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$*$-Miao-Tam Equation on Contact Geometry

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Abstract. In this article, we classify $*$-Miao-Tam equations in contact geometry. In the beginning, it is demonstrated that if a Sasakian manifold satisfies $*$-Miao-Tam equation with the potential function λ, then $\lambda = c_1 t + c_2$, $c_1 \neq 0$, provided the scalar curvature is invariant under ζ. Also, we show that if a Sasakian 3-manifold satisfies $*$-Miao-Tam equation with non-constant potential function, then the manifold is $*$-Ricci flat and becomes a Sasakian space-form. Next, we characterize $*$-Miao-Tam equation on (k,μ)-contact manifolds.

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1. Introduction

The metric g is considered to be a crtical metric of a compact Riemannian manifold $(\mathcal{M}, g)$ of dimension $(2n+1)$ with boundary $\partial\mathcal{M}$ if the following equation holds

$$-(\triangle_g \lambda)g + \nabla_g^2 \lambda - \lambda S = g, \tag{1}$$

$\lambda : \mathcal{M} \to \mathbb{R}$ is a smooth function (known as the potential function) and λ vanishes on $\partial\mathcal{M}$, $\triangle_g$, ∇_g^2 are the Laplacian, Hessain operator with respect to the metric g, respectively and S denotes the Ricci tensor. The equation (1) is named Miao-Tam equation (shortly, **MTE**). The metrics fulfilling (1) are known as the Miao-Tam critical metrics [13].

Equation (1) implies

$$\nabla_g^2 \lambda - \lambda S_g + (\frac{r\lambda + 1}{2n})g = 0. \tag{2}$$

Recent research shows that Patra-Ghosh examined equation (1) for contact metric manifolds and almost Kenmotsu manifolds ([15], [16]). It was demonstrated in particular that a complete K-contact metric that solves the **MTE** is isometric to the unit sphere S^{2n+1}. An almost Kenmotsu manifold with the **MTE** was also taken to be considered by Wang and Wang [20]. The physical and geometrical characteristics of the **MTE** have been studied by a number of researchers ([6], [9], [10], [12], [16]).

In this study, we develop the $*$-**MTE** on contact geometry, which is given by

$$\nabla_g^2 \lambda - \lambda S_g^* + (\frac{r^*\lambda + 1}{2n})g = 0, \tag{3}$$

where we have replaced the Ricci tensor S and the scalar curvature r, respectively by $*$-Ricci tensor S^* and $*$-scalar curvature r^* in (2).

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A variety of works have been carried out on $*$-Ricci tensors over the past few years. Various solitons have been carried out on the $*$-Ricci tensor ([8], [11], [14], [17], [18], [19], [21]). Inspired by these works, we study the $*$-**MTE** in contact geometry. To be more detailed, the following outcomes are obtained:

Theorem 1.1. *If a Sasakian manifold satisfies $*$-**MTE** with potential function λ, then the solution is $\lambda = c_1 t + c_2$, $c_1 \neq 0$, provided the scalar curvature is invariant under ζ where ζ is characteristic vector field.*

Theorem 1.2. *If a Sasakian 3-manifold satisfies $*$-**MTE** with non-constant potential function, then the manifold is $*$-Ricci flat and becomes a Sasakian space-form.*

Theorem 1.3. *If a $(k, \boldsymbol{\mu})$-contact manifold satisfies $*$-**MTE** with potential function λ, then one of the cases holds:*
(i) either the manifold is flat for $n = 1$ and for $n > 1$ it is locally isometric to $E^{n+1} \times S^n(4)$ or grad λ is pointwise collinear with ζ.
(ii) The equation has constant solution, provided $(2k + n\boldsymbol{\mu})^2 + \boldsymbol{\mu}^2(k-1) \neq 0$.

2. Preliminaries

A manifold $\mathcal{M}$ of dimension $(2n+1)$ is an almost contact manifold with almost contact structure (φ, ζ, η), ζ(characteristic vector field) is a unit vector field, φ being a $(1,1)$-tensor field and η is a 1-form fulfilling

$$\varphi^2 \Lambda_1 = -\Lambda_1 + \eta(\Lambda_1)\zeta, \;\; \eta(\zeta) = 1. \tag{4}$$

An almost contact metric (shortly, **acm**) structure $(\varphi, \zeta, \eta, g)$ is an almost contact structure whose metric g fulfills

$$g(\varphi\Lambda_1, \varphi\Lambda_2) = g(\Lambda_1, \Lambda_2) - \eta(\Lambda_1)\eta(\Lambda_2), \;\; g(\Lambda_1, \zeta) = \eta(\Lambda_1) \tag{5}$$

for any $\Lambda_1, \Lambda_2 \in \mathfrak{X}(M)$. An **acm** structure $(\varphi, \zeta, \eta, g)$ is said to be normal if the associated complex structure J on $\mathcal{M} \times \mathbb{R}$ is integrable.

If a Riemannian manifold $\mathcal{M}$ and its **acm** structure $(\varphi, \zeta, \eta, g)$ satisfies the condition ([2], p. 47) $d\eta(\Lambda_2, \Lambda_3) = g(\Lambda_2, \varphi\Lambda_3)$ for all vector fields Λ_2, Λ_3, it is said to be a contact metric manifold (shortly, **cmm**). The contact metric structure's associated metric is g. Two self-adjoint operators, $h = \frac{1}{2}\pounds_\zeta \varphi$ and $l = R(., \zeta)\zeta$, are taken into consideration on the **cmm** $\mathcal{M}(\varphi, \zeta, \eta, g)$, $\pounds_\zeta$ is the Lie-derivative along ζ. The two operators h and l satisfy ([2], p. 84-85)

$$tr\, h = tr\, h\varphi = 0, \;\; h\zeta = l\zeta = 0, \;\; h\varphi = -\varphi h. \tag{6}$$

Lemma 2.1. [5] *On a* **cmm** $\mathcal{M}(\varphi, \zeta, \eta, g)$

$$\nabla_{\Lambda_2}\zeta = -\varphi\Lambda_2 - \varphi h \Lambda_2, \tag{7}$$

$$S(\zeta, \zeta) = g(Q\zeta, \zeta) = tr\, l = 2n - tr\, h^2, \tag{8}$$

$$(div(h\varphi))\Lambda_2 = g(Q\zeta, \Lambda_2) - 2n\eta(\Lambda_2) \tag{9}$$

hold for any vector field Λ_2.

If ζ is a Killing vector field, or alternatively if $h = 0$ ([2], p. 87), a **cmm** is called a K-contact manifold. Therefore, on a K-contact manifold equation (7) turns into

$$\nabla_{\Lambda_2}\zeta = -\varphi\Lambda_2. \tag{10}$$

A Sasakian manifold always represents a K-contact manifold. When the dimension is three, the reverse also holds true, however this may not be true in higher dimensions. The following equations are widely known on the Sasakian manifold:

$$\nabla_{\Lambda_1}\zeta = -\varphi\Lambda_1, \tag{11}$$

$$R(\Lambda_1, \Lambda_2)\zeta = \eta(\Lambda_2)\Lambda_1 - \eta(\Lambda_1)\Lambda_2, \tag{12}$$

$$R(\zeta, \Lambda_1)\Lambda_2 = g(\Lambda_1, \Lambda_2)\zeta - \eta(\Lambda_2)\Lambda_1, \tag{13}$$

$$S(\Lambda_1, \zeta) = 2n\eta(\Lambda_1),\ Q\zeta = 2n\zeta \tag{14}$$

hold for any vector field Λ_1, Λ_2 and the Ricci tensor, curvature tensor and Ricci operator are denoted by S, R and Q, respectively.

Let $\mathcal{M}(\varphi, \zeta, \eta, g)$ represent an almost contact metric manifold with Ricci tensor S. The $*$-Ricci tensor and $*$-scalar curvature of $\mathcal{M}$ respectively, are given by

$$S^*(\Lambda_1, \Lambda_2) = \sum_{j=1}^{2n+1} R(\Lambda_1, \mathfrak{m}_j, \varphi\mathfrak{m}_j, \varphi\Lambda_2),\ \ r^* = \sum_{j=1}^{2n+1} S^*(\mathfrak{m}_j, \mathfrak{m}_j). \tag{15}$$

Lemma 2.2. [14] *In a Sasakian manifold $\mathcal{M}$ with dimension $(2n+1)$, the $*$-Ricci tensor and $*$-scalar curvature are given by*

$$S^*(\Lambda_1, \Lambda_2) = S(\Lambda_1, \Lambda_2) - (2n-1)g(\Lambda_1, \Lambda_2) - \eta(\Lambda_1)\eta(\Lambda_2) \tag{16}$$

and

$$r^* = r - 4n^2. \tag{17}$$

Lemma 2.3. [5] *In a Sasakian 3-manifold, the Ricci tensor S is given by*

$$S(\Lambda_1, \Lambda_2) = (\frac{r}{2} - 1)g(\Lambda_1, \Lambda_2) + (3 - \frac{r}{2})\eta(\Lambda_1)\eta(\Lambda_2). \tag{18}$$

Blair et al. [4] investigated a novel class of **cmm**s called a $(k, \boldsymbol{\mu})$-contact manifold (shortly, $(k, \boldsymbol{\mu})$-**cm**). In [7], Boeckx properly characterized these manifolds. A $(k, \boldsymbol{\mu})$-**cm** is a $\mathcal{M}(\varphi, \zeta, \eta, g)$ **cmm**. Its curvature tensor fulfills

$$R(\Lambda_2, \Lambda_3)\zeta = k[\eta(\Lambda_3)\Lambda_2 - \eta(\Lambda_2)\Lambda_3] + \boldsymbol{\mu}[\eta(\Lambda_3)h\Lambda_2 - \eta(\Lambda_2)h\Lambda_3] \tag{19}$$

for all Λ_2, Λ_3 and k, $\boldsymbol{\mu} \in \mathbb{R}$. A $(k, \boldsymbol{\mu})$-contact manifold is called an $N(k)$-**cmm** if $\boldsymbol{\mu} = 0$ ([1], [3]).

The following formulas are used for $(k, \boldsymbol{\mu})$-contact manifolds [4]:

$$\begin{aligned} S(\Lambda_2, \Lambda_3) \ &= \ [2(n-1) - n\boldsymbol{\mu}]g(\Lambda_2, \Lambda_3) + [2(n-1) + \boldsymbol{\mu}]g(h\Lambda_2, \Lambda_3) \\ &\quad +[2(1-n) + n(2k + \boldsymbol{\mu})]\eta(\Lambda_2)\eta(\Lambda_3), \end{aligned} \tag{20}$$

$$Q\zeta = 2nk\zeta, \tag{21}$$

$$h^2 = (k-1)\varphi^2,\ k \leq 1, \tag{22}$$

when $k = 1$ (equivalently, $h = 0$), equality holds, that is, $\mathcal{M}$ is Sasakian. The $(k, \boldsymbol{\mu})$-nullity condition completely determines the curvature of $\mathcal{M}$ in the non-Sasakian case, that is, $k < 1$. Furthermore, the scalar curvature r is obtained by

$$r = 2n(2(n-1) + k - n\boldsymbol{\mu}). \tag{23}$$

Lemma 2.4. *In a $(k,\boldsymbol{\mu})$-contact manifold*

$$R(\zeta,\Lambda_2)\Lambda_3 = k[g(\Lambda_2,\Lambda_3)\zeta-\eta(\Lambda_3)\Lambda_2]+\boldsymbol{\mu}[g(h\Lambda_2,\Lambda_3)\zeta-\eta(\Lambda_3)h\Lambda_2], \tag{24}$$
$$(\nabla_\zeta Q)\Lambda_2 = \boldsymbol{\mu}[2(n-1)+\boldsymbol{\mu}]h\varphi\Lambda_2, \tag{25}$$
$$(\nabla_\zeta h)\Lambda_2 = \boldsymbol{\mu}h\varphi\Lambda_2, \tag{26}$$
$$(\nabla_{\Lambda_2}Q)\zeta = (\varphi+\varphi h)Q\Lambda_2-2nk(\varphi+\varphi h)\Lambda_2 \tag{27}$$

hold.

Lemma 2.5. [14] *In a $(k,\boldsymbol{\mu})$-contact manifold, the $*$-Ricci tensor and $*$-scalar curvature are given by*

$$S^*(\Lambda_1,\Lambda_2) = (n\boldsymbol{\mu}+k)[-g(\Lambda_1,\Lambda_2)+\eta(\Lambda_1)\eta(\Lambda_2)] \tag{28}$$

and

$$r^* = -2n(n\boldsymbol{\mu}+k). \tag{29}$$

Lemma 2.6. [2] *If $k=\boldsymbol{\mu}=0$ in a $(k,\boldsymbol{\mu})$-contact manifold, then the manifold is flat for $n=1$ and $n>1$ it is locally isometric to $E^{n+1}\times S^n(4)$.*

3. Proof of the Main Results

Proof of Theorem 1.1. From equation (3), we obtain

$$\nabla^2\lambda-\lambda S^*+(\frac{r^*\lambda+1}{2n})g=0. \tag{30}$$

Using (16) in (30), we obtain

$$\nabla_{\Lambda_1}D\lambda = \lambda Q\Lambda_1-\psi\Lambda_1-\lambda\eta(\Lambda_1)\zeta, \tag{31}$$

where $\psi=\frac{(r-2n)\lambda+1}{2n}$.
Differentiating (31), we infer

$$\begin{aligned}\nabla_{\Lambda_2}\nabla_{\Lambda_1}D\lambda &= (\Lambda_2\lambda)Q\Lambda_1+\lambda\nabla_{\Lambda_2}Q\Lambda_1-(\Lambda_2\psi)\Lambda_1-\psi\nabla_{\Lambda_2}\Lambda_1\\ &\quad-(\Lambda_2\lambda)\eta(\Lambda_1)\zeta-\lambda[\nabla_{\Lambda_2}\eta(\Lambda_1)\zeta-\eta(\Lambda_1)\varphi\Lambda_2].\end{aligned} \tag{32}$$

Interchanging Λ_1 and Λ_2 in (32) gives

$$\begin{aligned}\nabla_{\Lambda_1}\nabla_{\Lambda_2}D\lambda &= (\Lambda_1\lambda)Q\Lambda_2+\lambda\nabla_{\Lambda_1}Q\Lambda_2-(\Lambda_1\psi)\Lambda_2-\psi\nabla_{\Lambda_1}\Lambda_2\\ &\quad-(\Lambda_1\lambda)\eta(\Lambda_2)\zeta-\lambda[\nabla_{\Lambda_1}\eta(\Lambda_2)\zeta-\eta(\Lambda_2)\varphi\Lambda_1].\end{aligned} \tag{33}$$

From (31), we get

$$\nabla_{[\Lambda_1,\Lambda_2]}D\lambda = \lambda Q([\Lambda_1,\Lambda_2])-\psi([\Lambda_1,\Lambda_2])-\lambda\eta([\Lambda_1,\Lambda_2])\zeta, \tag{34}$$

Equations (32)-(34) together imply

$$\begin{aligned}R(\Lambda_1,\Lambda_2)D\lambda &= (\Lambda_1\lambda)Q\Lambda_2-(\Lambda_2\lambda)Q\Lambda_1+\lambda[(\nabla_{\Lambda_1}Q)\Lambda_2-(\nabla_{\Lambda_2}Q)\Lambda_1]\\ &\quad-(\Lambda_1\psi)\Lambda_2+(\Lambda_2\psi)\Lambda_1-(\Lambda_1\lambda)\eta(\Lambda_2)\zeta+(\Lambda_2\lambda)\eta(\Lambda_1)\zeta\\ &\quad-\lambda[2g(\Lambda_1,\varphi\Lambda_2)\zeta-\eta(\Lambda_2)\varphi\Lambda_1+\eta(\Lambda_1)\varphi\Lambda_2].\end{aligned} \tag{35}$$

Contracting Λ_1 in (35), we obtain

$$\begin{aligned}S(\Lambda_2,D\lambda) &= g(Q\Lambda_1,D\lambda)-r(\Lambda_2\lambda)-\frac{1}{2}\lambda(\Lambda_2 r)\\ &\quad+2n(\Lambda_2\psi)-(\zeta\lambda)\eta(\Lambda_2)+(\Lambda_2\lambda).\end{aligned} \tag{36}$$

Again, we aasume

$$\psi = \frac{(r-2n)\lambda + 1}{2n},$$

which implies

$$2n(\Lambda_2\psi) = (r-2n)\Lambda_2\lambda + (\Lambda_2 r)\lambda. \tag{37}$$

Using (37) in (36) entails that

$$(2n-1)\Lambda_2\lambda + (\zeta\lambda)\eta(\Lambda_2) = \frac{\lambda}{2}(\Lambda_2 r). \tag{38}$$

Putting $\Lambda_2 = \zeta$ in (38), we get

$$\zeta\lambda = \frac{\lambda}{4n}(\zeta r). \tag{39}$$

If we taking $\zeta r = 0$, then (39) implies $\zeta\lambda = 0$. Hence $\zeta(\zeta\lambda) = 0$. If we take $\zeta = \frac{\partial}{\partial t}$, then

$$\frac{\partial^2\lambda}{\partial t^2} = 0.$$

Solution of the above equation is $\lambda = c_1 t + c_2$, $c_1 \neq 0$. As a result, the proof is completed. □

Proof of Theorem 1.2. For 3-dimension, equations (16), (17) and (18) together imply

$$S^*(\Lambda_1, \Lambda_2) = (\frac{r}{2} - 2)[g(\Lambda_1, \Lambda_2) - \eta(\Lambda_1)\eta(\Lambda_2)] \tag{40}$$

and

$$r^* = r - 4. \tag{41}$$

In view of (3), (40) and (41) gives

$$\nabla_{\Lambda_1} D\lambda = -\frac{1}{2}\Lambda_1 - \frac{\lambda}{2}(r-4)\eta(\Lambda_1)\zeta. \tag{42}$$

Differentiating (42), we get

$$\begin{aligned}\nabla_{\Lambda_2}\nabla_{\Lambda_1} D\lambda &= -\frac{1}{2}\nabla_{\Lambda_2}\Lambda_1 - \frac{1}{2}[(\Lambda_2\lambda)(r-4)\eta(\Lambda_1)\zeta + \lambda(\Lambda_2 r)\eta(\Lambda_1)\zeta \\ &\quad +\lambda(r-4)\{\nabla_{\Lambda_2}\eta(\Lambda_1)\zeta - \eta(\Lambda_1)\varphi\Lambda_2\}].\end{aligned} \tag{43}$$

Interchanging Λ_1 and Λ_2 in (43) entails that

$$\begin{aligned}\nabla_{\Lambda_1}\nabla_{\Lambda_2} D\lambda &= -\frac{1}{2}\nabla_{\Lambda_1}\Lambda_2 - \frac{1}{2}[(\Lambda_1\lambda)(r-4)\eta(\Lambda_2)\zeta + \lambda(\Lambda_1 r)\eta(\Lambda_2)\zeta \\ &\quad +\lambda(r-4)\{\nabla_{\Lambda_1}\eta(\Lambda_2)\zeta - \eta(\Lambda_2)\varphi\Lambda_1\}].\end{aligned} \tag{44}$$

From (42), we infer

$$\nabla_{[\Lambda_1,\Lambda_2]} D\lambda = -\frac{1}{2}([\Lambda_1, \Lambda_2]) - \frac{\lambda}{2}(r-4)\eta([\Lambda_1, \Lambda_2])\zeta. \tag{45}$$

Equations (43)-(45) together imply

$$\begin{aligned}R(\Lambda_1, \Lambda_2)D\lambda &= \frac{1}{2}[\{(\Lambda_2\lambda)(r-4) + \lambda(\Lambda_2 r)\}\eta(\Lambda_1)\zeta \\ &\quad -\{(\Lambda_1\lambda)(r-4) + \lambda(\Lambda_1 r)\}\eta(\Lambda_2)\zeta] \\ &\quad -\lambda(r-4)g(\Lambda_1, \varphi\Lambda_2)\zeta.\end{aligned} \tag{46}$$

Contracting Λ_1 in (46), we obtain

$$S(\Lambda_2, D\lambda) = \frac{1}{2}[(\Lambda_2\lambda)(r-4) + \lambda(\Lambda_2 r) - \{(\zeta\lambda)(r-4) + \lambda(\zeta r)\}\eta(\Lambda_2)]. \tag{47}$$

Replacing Λ_1 by $D\lambda$ in (18) gives

$$S(\Lambda_2, D\lambda) = (\frac{r}{2} - 1)(\Lambda_2\lambda) + (3 - \frac{r}{2})(\zeta\lambda)\eta(\Lambda_2). \tag{48}$$

In view of the above two equations, we get

$$2[(\Lambda_2\lambda) + (\zeta\lambda)\eta(\Lambda_2)] = \lambda[(\Lambda_2 r) - (\zeta r)\eta(\Lambda_2)]. \tag{49}$$

Again, from (18) we get

$$Q\Lambda_1 = (\frac{r}{2} - 1)\Lambda_1 + (3 - \frac{r}{2})\eta(\Lambda_1)\zeta. \tag{50}$$

Equation (50) implies

$$(\nabla_{\Lambda_2} Q)\Lambda_1 = \frac{\Lambda_2 r}{2}[\Lambda_1 - \eta(\Lambda_1)\zeta] + (3 - \frac{r}{2})[g(\varphi\Lambda_1, \Lambda_2)\zeta - \eta(\Lambda_1)\varphi\Lambda_2]. \tag{51}$$

Contracting Λ_1 in (51) and using $(div\, Q)\Lambda_1 = \frac{1}{2}(\Lambda_1 r)$, we get

$$\zeta r = 0. \tag{52}$$

Using (52) in (49), we get

$$(\Lambda_2\lambda) + (\zeta\lambda)\eta(\Lambda_2) = \frac{\lambda}{2}(\Lambda_2 r). \tag{53}$$

Considering the inner product of (46) with ζ, we infer

$$\begin{aligned}(\Lambda_2\lambda)\eta(\Lambda_1) - (\Lambda_1\lambda)\eta(\Lambda_2) \quad = \quad & \frac{1}{2}[\{(\Lambda_2\lambda)(r-4) + \lambda(\Lambda_2 r)\}\eta(\Lambda_1) \\ & -\{(\Lambda_1\lambda)(r-4) + \lambda(\Lambda_1 r)\}\eta(\Lambda_2)] \\ & -\lambda(r-4)g(\Lambda_1, \varphi\Lambda_2).\end{aligned} \tag{54}$$

Setting $\Lambda_1 = \zeta$ in (54), we obtain

$$\frac{(6-r)}{2}[\Lambda_2\lambda - (\zeta\lambda)\eta(\Lambda_2)] = \frac{\lambda}{2}(\Lambda_2 r). \tag{55}$$

From (53) and (55), we infer

$$(\Lambda_2\lambda) + (\zeta\lambda)\eta(\Lambda_2) = \frac{(6-r)}{2}[\Lambda_2\lambda - (\zeta\lambda)\eta(\Lambda_2)]. \tag{56}$$

Putting $\Lambda_2 = \zeta$ in (53) gives

$$\zeta\lambda = 0. \tag{57}$$

The above two equations together imply

$$(r-4)\Lambda_2\lambda = 0, \tag{58}$$

which implies either $r = 4$ or $r \neq 4$.

If $r = 4$, then (40) implies $S^*(\Lambda_1, \Lambda_2) = 0$, that is, $*$-Ricci tensor vanishes, which implies r^* vanishes. It can be easily seen that for a 3-dimensional Sasakian manifold the φ-sectional curvature is equal to $\frac{r-4}{2}$. Since $r =$ constant, the φ-sectional curvature is a constant and thus the manifold becomes a 3-dimensional Sasakian space-form [2], provided the potential function is non-constant. $r \neq 4$ implies $\lambda =$ constant, which is a contradiction. Thus, the proof is finished. □

Proof of Theorem 1.3. From (3) and (28), we get

$$\nabla_{\Lambda_1} D\lambda = \lambda(n\boldsymbol{\mu} + k)\eta(\Lambda_1)\zeta - \frac{1}{2n}\Lambda_1. \tag{59}$$

Differetiating (59), we get

$$\begin{aligned}\nabla_{\Lambda_2}\nabla_{\Lambda_1} D\lambda &= (\Lambda_2\lambda)(n\boldsymbol{\mu} + k)\eta(\Lambda_1)\zeta + \lambda(n\boldsymbol{\mu} + k)[\nabla_{\Lambda_2}\eta(\Lambda_1)\zeta \\ &\quad -(\varphi\Lambda_2 + \varphi h\Lambda_2)\eta(\Lambda_1)] - \frac{1}{2n}\nabla_{\Lambda_2}\Lambda_1.\end{aligned} \tag{60}$$

Interchanging Λ_1 and Λ_2 in (60), we get

$$\begin{aligned}\nabla_{\Lambda_1}\nabla_{\Lambda_2} D\lambda &= (\Lambda_1\lambda)(n\boldsymbol{\mu} + k)\eta(\Lambda_2)\zeta + \lambda(n\boldsymbol{\mu} + k)[\nabla_{\Lambda_1}\eta(\Lambda_2)\zeta \\ &\quad -(\varphi\Lambda_1 + \varphi h\Lambda_1)\eta(\Lambda_2)] - \frac{1}{2n}\nabla_{\Lambda_1}\Lambda_2.\end{aligned} \tag{61}$$

From (59), we obtain

$$\nabla_{[\Lambda_1,\Lambda_2]} D\lambda = \lambda(n\boldsymbol{\mu} + k)\eta([\Lambda_1, \Lambda_2])\zeta - \frac{1}{2n}([\Lambda_1, \Lambda_2]). \tag{62}$$

Equations (60)-(62) together imply

$$\begin{aligned}R(\Lambda_1, \Lambda_2)D\lambda &= (\Lambda_1\lambda)(n\boldsymbol{\mu} + k)\eta(\Lambda_2)\zeta - (\Lambda_2\lambda)(n\boldsymbol{\mu} + k)\eta(\Lambda_1)\zeta \\ &\quad +\lambda(n\boldsymbol{\mu} + k)[g(\Lambda_1, \varphi\Lambda_2)\zeta + (\varphi\Lambda_2 + \varphi h\Lambda_2)\eta(\Lambda_1) \\ &\quad -(\varphi\Lambda_1 + \varphi h\Lambda_1)\eta(\Lambda_2)].\end{aligned} \tag{63}$$

Taking inner product in (63) with ζ, we infer

$$\begin{aligned}&k[(\Lambda_2\lambda)\eta(\Lambda_1) - (\Lambda_1\lambda)\eta(\Lambda_2)] + \boldsymbol{\mu}[g(h\Lambda_2, D\lambda)\eta(\Lambda_1) - g(h\Lambda_1, D\lambda)\eta(\Lambda_2)] \\ &= (n\boldsymbol{\mu} + k)[(\Lambda_1\lambda)\eta(\Lambda_2) - (\Lambda_2\lambda)\eta(\Lambda_1) + 2\lambda g(\Lambda_1, \varphi\Lambda_2)].\end{aligned} \tag{64}$$

Setting $\Lambda_1 = \zeta$ in (64) gives

$$(2k + n\boldsymbol{\mu})[\Lambda_2\lambda - (\zeta\lambda)\eta(\Lambda_2)] + \boldsymbol{\mu}g(h\Lambda_2, D\lambda) = 0. \tag{65}$$

Replacing Λ_2 by $h\Lambda_2$ in (65) entails that

$$(2k + n\boldsymbol{\mu})g(h\Lambda_2, D\lambda) - \boldsymbol{\mu}(k - 1)[\Lambda_2\lambda - (\zeta\lambda)\eta(\Lambda_2)] = 0. \tag{66}$$

In view of (65)and (66), we obtain

$$[(2k + n\boldsymbol{\mu})^2 + \boldsymbol{\mu}^2(k - 1)][\Lambda_2\lambda - (\zeta\lambda)\eta(\Lambda_2)] = 0. \tag{67}$$

Contracting Λ_1 in (63), we get

$$S(\Lambda_2, D\lambda) = (n\boldsymbol{\mu} + k)[(\zeta\lambda)\eta(\Lambda_2) - \Lambda_2\lambda]. \tag{68}$$

Again, replacing Λ_1 by $D\lambda$ in (20) and comparing with (68) entails that

$$\begin{aligned}&[2(n - 1) - n\boldsymbol{\mu}](\Lambda_2\lambda) + [2(n - 1) + \boldsymbol{\mu}]g(h\Lambda_2, D\lambda) \\ &+[2(1 - n) + n(2k + \boldsymbol{\mu})](\zeta\lambda)\eta(\Lambda_2) \\ &= (n\boldsymbol{\mu} + k)[(\zeta\lambda)\eta(\Lambda_2) - \Lambda_2\lambda].\end{aligned} \tag{69}$$

Setting $\Lambda_2 = \zeta$ in (69), we get

$$k(\zeta\lambda) = 0, \tag{70}$$

which means either $k = 0$ or $k \neq 0$.

Case I: If $k=0$, then (67) implies

$$\boldsymbol{\mu}[\Lambda_2\lambda-(\zeta\lambda)\eta(\Lambda_2)]=0, \tag{71}$$

which means either $\boldsymbol{\mu}=0$ or $\boldsymbol{\mu}\neq 0$.
Case i: If $\boldsymbol{\mu}=0$, then from Lemma 2.6 the manifold is flat for $n=1$ and for $n>1$ it is locally isometric to $E^{n+1}\times S^n(4)$.
Case ii: If $\Lambda_2\lambda-(\zeta\lambda)\eta(\Lambda_2)=0$, then $D\lambda=(\zeta\lambda)\zeta$ which means that $grad\,\lambda$ is pointwise collinear with ζ.

Case II: If $k\neq 0$, then (70) implies $\zeta\lambda=0$. Hence (67) implies $\Lambda_2\lambda=0$ which means that λ is constant for $(2k+n\boldsymbol{\mu})^2+\boldsymbol{\mu}^2(k-1)\neq 0$. Therefore the equation has constant solution.
As a result, our proof is completed. □

4. Conclusion

blair3 Critical point equations have been investigated in Riemannian as well as in semi-Riemannian manifolds. Several researchers have characterized **MTE** equations in Riemannian and semi-Riemannian manifolds but nobody is interested to investigate $*$-**MTE** in contact geometry. In this article, we introduce $*$-**MTE** in contact geometry and characterize Sasakian and $(k,\boldsymbol{\mu})$-contact metric manifolds admitting $*$-**MTE**.

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