get the result of Singh et al. [33, Corollary 3] which sharpens inequality (7) as well.

As an application of Corollary 2.3, we now prove the following result which deals with a subclass of polynomials having no zero in $|z| < k$, $k \leq 1$ and it gives an improved and generalized integral extension of inequality (16) of Theorem 1.7 in polar derivative. More precisely, we prove

Theorem 2.7. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k, k \leq 1$, then for every complex number α, β with $|\alpha| \geq 1$, $|\beta| < 1$, for each γ, $0 \leq \gamma < 2\pi$, and $r > 0$,*

$$\frac{k^n(|\alpha|-1)}{2} A' E'_r \left\| p(e^{i\theta}) + \frac{\bar{\beta} m e^{in\theta}}{k^n} \right\|_r \leq n|\alpha| \|p\|_\infty - \|D_\alpha p\|_\infty, \tag{27}$$

where

$$A' = \left\{ n + \frac{\sqrt{|a_0|} - \sqrt{|\bar{a}_n k^n + \beta m|}}{\sqrt{|a_0|}} \right\}, \qquad E'_r = \frac{\left\{ \int_0^{2\pi} \left| \frac{|a_0| + |a_n| k^{n+1} + |\beta| mk}{|a_0| k + k^n |a_n| + |\beta| m} + e^{i\gamma} \right|^r d\gamma \right\}^{\frac{1}{r}}}{\left\{ \int_0^{2\pi} |k^n + e^{i\gamma}|^r d\gamma \right\}^{\frac{1}{r}}}$$

and $m = \min_{|z|=k} |p(z)|$, provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Remark 2.3. Letting $r \to \infty$ on both sides of (27), we obtain

$$\frac{k^n(|\alpha|-1)}{2} A' \frac{\left\{ \frac{|a_0| + |a_n| k^{n+1} + |\beta| mk}{|a_0| k + k^n |a_n| + |\beta| m} + 1 \right\}}{1 + k^n} \max_{|z|=1} \left| p(z) + \frac{\bar{\beta} m z^n}{k^n} \right| \leq n|\alpha| \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)|. \tag{28}$$

Suppose z_0 on $|z| = 1$ be such that $\max_{|z|=1} |p(z)| = |p(z_0)|$. Then, in particular,

$$\left| p(z_0) + \frac{\bar{\beta} m z_0^n}{k^n} \right| \leq \max_{|z|=1} \left| p(z) + \frac{\bar{\beta} m z^n}{k^n} \right|. \tag{29}$$

Now to appropriate choice of the argument of β, we can get

$$\left| p(z_0) + \frac{\bar{\beta} m z_0^n}{k^n} \right| = |p(z_0)| + |\beta| \frac{m}{k^n}. \tag{30}$$

Using (30) in (29), we have

$$|p(z_0)| + |\beta| \frac{m}{k^n} \leq \max_{|z|=1} \left| p(z) + \frac{\bar{\beta} m z^n}{k^n} \right|. \tag{31}$$

On combining (28) and (31), we have

$$\frac{k^n(|\alpha|-1)}{2} A' \frac{\left\{ \frac{|a_0| + |a_n| k^{n+1} + |\beta| mk}{|a_0| k + k^n |a_n| + |\beta| m} + 1 \right\}}{1 + k^n} \left(\max_{|z|=1} |p(z)| + |\beta| \frac{m}{k^n} \right) \leq n|\alpha| \max_{|z|=1} |p(z)| - \max_{|z|=1} |D_\alpha p(z)|,$$

which on simplification, gives the following interesting result sharpening inequality (14) due to Mir and Dar [26], and it is an improved polar derivative generalization of Theorem 1.7.

Corollary 2.8. *If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z|<k$, $k\leq 1$, then for every complex number α with $|\alpha|\geq 1$, and a fixed complex number β with $|\beta|<1$,*

$$\begin{aligned}\max_{|z|=1}|D_\alpha p(z)| \leq\ & \frac{1}{1+k^n}\Bigg[n|\alpha|(1+k^n)-k^n(|\alpha|-1)\left(n+\frac{\sqrt{|a_0|}-\sqrt{|\bar{a}_n k^n+\beta m|}}{\sqrt{|a_0|}}\right)\\ &\times\left(1+\frac{(|a_0|-|a_n|k^n-|\beta|m)(1-k)}{2(|a_0|k+k^n|a_n|+|\beta|m)}\right)\Bigg]\max_{|z|=1}|p(z)|\\ &-\frac{(|\alpha|-1)}{1+k^n}\left(n+\frac{\sqrt{|a_0|}-\sqrt{|\bar{a}_n k^n+\beta m|}}{\sqrt{|a_0|}}\right)\\ &\times\left(1+\frac{(|a_0|-|a_n|k^n-|\beta|m)(1-k)}{2(|a_0|k+k^n|a_n|+|\beta|m)}\right)|\beta|m, \end{aligned}\tag{32}$$

where $m=\min_{|z|=k}|p(z)|$, provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z|=1$.

If we divide both sides of (27) of Theorem 2.7 by $|\alpha|$ and let $|\alpha|\to\infty$, we get an improved integral extension of Theorem 1.7 which further improves inequality (10).

Corollary 2.9. *If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z|<k, k\leq 1$, then for every complex number β with $|\beta|<1$, for each γ, $0\leq\gamma<2\pi$, and $r>0$,*

$$\frac{k^n A' E_r'}{2}\left\|p(e^{i\theta})+\frac{\bar{\beta} m e^{in\theta}}{k^n}\right\|_r \leq n\|p\|_\infty-\|p'\|_\infty, \tag{33}$$

where A', E_r' and m are as defined in Theorem 2.7, provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z|=1$.

Further, taking limit as $r\to\infty$ on both sides of (33) and following similar arguments of Remark 2.3 for routing inequality (31), namely, $\max_{|z|=1}|p(z)|+|\beta|\frac{m}{k^n}\leq \max_{|z|=1}\left|p(z)+\frac{\bar{\beta} m z^n}{k^n}\right|$ for some suitable β, we get the following improvement of Theorem 1.7 and it sharpens inequality (8).

Corollary 2.10. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then for a fixed complex number β with $|\beta| < 1$,*

$$\begin{aligned}
\max_{|z|=1}|p'(z)| \leq & \frac{1}{1+k^n}\Bigg[n|\alpha|(1+k^n) - k^n(|\alpha|-1)\left(n + \frac{\sqrt{|a_0|} - \sqrt{|\bar{a}_n k^n + \beta m|}}{\sqrt{|a_0|}}\right) \\
& \times \left(1 + \frac{(|a_0| - |a_n|k^n - |\beta| m)(1-k)}{2(|a_0|k + k^n|a_n| + |\beta| m)}\right)\Bigg] \max_{|z|=1}|p(z)| \\
& - \frac{(|\alpha|-1)}{1+k^n}\left(n + \frac{\sqrt{|a_0|} - \sqrt{|\bar{a}_n k^n + \beta m|}}{\sqrt{|a_0|}}\right) \\
& \times \left(1 + \frac{(|a_0| - |a_n|k^n - |\beta| m)(1-k)}{2(|a_0|k + k^n|a_n| + |\beta| m)}\right)|\beta| m, \qquad (34)
\end{aligned}$$

where $m = \min_{|z|=k}|p(z)|$, provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Putting $\beta = 0$ in (27) of Theorem 2.7, we further get the following interesting result which is the generalized integral extension of Theorem 1.7 in the polar derivative.

Corollary 2.11. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k, k \leq 1$, then for every complex number α with $|\alpha| \geq 1$, for each γ, $0 \leq \gamma < 2\pi$, and $r > 0$,*

$$\begin{aligned}
\frac{k^n(|\alpha|-1)}{2}\left\{n + \frac{\sqrt{|a_0|} - \sqrt{|a_n|k^n}}{\sqrt{|a_0|}}\right\} \frac{\left\{\int_0^{2\pi}\left|\frac{|a_0|+|a_n|k^{n+1}}{|a_0|k+k^n|a_n|} + e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}|k^n + e^{i\gamma}|^r d\gamma\right\}^{\frac{1}{r}}}\|p\|_r \\
\leq n|\alpha|\|p\|_\infty - \|D_\alpha p\|_\infty, \qquad (35)
\end{aligned}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

Letting $r \to \infty$ on both sides of (35), we get the following result which is the polar derivative generalization of Theorem 1.7.

Corollary 2.12. *If $p(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z| < k$, $k \leq 1$, then for every complex number α with $|\alpha| \geq 1$,*

$$\begin{aligned}
\max_{|z|=1}|D_\alpha p(z)| \leq & \frac{1}{1+k^n}\Bigg[n|\alpha|(1+k^n) - k^n(|\alpha|-1)\left(n + \frac{\sqrt{|a_0|} - \sqrt{|a_n|k^n}}{\sqrt{|a_0|}}\right) \\
& \times \left(1 + \frac{(|a_0| - |a_n|k^n)(1-k)}{2(|a_0|k + k^n|a_n|)}\right)\Bigg] \max_{|z|=1}|p(z)|, \qquad (36)
\end{aligned}$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$.

If we divide both sides of (35) of Corollary 2.11 by $|\alpha|$ and let $|\alpha| \to \infty$, we get the integral analog of Theorem 1.7 which also improves inequality (10).

Corollary 2.13. *If $p(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zero in $|z|<k, k\leq 1$, then for each γ, $0\leq\gamma<2\pi$, and $r>0$,*

$$\frac{k^n}{2}\left\{n+\frac{\sqrt{|a_0|}-\sqrt{|a_n|k^n}}{\sqrt{|a_0|}}\right\}\frac{\left\{\int_0^{2\pi}\left|\frac{|a_0|+|a_n|k^{n+1}}{|a_0|k+k^n|a_n|}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}\left|k^n+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}\|p\|_r\leq n\|p\|_\infty-\|p'\|_\infty,$$

provided $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z|=1$.

3. Lemmas

For the proofs of the theorems, we require the following lemmas.

The first lemma is due to Singh and Chanam [32].

Lemma 3.1. *If $p(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having all its zeros in $|z|\leq 1$, then for $|z|=1$,*

$$|p'(z)|\geq\frac{1}{2}\left\{n+\frac{\sqrt{|a_n|}-\sqrt{|a_0|}}{\sqrt{|a_n|}}\right\}|p(z)|.$$

The next lemma is due to Singha and Chanam [34].

Lemma 3.2. *If $p(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\geq 1$, then for each γ, $0\leq\gamma<2\pi$, and $r>0$,*

$$\left\{\int_0^{2\pi}\left|p(ke^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}\geq k^n\frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0|}{|a_n|k^n+k|a_0|}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}\left|1+e^{i\gamma}k^n\right|^r d\gamma\right\}^{\frac{1}{r}}}\left\{\int_0^{2\pi}\left|p(e^{i\theta})\right|^r d\theta\right\}^{\frac{1}{r}}.\quad(37)$$

Lemma 3.3. *If $p(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n having all its zeros in $|z|\leq k$, $k\leq 1$, then for $|z|=1$,*

$$|q'(z)|\leq k|p'(z)|.$$

The above result is due to Malik [21].

The next lemma is a simple deduction from the Maximum Modulus Principle which is due to G. Pólya and G. Szegö [27] (or see [25]).

Lemma 3.4. *If $p(z)$ is a polynomial of degree n, then for every $k\geq 1$,*

$$\max_{|z|=k}|p(z)|\leq k^n\max_{|z|=1}|p(z)|.$$

The following lemma is again due to Singha and Chanam [34].

Lemma 3.5. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leq k$, $k > 0$, *then for every complex number* β *with* $|\beta| < 1$, *and* $m = \min_{|z|=k} |p(z)|$,

$$k^n |a_n| \geq |\beta| m + |a_0|.$$

The last lemma is due to Govil and Rahman [15].

Lemma 3.6. *If* $p(z)$ *is a polynomial of degree* n, *then on* $|z| = 1$,

$$|p'(z)| + |q'(z)| \leq n \max_{|z|=1} |p(z)|.$$

4. Proofs of the main results

Proof of Theorem 2.1. By hypothesis, $p(z)$ has all its zeros in $|z| \leq k$, $k \geq 1$. In case when $m = \min_{|z|=k} |p(z)| \neq 0$, consider the polynomial $R(z) = p(z) + \beta m$, where β is a complex number with $|\beta| < 1$.

Now, on $|z| = k$,

$$|\beta m| < m \leq |p(z)|.$$

Then by Rouche's theorem [7], it follows that $R(z)$ has all its zeros in $|z| < k$, and in case $m = 0$, $R(z) = p(z)$. Thus in any case $R(z)$ has all its zeros in $|z| \leq k$. And so the polynomial $P(z) = R(kz)$ has all its zeros in $|z| \leq 1$.

Applying Lemma 3.1 to $P(z)$, we have for $|z| = 1$,

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{2} \left\{ n + \frac{\sqrt{|a_n| k^n} - \sqrt{|a_0 + \beta m|}}{\sqrt{|a_n| k^n}} \right\} |P(z)|,$$

which is equivalent to

$$k \max_{|z|=k} |p'(z)| \geq \frac{1}{2} \left\{ n + \frac{\sqrt{|a_n| k^n} - \sqrt{|a_0 + \beta m|}}{\sqrt{|a_n| k^n}} \right\} |p(kz) + \beta m|.$$

Therefore for any $r > 0$, we have

$$k^r \left\{ \max_{|z|=k} |p'(z)| \right\}^r \geq \frac{1}{2^r} \left\{ n + \frac{\sqrt{|a_n| k^n} - \sqrt{|a_0 + \beta m|}}{\sqrt{|a_n| k^n}} \right\}^r \left| p(ke^{i\theta}) + \beta m \right|^r, \quad 0 \leq \theta < 2\pi,$$

and hence

$$k \max_{|z|=k} |p'(z)| \geq \frac{1}{2} \left\{ n + \frac{\sqrt{|a_n| k^n} - \sqrt{|a_0 + \beta m|}}{\sqrt{|a_n| k^n}} \right\} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(ke^{i\theta}) + \beta m \right|^r d\theta \right\}^{\frac{1}{r}}. \tag{38}$$

Applying Lemma 3.2 to $R(z)$, we get

$$\left\{ \int_0^{2\pi} \left| R(ke^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}} \geq k^n \frac{\left\{ \int_0^{2\pi} \left| \frac{|a_n| k^{n+1} + |a_0 + \beta m|}{|a_n| k^n + k |a_0 + \beta m|} + e^{i\gamma} \right|^r d\gamma \right\}^{\frac{1}{r}}}{\left\{ \int_0^{2\pi} \left| 1 + e^{i\gamma} k^n \right|^r d\gamma \right\}^{\frac{1}{r}}} \left\{ \int_0^{2\pi} \left| R(e^{i\theta}) \right|^r d\theta \right\}^{\frac{1}{r}}$$

or

$$\left\{\int_0^{2\pi}\left|p(ke^{i\theta})+\beta m\right|^r d\theta\right\}^{\frac{1}{r}} \geq k^n\frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{\frac{1}{r}}} \times\left\{\int_0^{2\pi}\left|p(e^{i\theta})+\beta m\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{39}$$

Using (39) in (38), we get

$$k\max_{|z|=k}|p'(z)| \geq \frac{k^n}{2}\left\{n+\frac{\sqrt{|a_n|k^n}-\sqrt{|a_0+\beta m|}}{\sqrt{|a_n|k^n}}\right\} \times\frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}|1+e^{i\gamma}k^n|^r d\gamma\right\}^{\frac{1}{r}}} \times\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|p(e^{i\theta})+\beta m\right|^r d\theta\right\}^{\frac{1}{r}}. \tag{40}$$

Also, applying Lemma 3.3 to $P(z)$, we have for $|z|=1$,

$$|Q'(z)|\leq|P'(z)|, \tag{41}$$

where $Q(z)=z^n\overline{P\left(\frac{1}{\bar{z}}\right)}$.

Using (41), we have for $\left|\frac{\alpha}{k}\right|\geq 1$ and $|z|=1$,

$$\begin{aligned}\left|D_{\frac{\alpha}{k}}P(z)\right| &= \left|nP(z)+\left(\frac{\alpha}{k}-z\right)P'(z)\right|\\ &\geq \left|\frac{\alpha}{k}\right||P'(z)|-|nP(z)-zP'(z)|\\ &= \left|\frac{\alpha}{k}\right||P'(z)|-|Q'(z)| \quad (\because\ |Q'(z)|=|nP(z)-zP'(z)| \text{ for } |z|=1)\\ &\geq \left(\left|\frac{\alpha}{k}\right|-1\right)|P'(z)|.\end{aligned}$$

Replacing $P(z)$ by $R(kz)$ in the above inequality, we obtain

$$\left|nR(kz)+\left(\frac{\alpha}{k}-z\right)kR'(kz)\right|\geq\left(\left|\frac{\alpha}{k}\right|-1\right)k|R'(kz)|$$

which implies

$$|nR(kz)+(\alpha-kz)R'(kz)|\geq(|\alpha|-k)|R'(kz)|,$$

so that we obtain

$$\max_{|z|=1}|D_\alpha R(kz)|\geq(|\alpha|-k)\max_{|z|=1}|R'(kz)|$$

which implies

$$\max_{|z|=k}|D_\alpha R(z)|\geq(|\alpha|-k)\max_{|z|=k}|R'(z)|,$$

which is equivalent to

$$\max_{|z|=k} |D_\alpha\{p(z)+\beta m\}| \geq (|\alpha|-k) \max_{|z|=k} |p'(z)|. \tag{42}$$

Using (42) in (40), we get

$$k \max_{|z|=k} |D_\alpha\{p(z)+\beta m\}| \geq k^n \frac{|\alpha|-k}{2} \left\{ n + \frac{\sqrt{|a_n|k^n} - \sqrt{|a_0+\beta m|}}{\sqrt{|a_n|k^n}} \right\}$$
$$\times \frac{\left\{ \int_0^{2\pi} \left| \frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|} + e^{i\gamma} \right|^r d\gamma \right\}^{\frac{1}{r}}}{\left\{ \int_0^{2\pi} |1+e^{i\gamma}k^n|^r \, d\gamma \right\}^{\frac{1}{r}}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) + \beta m \right|^r d\theta \right\}^{\frac{1}{r}}. \tag{43}$$

Also, since $D_\alpha\{p(z)+\beta m\}$ is a polynomial of degree at most $n-1$, applying Lemma 3.4 to $D_\alpha\{p(z)+\beta m\}$ for $k \geq 1$, we have

$$\max_{|z|=k} |D_\alpha\{p(z)+\beta m\}| \leq k^{n-1} \max_{|z|=1} |D_\alpha\{p(z)+\beta m\}|. \tag{44}$$

Using (44) in (43), we get

$$\max_{|z|=1} |D_\alpha\{p(z)+\beta m\}| \geq \frac{|\alpha|-k}{2} \left\{ n + \frac{\sqrt{|a_n|k^n} - \sqrt{|a_0+\beta m|}}{\sqrt{|a_n|k^n}} \right\}$$
$$\times \frac{\left\{ \int_0^{2\pi} \left| \frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|} + e^{i\gamma} \right|^r d\gamma \right\}^{\frac{1}{r}}}{\left\{ \int_0^{2\pi} |1+e^{i\gamma}k^n|^r \, d\gamma \right\}^{\frac{1}{r}}} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) + \beta m \right|^r d\theta \right\}^{\frac{1}{r}}. \tag{45}$$

For every $\beta \in \mathbb{C}$, we have

$$|a_0+\beta m| \leq |a_0| + |\beta| m,$$

and since the function

$$x \mapsto \frac{|a_n|k^{n+1}+x}{|a_n|k^n+kx},$$

is non-increasing for $x \geq 0$ and for every $k \geq 1$, it follows that

$$\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|} \geq \frac{|a_n|k^{n+1}+|a_0|+|\beta| m}{|a_n|k^n+k|a_0|+|\beta| mk}, \tag{46}$$

where

$$\frac{|a_n|k^{n+1}+|a_0|+|\beta| m}{|a_n|k^n+k|a_0|+|\beta| mk} \geq 1. \tag{47}$$

(47) follows readily in view of Lemma 3.5.

It can be easily verified that for every real number γ and $L \geq l \geq 1$,

$$|L+e^{i\gamma}| \geq |l+e^{i\gamma}|.$$

This implies for each $r > 0$,

$$\int_0^{2\pi} |L+e^{i\gamma}|^r d\gamma \geq \int_0^{2\pi} |l+e^{i\gamma}|^r d\gamma. \tag{48}$$

Now, we take $L=\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}$ and $l=\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}$, then by (46) and (47), $L\geq l\geq 1$, and from (48), we get for each $r>0$,

$$\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0+\beta m|}{|a_n|k^n+k|a_0+\beta m|}+e^{i\gamma}\right|^r d\gamma\geq\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}+e^{i\gamma}\right|^r d\gamma. \tag{49}$$

From (49) and (45), we get

$$\max_{|z|=1}|D_\alpha p(z)+\beta mn|\geq\frac{|\alpha|-k}{2}\left\{n+\frac{\sqrt{|a_n|k^n}-\sqrt{|a_0+\beta m|}}{\sqrt{|a_n|k^n}}\right\}$$
$$\times\frac{\left\{\int_0^{2\pi}\left|\frac{|a_n|k^{n+1}+|a_0|+|\beta|m}{|a_n|k^n+k|a_0|+|\beta|mk}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}\left|1+e^{i\gamma}k^n\right|^r d\gamma\right\}^{\frac{1}{r}}}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|p(e^{i\theta})+\beta m\right|^r d\theta\right\}^{\frac{1}{r}}.$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.7. Since $p(z)$ has no zero in $|z|<k$, $k\leq 1$, the polynomial $q(z)$ of degree n has all its zeros in $|z|\leq\frac{1}{k}$, $\frac{1}{k}\geq 1$. Applying Corollary 2.3 to $q(z)$, we get for $|\beta|<1$,

$$\max_{|z|=1}|q'(z)|\geq\frac{A'E''_r}{2}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|q(e^{i\theta})+\beta m'\right|^r d\theta\right\}^{\frac{1}{r}}, \tag{50}$$

where

$$m'=\min_{|z|=\frac{1}{k}}|q(z)|=\min_{|z|=\frac{1}{k}}\left|z^n\overline{p\left(\frac{1}{\bar z}\right)}\right|=\frac{1}{k^n}\min_{|z|=k}|p(z)|=\frac{m}{k^n},$$

$$A'=\left\{n+\frac{\sqrt{|a_0|\frac{1}{k^n}}-\sqrt{|\bar a_n+\beta\frac{m}{k^n}|}}{\sqrt{|a_0|\frac{1}{k^n}}}\right\}=\left\{n+\frac{\sqrt{|a_0|}-\sqrt{|\bar a_nk^n+\beta m|}}{\sqrt{|a_0|}}\right\}$$

and

$$\begin{aligned}E''_r&=\frac{\left\{\int_0^{2\pi}\left|\frac{|a_0|\frac{1}{k^{n+1}}+|a_n|+|\beta|\frac{m}{k^n}}{|a_0|\frac{1}{k^n}+\frac{1}{k}|a_n|+|\beta|\frac{m}{k^{n+1}}}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}\left|1+e^{i\gamma}\frac{1}{k^n}\right|^r d\gamma\right\}^{\frac{1}{r}}}\\&=k^n\frac{\left\{\int_0^{2\pi}\left|\frac{|a_0|+|a_n|k^{n+1}+|\beta|mk}{|a_0|k+k^n|a_n|+|\beta|m}+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}{\left\{\int_0^{2\pi}\left|k^n+e^{i\gamma}\right|^r d\gamma\right\}^{\frac{1}{r}}}=k^nE'_r.\end{aligned}$$

We have

$$|q(z)+\beta m'|=\left|z^n\overline{p\left(\frac{1}{\bar z}\right)}+\beta\frac{m}{k^n}\right|=|z|^n\left|\overline{p\left(\frac{1}{\bar z}\right)}+\frac{\beta m}{k^nz^n}\right|=|z|^n\left|p\left(\frac{1}{\bar z}\right)+\frac{\bar\beta m}{k^n\bar z^n}\right|.$$

In particular, for each θ, $0 \le \theta < 2\pi$,

$$\left|q(e^{i\theta}) + \beta m'\right| = \left|p(e^{i\theta}) + \frac{\bar{\beta} m e^{in\theta}}{k^n}\right|. \tag{51}$$

Using (51) in (50), we get

$$\max_{|z|=1} |q'(z)| \ge \frac{k^n A' E_r'}{2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) + \frac{\bar{\beta} m e^{in\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}}. \tag{52}$$

By Lemma 3.6, we have for $|z| = 1$,

$$|p'(z)| + |q'(z)| \le n \max_{|z|=1} |p(z)|. \tag{53}$$

Since $|p'(z)|$ and $|q'(z)|$ attain their maxima at the same point on $|z| = 1$, let z_0 on $|z| = 1$ be such that $\max\limits_{|z|=1} |q'(z)| = |q'(z_0)|$, then

$$\max_{|z|=1} |p'(z)| = |p'(z_0)|.$$

Now, in particular, (53) gives

$$|p'(z_0)| + |q'(z_0)| \le n \max_{|z|=1} |p(z)|,$$

which implies

$$\max_{|z|=1} |q'(z)| \le n \max_{|z|=1} |p(z)| - \max_{|z|=1} |p'(z)|. \tag{54}$$

Using (54) in (52), we get

$$n \max_{|z|=1} |p(z)| \ge \max_{|z|=1} |p'(z)| + \frac{k^n A' E_r'}{2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) + \frac{\bar{\beta} m e^{in\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}}. \tag{55}$$

Also, by using Lemma 3.6, we have for $|\alpha| \ge 1$ and $|z| = 1$,

$$\begin{aligned} |D_\alpha p(z)| &= |np(z) + (\alpha - z)p'(z)| \\ &\le |np(z) - zp'(z)| + |\alpha||p'(z)| \\ &= |q'(z)| + |\alpha||p'(z)| \quad (\because |q'(z)| = |np(z) - zp'(z)| \text{ for } |z| = 1) \\ &= |q'(z)| + |p'(z)| - |p'(z)| + |\alpha||p'(z)| \\ &\le n \max_{|z|=1} |p(z)| + (|\alpha| - 1)|p'(z)| \\ &\le n \max_{|z|=1} |p(z)| + (|\alpha| - 1) \max_{|z|=1} |p'(z)|. \end{aligned} \tag{56}$$

Combining (56) and (55), we get

$$\begin{aligned} \max_{|z|=1} |D_\alpha p(z)| \le\ & n \max_{|z|=1} |p(z)| + (|\alpha| - 1) \Bigg[n \max_{|z|=1} |p(z)| \\ & - \frac{k^n A' E_r'}{2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| p(e^{i\theta}) + \frac{\bar{\beta} m e^{in\theta}}{k^n} \right|^r d\theta \right\}^{\frac{1}{r}} \Bigg], \end{aligned}$$

which is equivalent to

$$\frac{k^n(|\alpha|-1)}{2}A'E'_r\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|p(e^{i\theta})+\frac{\bar{\beta}me^{in\theta}}{k^n}\right|^r d\theta\right\}^{\frac{1}{r}} \leq n|\alpha|\max_{|z|=1}|p(z)|-\max_{|z|=1}|D_\alpha p(z)|.$$

This completes the proof of Theorem 2.7. □

5. Conclusion

Studying the extremal problems of functions of a complex variable and generalizing the classical polynomial inequalities are topical in geometric function theory. In the past few years, a series of papers related both to Bernstein and Turán-type inequalities have been published and significant advances in terms of extension, improvement as well as generalization have been achieved in different directions. One such generalization is replacing the sup-norm by a factor involving integral mean. These types of inequalities are of interest both in mathematics and in the application areas such as physical systems. More precisely, the authors contribute a rare work in establishing integral mean extensions of some Turán and Bernstein-type inequalities for the polar derivatives of some classes of polynomials by following some new approach.

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