Annals of the University of Craiova, Mathematics and Computer Science Series
Volume 51(2), 2024, Pages 488–504, DOI: 10.52846/ami.v51i2.1888
ISSN: 1223-6934

Novel Ostrowski type inequalities via exponentially $(m_1, m_2)-$convex functions and their applications

Saad Ihsan Butt, Bahtiyar Bayraktar, and Jamshed Nasir

Abstract. In this work, we introduce and explore the new concept of convexity class, namely exponential $(m_1, m_2)-$convex functions along with some examples. In extension, we perform an artistic analysis of the properties of this class. We give the formulation of the new quadrature type identity. Based on this identity, we obtain some integral inequalities for the introduced convex functions via fractional operators tool. In addition, specific means of different positive real numbers and some new limits for the $\mathbf{q}-$digamma function are also presented.

2020 *Mathematics Subject Classification.* Primary: 26A51; Secondary: 26A33, 26D07, 26D10, 26D15.
Key words and phrases. convex function; Ostrovski type inequality; Hermite–Hadamard inequality; Hölder inequality; Power-Mean inequality; Fractional integral.

1. Introduction

Ostrowski integral inequalities are a type of inequalities that relate the values of a function to its derivatives over a given interval. Specifically, these inequalities provide bounds on the difference between the integral of a function and its average value, in terms of the maximum value of its derivative over the same interval. These inequalities were introduced by Alexander Ostrowski in the year 1930 and have since been studied extensively in mathematical analysis and its applications.

Ostrowski integral inequalities have many applications in various fields of mathematics, such as approximation theory, numerical analysis, and optimization. For example, they can be used to derive error estimates for numerical integration methods or to establish convergence rates of approximation algorithms. Additionally, they have important applications in probability theory, where they are used to derive tail estimates for random variables and to establish concentration inequalities for empirical measures.

There are many variants of Ostrowski integral inequalities, each with its own set of conditions and assumptions. Some of the most well-known variants include the Grüss inequality, the Lupas-type inequality, and the Chebyshev inequality. These inequalities have been studied extensively by mathematicians and have led to many important results in the theory of inequalities and its applications.

In 1938, Ostrowski established the following valuable and attractive integral inequality (see [15], page 468).

Received December 12, 2023. Accepted November 11, 2024.

Let $g:[a_1,a_2]\to\mathbb{R}$, $g\in C^1(a_1,a_2)$, such that $g\in\mathcal{L}[a_1,a_2]$. If $\underset{\forall\, z\in[a_1,a_2]}{|g'(z)|}\le\mathcal{M}$, then the inequality holds:

$$\left|g(z)-\frac{1}{a_2-a_1}\int_{a_1}^{a_2}g(u)\,du\right|\le\mathcal{M}(a_2-a_1)\left[\frac{1}{4}+\frac{\left(z-\frac{a_1+a_2}{2}\right)^2}{(a_2-a_1)^2}\right] \tag{1}$$

Ostrovski inequality expresses the boundaries of the deviation of a function from its mean integral value.

For recent result and the related generalizations, variants and extensions concerning (1) (see [1, 2, 6, 9, 19, 21, 22, 28]).

Alomari et al, in [1] for $s-$convex functions in the second sense was obtained some new Ostrowski-type inequalities. For the quasi-convex functions obtained this inequality by Alomari and Darus in [2]. An Ostrowski-type inequality for mappings with bounded variation and its application to quadrature formulas is given by Dragomir in [9]. In [21] Pachpatte established a new Ostrowski type inequality for real functions of three variables, Set et al in [25] for $(\alpha,m)-$convex. In [28], Xie et al derived new Hermite-Hadamard, Simpson and Ostrowski type inequalities for differentiable convex transformations with respect to a pair of functions via Riemann-Liouville fractional integrals.

The theory of inequalities is closely related to the concept of convexity. Numerous classes of convex function described in the literature are collected in [18]. Some of them are: $m-$convex functions, $(\alpha,m)-$convex functions, $(s,m-$convex functions, $(s,m_1,m_2)-$convex functions, $(h,m)-$convex functions in generalized context. These classes and others are studied in various articles, for example (see [4, 5, 25]) and referenced therein.

One of the significant and new class of convex function produced by exponentially convex functions has vital applications in technology, information science, and statistics.

Definition 1.1. ([11]) For $(m_1,m_2)\in(0,1]$, a function $g:[0,v]\to\mathbb{R}$ is said to be $(m_1,m_2)-$convex, if

$$g(m_1\omega x+m_2(1-\omega)y)\le m_1\omega g(x)+m_2(1-\omega)g(y), \tag{2}$$

for all $x,y\in[0,v]$ and $\omega\in[0,1]$.

In [3], Awan et al. elaborated the following definition for exponentially convex function.

Definition 1.2. For $\lambda\in\mathbb{R}$ a function $g:\mathcal{I}\subseteq\mathbb{R}\to\mathbb{R}$ is said exponentially convex, if

$$g(\omega x+(1-\omega)y)\le\omega\,\frac{g(x)}{e^{\lambda a_1}}+(1-\omega)\,\frac{g(y)}{e^{\lambda a_2}}, \tag{3}$$

for all $a_1,a_2\in\mathcal{I}$, $\omega\in[0,1]$.

The Işcan-Hölder improved integral inequality is given below:

Theorem 1.1. ([10]) *Let* $f,g:[a_1,a_2]\to\mathbb{R}$. *If* $|f|^p,|g|^p\in\mathcal{L}[a_1,a_2]$ *with* $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}&\int_{a_1}^{a_2}|f(\omega)g(\omega)|\,d\omega\leq\\&\leq\frac{1}{a_2-a_1}\left\{\left(\int_{a_1}^{a_2}(a_2-\omega)|f(\omega)|^p\,d\omega\right)^{\frac{1}{p}}\left(\int_{a_1}^{a_2}(a_2-\omega)|g(\omega)|^q\,d\omega\right)^{\frac{1}{q}}\right.\\&\left.+\frac{1}{a_2-a_1}\left(\int_{a_1}^{a_2}(\omega-a_1)|f(\omega)|^p\,d\omega\right)^{\frac{1}{p}}\left(\int_{a_1}^{a_2}(\omega-a_1)|g(\omega)|^q\,d\omega\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{4}$$

The power-mean integral inequality has been improved upon and is further developed in the following ways:

Theorem 1.2. *([12]) Let* $f,g:[a_1,a_2]\to\mathbb{R}$. *and if* $|f|,|f||g|^p\in\mathcal{L}[a_1,a_2]$ *with* $q\geq 1$, *then*

$$\begin{aligned}&\int_{a}^{b}|f(\omega)g(\omega)|\,d\omega\leq\\&\leq\frac{1}{a_2-a_1}\left\{\left(\int_{a_1}^{a_2}(a_2-\omega)|f(\omega)|\,d\omega\right)^{1-\frac{1}{q}}\left(\int_{a_1}^{a_2}(a_2-\omega)|g(\omega)|^q\,d\omega\right)^{\frac{1}{q}}\right.\\&\left.+\left(\int_{a_1}^{a_2}(\omega-a_1)|f(\omega)|\,d\omega\right)^{1-\frac{1}{q}}\left(\int_{a_1}^{a_2}(\omega-a_1)|g(\omega)|^q\,d\omega\right)^{\frac{1}{q}}\right\}.\end{aligned}\tag{5}$$

This paper aims to define a new class of convex functions and to deduce new extended Ostrowski-like inequalities, spurred by recent advances in this important area of research.

2. Main Results

Here, we yield a new definition, which is known as exponentially $(m_1,m_2)-$convex.

Definition 2.1. A mapping $g:\mathcal{I}=[0,v]\to\mathbb{R}, v>0$ is called an exponentially $(m_1,m_2)-$convex, if

$$g(m_1\omega a_1+m_2(1-\omega)a_2)\leq m_1\omega\,\frac{g(a_1)}{e^{\lambda a_1}}\;+\;m_2(1-\omega)\,\frac{g(a_2)}{e^{\lambda a_2}},\tag{6}$$

for all $m_1a_1, m_2a_2\in I$, $\in[0,1]$ and $\lambda\in\mathbb{R}$, $m_1,m_2\in(0,1]$. In case inequality (6) is reversed, then g is termed as exponentially $(m_1,m_2)-$concave.

Example 2.1. The function $f(x)=x^2$ on any interval $[a_1,a_2]$ is $(m_1,m_2)-$convex.

Proof. Indeed, from the condition of $(m_1,m_2)-$convexity of the function, we can write:

$$[m_1\omega a_1+m_2(1-\omega)a_2]^2\leq m_1\omega a_1^2+m_2(1-\omega)a_2^2$$
$$\Longrightarrow m_1\omega(1-m_1\omega)a_1^2-2m_1m_2\omega(1-\omega)a_1a_2+m_2(1-\omega)(1-m_2(1-\omega))a_2^2\geq 0.$$

The discriminant of this quadratic form will be:

$$\begin{aligned}\Delta&=[m_1m_2\omega(1-\omega)]^2-m_1\omega(1-m_1\omega)m_2(1-\omega)\left[1-m_2(1-\omega)\right]\\&=[m_1m_2\omega(1-\omega)]^2-m_1m_2\omega(1-\omega)(1-m_1\omega)\left[1-m_2(1-\omega)\right]\\&=m_1m_2\omega(1-\omega)\left[m_1\omega+m_2(1-\omega)-1\right].\end{aligned}$$

Because $m_1, m_2 \in (0,1]$, then $m_1\omega+m_2(1-\omega) \in [m_1, m_2]$, i.e. $m_1\omega+m_2(1-\omega)-1 \leq 0$ and $\Delta \leq 0$. Thus, the quadratic form is positive definite, i.e. the convexity condition is satisfied. □

Example 2.2. The function $f(x) = x^2-k^2$ on any interval $[a_1, a_2]$ is $(m_1, m_2)-$convex.

Proof. Indeed, from the condition of $(m_1, m_2)-$convexity of the function, we can write:

$$[m_1\omega a_1 + m_2(1-\omega)b]^2 - k^2 \leq m_1\omega\left(a_1^2 - k^2\right) + m_2(1-\omega)\left(a_2^2 - k^2\right)$$
$$\Longrightarrow [m_1\omega a_1 + m_2(1-\omega)a_2]^2 - k^2 \leq m_1\omega a_1^2 + m_2(1-\omega)a_2^2 - k^2\left[m_1\omega + m_2(1-\omega)\right]$$
$$\Longrightarrow [m_1\omega a_1 + m_2(1-\omega)a_2]^2 \leq m_1\omega a_1^2 + m_2(1-\omega)a_2^2 + k^2\left[1 - m_1\omega - m_2(1-\omega)\right].$$

Considering Example 2.1. the last inequality is true for all (a_1, a_2). □

Two of the following statements are self-evident:

Proposition 2.1. *Let $f : [0, v] \to \mathbb{R}$ and function f is positive definite, then for all $\lambda \leq 0$ the following statement is true: any $(m_1, m_2)-$convex function is exponentially $(m_1, m_2)-$ convex.*

Proposition 2.2. *Let $f : [0, v] \to \mathbb{R}$ and function f is non positive definite, then for all $\lambda \geq 0$ the following statement is true: any $(m_1, m_2)-$ convex function is exponentially $(m_1, m_2)-$ convex.*

Remark 2.1. It follows from Definition 2.1. that

1. if $m_1 = m_2 = 1$ and $\lambda = 0$, then $f(x)$ is an classical convex function on $[a_1, a_2]$;
2. if $\lambda = 0$, then $f(x)$ is an (m_1, m_2)- convex function on $[a_1, a_2]$ see [11];
3. if $\lambda = 0$ and $m_1 = 1$ and $m = m_2$, then $f(x)$ is an m-convex function on $[a_1, a_2]$ see [26];
4. if $m_1 = m_2 = 1$, then $f(x)$ is an exponentially convex function on $[a_1, a_2]$ see [3];
5. if $m_1 = 1$ and $m = m_2$, then $f(x)$ is an exponentially m-convex function on $[a_1, a_2]$ see [23].

We will denote by $K_{m_1,m_2}(\mathcal{I})$ the class of all exponentially $(m_1, m_2)-$convex functions on the interval I for which $f(0) \leq 0$. The subsequent are some algebraic properties for the functions of this class.

Theorem 2.3. *Let $f, g : [0, v] \to \mathbb{R}$. If $f, g \in K_{m_1,m_2}[0, v]$,*
(i) $f + g$ is exponentially $(m_1, m_2)-$ convex,
(ii) For $c \in \mathbb{R}$ $(c \geq 0)$, cf is an exponentially $(m_1, m_2)-$convex.

Proof. (i) For $a_1, a_2 \in [0, v]$ and $\omega \in [0, 1]$, we have

$$\begin{aligned}
&(f+g)\left(m_1\omega a_1 + m_2\left(1-\omega\right)a_2\right)\\
=\ &f\left(m_1\omega a_1 + m_2\left(1-\omega\right)a_2\right) + g\left(m_1\omega a_1 + m_2\left(1-\omega\right)a_2\right)\\
\leq\ &\left(m_1\omega\frac{1}{e^{\lambda a_1}}f\left(a_1\right) + m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}f\left(a_2\right)\right)\\
&\quad + \left(m_1\omega\frac{1}{e^{\lambda a_1}}g\left(a_1\right) + m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}g\left(a_2\right)\right)\\
=\ &m_1\omega\frac{1}{e^{\lambda a_1}}\left(f+g\right)\left(a_1\right) + m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}\left(f+g\right)\left(a_2\right).
\end{aligned}$$

(ii) For $c\in\mathbb{R}\ \ (c\geq 0)$, we have

$$(cf)\left(m_1\omega a_1+m_2\left(1-\omega\right)a_2\right)\leq\ c\left(m_1\omega\frac{1}{e^{\lambda a_1}}f\left(a_1\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}f\left(a_2\right)\right)$$
$$\leq m_1\omega\frac{1}{e^{\lambda a_1}}\left(cf\right)\left(a_1\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}\left(cf\right)\left(a_2\right).$$

Hence proved. □

Theorem 2.4. *Let $f:[0,v]\to\mathbb{R}$ be a $(m_1,m_2)-$ convex function. If $g\in K_{m_1,m_2}[0,v]$ and increasing, then the function $gof\in K_{m_1,m_2}[0,v]$.*

Proof. For $a_1,a_2\in[0,v]$ and $\omega\in[0,1]$, we have

$$(gof)\left(m_1\omega a+m_2\left(1-\omega\right)a_2\right)=\ g\left(f\left(\left(m_1\omega a_1+m_2\left(1-\omega\right)a_2\right)\right)\right)$$
$$\leq g\left(m_1tf\left(a_1\right)+m_2\left(1-\omega\right)f\left(a_2\right)\right)$$
$$\leq\ m_1\omega\frac{1}{e^{\lambda a_1}}g\left(f\left(a_1\right)\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}g\left(f\left(a_2\right)\right)$$
$$\leq\ m_1\omega\frac{1}{e^{\lambda a_1}}\left(gof\right)\left(a_1\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}\left(gof\right)\left(b\right).$$

Hence proved. □

Theorem 2.5. *Let $m_1,m_2\in[0,1]$, $v>0$ and $f_\beta:[0,v]\to\mathbb{R}$ be an arbitrary family exponentially $(m_1,m_2)-$ convex functions and let $f(x)=Sup_\beta f_\beta(x)$. If $\mathcal{J}=\{u\in[0,v]:\frac{u}{m_1},\frac{u}{m_2}\in[0,v]$ and $f\left(u\right),\ f\left(\frac{u}{m_1}\right),\ f\left(\frac{u}{m_2}\right)<\infty\}$ is non-empty, then $\mathcal{J}$ is an interval and f is an exponentially $(m_1,m_2)-$convex on $\mathcal{J}$.*

Proof. Let $\omega\in[0,1]$ and $a_1,a_2\in\mathcal{J}$ be arbitrary. Then

$$f\left(\omega a_1+\left(1-\omega\right)a_2\right)=Sup_\beta f_\beta\left(m_1\omega\frac{a_1}{m_1}+m_2\left(1-\omega\right)\frac{a_2}{m_2}\right)$$
$$\leq\ Sup_\beta\left[m_1\omega\frac{1}{e^{\lambda a_1}}f_\beta\left(\frac{a_1}{m_1}\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}f_\beta\left(\frac{a_2}{m_2}\right)\right]$$
$$\leq\ m_1\omega\frac{1}{e^{\lambda a_1}}Sup_\beta f_\beta\left(\frac{a_1}{m_1}\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}Sup_\beta f_\beta\left(\frac{a_2}{m_2}\right)$$
$$\leq\ m_1\omega\frac{1}{e^{\lambda a_1}}f\left(\frac{a_1}{m_1}\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}f\left(\frac{a_2}{m_2}\right)<\infty.$$

This simultaneously shows that $\mathcal{J}$ is an interval because it contains every point between any two of its points.

Now we show that the function f is exponentially $(m_1,m_2)-$convex on $\mathcal{J}$.

If $\omega\in[0,1]$ and $a_1,a_2\in\mathcal{J}$,

$$f\left(m_1\omega a_1+m_2\left(1-\omega\right)a_2\right)=Sup_\beta f_\beta\left(m_1\omega a_1+m_2\left(1-\omega\right)a_2\right)$$
$$\leq\ Sup_\beta\left[m_1\omega\frac{1}{e^{\lambda a_1}}f_\beta\left(a_1\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}f_\beta\left(a_2\right)\right]$$
$$\leq\ m_1\omega\frac{1}{e^{\lambda a}}Sup_\beta f_\beta\left(a_1\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}Sup_\beta f_\beta\left(a_2\right)$$
$$\leq\ m_1\omega\frac{1}{e^{\lambda a}}f\left(a_1\right)+m_2\left(1-\omega\right)\frac{1}{e^{\lambda a_2}}f\left(a_2\right),$$

and that the function f exponentially $(m_1, m_2)-$convex on J.
Hence proved. □

Suppose $n \in \mathcal{N}$ and $a_1, a_2 \in \mathbb{R}$ and $a_2 > a_1$, $n \geq 1$. The interval $[a_1, a_2]$ with uniform step $\mathtt{h} = \frac{a_2 - a_1}{n}$ is divided into n sub intervals $[a_1, a_2] = \overset{n}{\underset{k=1}{U}} [\xi_{k-1}, \xi_k]$, where $\xi_k = a_1 + kh, k = 0, 1, 2, ..., n$. Throughout the article, we take $\alpha > 0$.

Lemma 2.6. *Suppose a mapping* $g : [a_1, a_2] \to \mathbb{R}$ *and* $g \in C^1(a_1, a_2)$. *If* $g' \in \mathcal{L}[a_1, a_2]$, *then* $\forall$ $x_k \in (\xi_{k-1}, \xi_k)$, *the following identity*

$$\sum_{k=1}^{n} \frac{(x_k - \xi_{k-1})^{a_2} + (\xi_k - x_k)^{\alpha}}{h} g(x_k) - \frac{\Gamma(\alpha+1)}{h} \sum_{k=1}^{n} \left(J^{\alpha}_{x_k^-} g(\xi_{k-1}) + J^{\alpha}_{x_k^+} g(\xi_k) \right)$$
$$= \sum_{k=1}^{n} (\mathbf{I}_{1k} - \mathbf{I}_{2k}), \tag{7}$$

where

$$\mathbf{I}_{1k} = \frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} g'(\omega x_k + (1-\omega)\xi_{k-1}) \, d\omega,$$
$$\mathbf{I}_{2k} = \frac{(\xi_k - x_k)^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} g'(\omega x_k + (1-\omega)\xi_k) \, d\omega$$

satisfies for $\omega \in [0, 1]$.

Proof. By employing the technique of integration by parts on the integrals under the sum and take into account that $h = \xi_k - \xi_{k-1}$, we get

$$\mathbf{I}_{1k} = \frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} g'(\omega x_k + (1-\omega)\xi_{k-1}) \, d\omega$$
$$= \frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \left[\frac{g(x_k)}{x_k - \xi_{k-1}} - \frac{\alpha}{x_k - \xi_{k-1}} \int_{\xi_{k-1}}^{x_k} \frac{(u - \xi_{k-1})^{\alpha-1}}{(x_k - \xi_{k-1})^{\alpha-1}} \frac{g(u)\, du}{x_k - \xi_{k-1}} \right].$$

Thus

$$\mathbf{I}_{1k} = \frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \left[\frac{g(x_k)}{x_k - \xi_{k-1}} - \frac{\Gamma(\alpha+1)}{(x_k - \xi_{k-1})^{\alpha+1}} J^{\alpha}_{x_k^-} g(\xi_{k-1}) \right]$$

and

$$\mathbf{I}_{2k} = \frac{(\xi_k - x_k)^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} g'(\omega x_k + (1-\omega)\xi_k) \, d\omega$$
$$= \frac{(\xi_k - x_k)^{\alpha+1}}{h} \left[\frac{\omega^{\alpha} g(\omega x_k + (1-\omega)\xi_k)}{x_k - \xi_k} \bigg|_0^1 - \int_0^1 \frac{\alpha t^{\alpha-1} g(\omega x_k + (1-\omega)\xi_k)}{x_k - \xi_k} d\omega \right]$$
$$= \frac{(\xi_k - x_k)^{\alpha+1}}{h} \left[-\frac{g(x_k)}{\xi_k - x_k} + \frac{\alpha}{\xi_k - x_k} \int_{x_k}^{\xi_k} \frac{(\xi_k - u)^{\alpha-1}}{(\xi_k - x_k)^{\alpha-1}} \frac{g(u)\, du}{\xi_k - x_k} \right].$$

So

$$\mathbf{I}_{2k} = \frac{(\xi_k - x_k)^{\alpha+1}}{h} \left\{ -\frac{g(x_k)}{\xi_k - x_k} + \frac{\Gamma(\alpha+1)}{(\xi_k - x_k)^{\alpha+1}} J^{\alpha}_{x_k^+} g(\xi_k) \right\}.$$

Combining I_{1k} and I_{2k} we get (7). □

Remark 2.2. By setting $n = \alpha = 1$, then from (7), we get equality form in [1](see Lemma 1).

Remark 2.3. By setting $n = 1$, then from Lemma 2.1. we get equality (2.1) in [24].

Theorem 2.7. *Let us consider* $g : [0, v] \to \mathbb{R}$ *and* $g \in C^1(0, v)$, *and* $\frac{a_1}{m_1}, \frac{a_2}{m_2} \in (0, v)$, *with* $\frac{a_1}{m_1} < \frac{a_2}{m_2}$. *If* $g' \in \mathcal{L}[\frac{a_1}{m_1}, \frac{a_2}{m_2}]$ *and* $|g'| \in K_{m_1,m_2}[\frac{a_1}{m_1}, \frac{a_2}{m_2}]$, *then the following inequality*

$$\left|\sum_{k=1}^{n} \frac{(x_k - \xi_{k-1})^{\alpha} + (\xi_k - x_k)^{\alpha}}{h} g(x_k) - \frac{\Gamma(\alpha+1)}{h} \sum_{k=1}^{n} \left(J^{\alpha}_{x_k^-} g(\xi_{k-1}) + J^{\alpha}_{x_k^+} g(\xi_k)\right)\right| \leq \frac{1}{\alpha+2} \sum_{k=1}^{n} \left[\frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \mathbf{G}_{1k} + \frac{(\xi_k - x_k)^{\alpha+1}}{h} \mathbf{G}_{2k}\right], \tag{8}$$

satisfies for $\omega \in [0, 1]$ *and* $m_1, m_2 \in (0, 1]$, *where*

$$\mathbf{G}_{1k} = \frac{m_1 \left|g'\left(\frac{x_k}{m_1}\right)\right|}{e^{\lambda \frac{x_k}{m_1}}} + \frac{m_2 \left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|}{(\alpha+1)\, e^{\lambda \frac{\xi_{k-1}}{m_2}}} \text{ and } \mathbf{G}_{2k} = \frac{m_1 \left|g'\left(\frac{x_k}{m_1}\right)\right|}{e^{\lambda \frac{x_k}{m_1}}} + \frac{m_2 \left|g'\left(\frac{\xi_k}{m_2}\right)\right|}{(\alpha+1)\, e^{\lambda \frac{\xi_k}{m_2}}}.$$

Proof. From (7), by using the triangle inequality and take into account that $h = \xi_k - \xi_{k-1}$, for $k = 1, 2, ..., n$, we obtain

$$\left|\sum_{k=1}^{n} \frac{(x_k - \xi_{k-1})^{\alpha} + (\xi_k - x_k)^{\alpha}}{h} g(x_k) - \frac{\Gamma(\alpha+1)}{h} \sum_{k=1}^{n} \left[J^{\alpha}_{x_k^-} g(\xi_{k-1}) + J^{\alpha}_{x_k^+} g(\xi_k)\right]\right| \leq \sum_{k=1}^{n} [|\mathbf{I}_{1k}| + |\mathbf{I}_{2k}|]. \tag{9}$$

Since $|g'|$ is exponentially $(m_1, m_2)-$convexity for the I_{1k}, we obtain

$$\begin{aligned} |\mathbf{I}_{1k}| &\leq \frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} \left|g'\left(m_1 \omega \frac{x_k}{m_1} + m_2 (1-\omega) \frac{\xi_{k-1}}{m_2}\right)\right| d\omega \\ &\leq \frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} \left[\omega\, m_1 \frac{\left|g'\left(\frac{x_k}{m_1}\right)\right|}{e^{\lambda \frac{x_k}{m_1}}} + (1-\omega)\, m_2 \frac{\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|}{e^{\lambda \frac{\xi_{k-1}}{m_2}}}\right] d\omega \\ &\leq \frac{(x_k - \xi_{k-1})^{\alpha+1}}{h} \left[\frac{m_1 \left|g'\left(\frac{x_k}{m_1}\right)\right|}{(\alpha+2)\, e^{\lambda \frac{x_k}{m_1}}} + \frac{m_2 \left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|}{(\alpha+1)(\alpha+2)\, e^{\lambda \frac{\xi_{k-1}}{m_2}}}\right] \end{aligned} \tag{10}$$

and, similarly for the second integral, we can write

$$\begin{aligned} |\mathbf{I}_{2k}| &\leq \frac{(\xi_k - x_k)^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} \left|g'(\omega x_k + (1-\omega)\xi_k)\right| d\omega \\ &\leq \frac{(\xi_k - x_k)^{\alpha+1}}{h} \int_0^1 \omega^{\alpha} \left[\omega\, m_1 \frac{\left|g'\left(\frac{x_k}{m_1}\right)\right|}{e^{\lambda \frac{x_k}{m_1}}} + (1-\omega)\, m_2 \frac{\left|g'\left(\frac{\xi_k}{m_2}\right)\right|}{e^{\lambda \frac{\xi_k}{m_2}}}\right] d\omega \\ &\leq \frac{(\xi_k - x_k)^{\alpha+1}}{h} \left[\frac{m_1 \left|g'\left(\frac{x_k}{m_1}\right)\right|}{(\alpha+2)\, e^{\lambda \frac{x_k}{m_1}}} + \frac{m_2 \left|g'\left(\frac{\xi_k}{m_2}\right)\right|}{(\alpha+1)(\alpha+2)\, e^{\lambda \frac{\xi_k}{m_2}}}\right]. \end{aligned} \tag{11}$$

Combining inequalities (10) and (11) with (9), we get (8). Hence proved. □

Corollary 2.8. *In Theorem 2.7 if* $\underset{\forall x\in[\frac{a_1}{m_1},\frac{a_2}{m_2}]}{|g'(x)|} \leq \mathcal{M}$, *then the inequality*

$$\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha}+(\xi_k-x_k)^{\alpha}}{h}g(x_k)-\frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left[J^{\alpha}_{x_k^-}g(\xi_{k-1})+J^{\alpha}_{x_k^+}g(\xi_k)\right]\right|$$
$$\leq\frac{\mathcal{M}}{\alpha+2}\left[\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}\left(\frac{m_1}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2}{(\alpha+1)e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)\right.$$
$$\left.+\sum_{k=1}^{n}\frac{(\xi_k-x_k)^{\alpha+1}}{h}\left(\frac{m_1}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2}{(\alpha+1)e^{\lambda\frac{\xi_k}{m_2}}}\right)\right]$$

is satisfied.

Remark 2.4. In the aforementioned inequality, by setting $\lambda=0, m_1=m_2=n=\alpha=1$ and in inequality (2.1) from [1] we choose $s=1$, then we obtain equivalent inequalities.

Corollary 2.9. *In Theorem 2.7, by letting* $\alpha=m_1=m_2=1$ *and* $\lambda=0$. *Also, if we choose* $x_k=\frac{\xi_{k-1}+\xi_k}{2}$, *then we get the mid-point inequality*

$$\left|\sum_{k=1}^{n}\left[g\left(\frac{\xi_{k-1}+\xi_k}{2}\right)-\frac{1}{\xi_k-\xi_{k-1}}\int_{\xi_{k-1}}^{\xi_k}g(u)\,du\right]\right|$$
$$=\left|\sum_{k=1}^{n}g\left(\frac{\xi_{k-1}+\xi_k}{2}\right)-\frac{n}{h}\int_{a_1}^{a_2}g(u)\,du\right|$$
$$\leq\frac{h}{3}\sum_{k=1}^{n}\left[\frac{|g'(\xi_{k-1})|+|g'(\xi_k)|}{2}+2g'\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right].$$

Theorem 2.10. *Let the considerations as defined in Theorem 2.7 are valid. If* $g'\in\mathcal{L}[\frac{a_1}{m_1},\frac{a_2}{m_2}]$ *and* $|g'|^q\in K_{m_1,m_2}[\frac{a_1}{m_1},\frac{a_2}{m_2}]$ *then the following inequality*

$$\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha}+(\xi_k-x_k)^{\alpha}}{h}g(x_k)-\frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left[J^{\alpha}_{x_k^-}g(\xi_{k-1})+J^{\alpha}_{x_k^+}g(\xi_k)\right]\right|$$
$$\leq\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\sum_{k=1}^{n}\left[\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}\cdot\mathbf{F}_{1k}+\frac{(\xi_k-x_k)^{\alpha+1}}{h}\cdot\mathbf{F}_{2k}\right]\tag{12}$$

holds for $\omega\in[0,1]$ *and* $m_1,m_2\in(0,1]$, p *and* q *are conjugate exponents,* $q>1$,

$$\mathbf{F}_{1k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{2e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{2e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}},$$

$$\mathbf{F}_{2k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{2e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{2e^{\lambda\frac{\xi_k}{m_2}}}\right)^{\frac{1}{q}}.$$

Proof. From (9), we have

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha}+(\xi_k-x_k)^{\alpha}}{h}g(x_k)-\frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left(J^{\alpha}_{x_k^-}g(\xi_{k-1})+J^{\alpha}_{x_k^+}g(\xi_k)\right)\right|\\
&\quad\le\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}\int_0^1\omega^{\alpha}\left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_{k-1}}{m_2}\right)\right|d\omega\\
&\quad+\sum_{k=1}^{n}\frac{(\xi_k-x_k)^{\alpha+1}}{h}\int_0^1\omega^{\alpha}\left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_k}{m_2}\right)\right|d\omega. \qquad (13)
\end{aligned}$$

By using the Hölder integral inequality and since $|g'|^q$ is exponentially $(m_1,m_2)-$convexity on $[\frac{a_1}{m_1},\frac{a_2}{m_2}]$, for the first integral on right-side of (13), we get

$$\begin{aligned}
&\int_0^1\omega^{\alpha}\left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_{k-1}}{m_2}\right)\right|d\omega\\
&\le\left(\int_0^1\omega^{\alpha p}d\omega\right)^{\frac{1}{p}}\left[\int_0^1\left(m_1\omega\frac{\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+m_2(1-\omega)\frac{\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)d\omega\right]^{\frac{1}{q}}\\
&\le\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{2e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{2e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right]^{\frac{1}{q}}=\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\cdot\mathbf{F}_{1k}. \qquad (14)
\end{aligned}$$

And for the second integral from (13), we can write

$$\begin{aligned}
&\int_0^1\omega^{\alpha}\left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_k}{m_2}\right)\right|d\omega\\
&\le\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{2e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{2e^{\lambda\frac{\xi_k}{m_2}}}\right]^{\frac{1}{q}}=\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\cdot\mathbf{F}_{2k}. \qquad (15)
\end{aligned}$$

From (14) and (15), taking into account (13), we obtain (9). Hence proved. □

Corollary 2.11. *In Theorem 2.10, if* $\underset{\forall x\in[\frac{a_1}{m_1},\frac{a_2}{m_2}]}{|g'(x)|}\le\mathcal{M}$, *then the inequality*

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha}+(\xi_k-x_k)^{\alpha}}{h}g(x_k)-\frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left[J^{\alpha}_{x_k^-}g(\xi_{k-1})+J^{\alpha}_{x_k^+}g(\xi_k)\right]\right|\\
&\le\frac{\mathcal{M}}{2^{\frac{1}{q}}}\cdot\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\sum_{k=1}^{n}\left[\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}\left(\frac{m_1}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2}{e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}\right.\\
&\quad\left.+\frac{(\xi_k-x_k)^{\alpha+1}}{h}\left(\frac{m_1}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2}{e^{\lambda\frac{\xi_k}{m_2}}}\right)^{\frac{1}{q}}\right]
\end{aligned}$$

is satisfied.

Remark 2.5. In the aforementioned corollary, by setting $\lambda = 0, m_1 = m_2 = n = \alpha = 1$ and in inequality (2.2) from [1] we choose $s = 1$, then we obtain equivalent inequalities.

Corollary 2.12. *In Theorem 2.10, by letting $\alpha = m_1 = m_2 = 1$ and $\lambda = 0$. Also, if we choose $x_k = \frac{\xi_{k-1}+\xi_k}{2}$, then we get the mid-point inequality*

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\left[g\left(\frac{\xi_{k-1}+\xi_k}{2}\right)-\frac{1}{\xi_k-\xi_{k-1}}\int_{\xi_{k-1}}^{\xi_k} g(u)\,du\right]\right| \\
&=\left|\sum_{k=1}^{n} g\left(\frac{\xi_{k-1}+\xi_k}{2}\right)-\frac{n}{h}\int_{a_1}^{a_2} g(u)\,du\right| \\
&\leq \frac{h}{2^{\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\sum_{k=1}^{n}\left\{\left[\left|g'\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right|^q+\left|g'(\xi_{k-1})\right|^q\right]^{\frac{1}{q}}\right. \\
&\qquad\left.+\left[\left|g'\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right|^q+\left|g'(\xi_k)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Theorem 2.13. *Let the considerations as defined in Theorem 2.7 are valid. If $g' \in \mathcal{L}[\frac{a_1}{m_1}, \frac{a_2}{m_2}]$ and $|g'|^q \in K_{m_1,m_2}[\frac{a_1}{m_1}, \frac{a_2}{m_2}]$, then the following inequality*

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha}+(\xi_k-x_k)^{\alpha}}{h}g(x_k)-\frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left[J_{x_k^-}^{\alpha}g(\xi_{k-1})+J_{x_k^+}^{\alpha}g(\xi_k)\right]\right| \\
&\leq \frac{1}{\alpha+1}\left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{1}{q}}\sum_{k=1}^{n}\left[\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}\mathbf{C}_{1k}+\frac{(\xi_k-x_k)^{\alpha+1}}{h}\mathbf{C}_{2k}\right], \qquad (16)
\end{aligned}$$

holds for $\omega \in [0,1]$ and $m_1, m_2 \in (0,1]$, p and q are conjugate exponents, $q \geq 1$,

$$\mathbf{C}_{1k} = \left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{(\alpha+1)\,e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}$$

$$\mathbf{C}_{2k} = \left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{(\alpha+1)\,e^{\lambda\frac{\xi_k}{m_2}}}\right)^{\frac{1}{q}}.$$

Proof. By using the power-mean integral inequality and since exponentially $(m_1, m_2)-$convexity of $|g'|^q$, for the first integral on right-side of (13), we obtain

$$\begin{aligned}
&\int_0^1 \omega^{\alpha}\left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_{k-1}}{m_2}\right)\right|d\omega \\
&\leq \left(\int_0^1 \omega^{\alpha}d\omega\right)^{1-\frac{1}{q}}\left[\int_0^1 \omega^{\alpha}\left(m_1\omega\frac{\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+m_2(1-\omega)\frac{\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)d\omega\right]^{\frac{1}{q}}
\end{aligned}$$

$$= \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{(\alpha+2)\, e^{\lambda\frac{x_k}{m_1}}} + \frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{(\alpha+1)(\alpha+2)\, e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}. \tag{17}$$

And for the second integral from (13), we can write

$$\int_0^1 \omega^\alpha \left|g'\left(m_1\omega\frac{x_k}{m_1} + m_2(1-\omega)\frac{\xi_k}{m_2}\right)\right| d\omega$$
$$\leq \left(\frac{1}{\alpha+1}\right)^{1-\frac{1}{q}} \left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{(\alpha+2)\, e^{\lambda\frac{x_k}{m_1}}} + \frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{(\alpha+1)(\alpha+2)\, e^{\lambda\frac{\xi_k}{m_2}}}\right)^{\frac{1}{q}}. \tag{18}$$

From (17) and (18), taking into account (13), we obtain (16). Hence proved. □

Corollary 2.14. *In Theorem 2.13, if* $\underset{\forall x\in[\frac{a_1}{m_1},\frac{a_2}{m_2}]}{|g'(x)|} \leq \mathcal{M}$*, then the inequality*

$$\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^\alpha + (\xi_k - x_k)^\alpha}{h} g(x_k) - \frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left[J^\alpha_{x_k^-} g(\xi_{k-1}) + J^\alpha_{x_k^+} g(\xi_k)\right]\right|$$
$$\leq \frac{\mathcal{M}}{\alpha+1}\left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{1}{q}} \left[\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}\left(\frac{m_1}{e^{\lambda\frac{x_k}{m_1}}} + \frac{m_2}{(\alpha+1)\, e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}\right.$$
$$\left. + \sum_{k=1}^{n}\frac{(\xi_k - x_k)^{\alpha+1}}{h}\left(\frac{m_1}{e^{\lambda\frac{x_k}{m_1}}} + \frac{m_2}{(\alpha+1)\, e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}\right]$$

is satisfied.

Remark 2.6. In Corollary 2.14 by setting $\lambda = 0, m_1 = m_2 = n = \alpha = 1$ and in inequality (2.3) from [1] we choose $s = 1$, then we obtain equivalent inequalities.

Corollary 2.15. *In Theorem 2.13, by setting* $\alpha = m_1 = m_2 = 1$ *and* $\lambda = 0$*. Also, if we choose* $x_k = \frac{\xi_{k-1}+\xi_k}{2}$*, then we get the mid-point inequality*

$$\left|\sum_{k=1}^{n}\left[g\left(\frac{\xi_{k-1}+\xi_k}{2}\right) - \frac{1}{\xi_k-\xi_{k-1}}\int_{\xi_{k-1}}^{\xi_k} g(u)\, du\right]\right|$$
$$\leq \frac{h}{4\cdot 3^{\frac{1}{q}}}\sum_{k=1}^{n}\left[\left(2\left|g'\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right|^q + |g'(\xi_{k-1})|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\left|2g'\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right|^q + |g'(\xi_k)|^q\right)^{\frac{1}{q}}\right].$$

Theorem 2.16. *Let the considerations as defined in Theorem 2.7 are valid. If* $g' \in \mathcal{L}[\frac{a_1}{m_1}, \frac{a_2}{m_2}]$ *and* $|g'|^q \in K_{m_1,m_2}[\frac{a_1}{m_1}, \frac{a_2}{m_2}]$*, then the following inequality*

$$\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^\alpha + (\xi_k - x_k)^\alpha}{h} g(x_k) - \frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left[J^\alpha_{x_k^-} g(\xi_{k-1}) + J^\alpha_{x_k^+} g(\xi_k)\right]\right| \leq$$

$$\leq \left(\frac{1}{\alpha p+2}\right)^{\frac{1}{p}} \sum_{k=1}^{n} \frac{(x_k-\xi_{k-1})^{\alpha+1}}{h} \left\{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \mathbf{A}_{1k}+\mathbf{A}_{2k}\right\}$$
$$+\left(\frac{1}{\alpha p+2}\right)^{\frac{1}{p}} \sum_{k=1}^{n} \frac{(\xi_k-x_k)^{\alpha+1}}{h} \left\{\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \mathbf{A}_{3k}+\mathbf{A}_{4k}\right\} \tag{19}$$

holds for $\omega\in[0,1]$ *and* $m_1,m_2\in(0,1]$, p *and* q *are conjugate exponents,* $q>1$,

$$\mathbf{A}_{1k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{6e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{3e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}},$$

$$\mathbf{A}_{2k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{3e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{6e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}},$$

$$\mathbf{A}_{3k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{6e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k}}{m_2}\right)\right|^q}{3e^{\lambda\frac{\xi_{k}}{m_2}}}\right)^{\frac{1}{q}},$$

$$\mathbf{A}_{4k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{3e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k}}{m_2}\right)\right|^q}{6e^{\lambda\frac{\xi_{k}}{m_2}}}\right)^{\frac{1}{q}}.$$

Proof. By using the Hölder-İşcan integral inequality and since exponentially $(m_1,m_2)-$convexity of $|g'|^q$, for the first integral on right-side of (13), we obtain

$$\int_0^1 \omega^\alpha \left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_{k-1}}{m_2}\right)\right| d\omega \leq \left(\int_0^1 (1-\omega)\,\omega^{\alpha p} d\omega\right)^{\frac{1}{p}}$$
$$\times\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}\int_0^1(1-\omega)\,\omega d\omega+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{e^{\lambda\frac{\xi_{k-1}}{m_2}}}\int_0^1(1-\omega)^2 d\omega\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 \omega^{\alpha p+1} d\omega\right)^{\frac{1}{p}}\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}\int_0^1\omega^2 d\omega+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{e^{\lambda\frac{\xi_{k-1}}{m_2}}}\int_0^1(1-\omega)\,\omega d\omega\right)^{\frac{1}{q}}$$
$$=\left(\frac{1}{(\alpha p+1)(\alpha p+2)}\right)^{\frac{1}{p}}\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{6e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{3e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\alpha p+2}\right)^{\frac{1}{p}}\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{3e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{6e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}.$$

And for the second integral from (13), we have

$$\int_0^1 \omega^\alpha \left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_k}{m_2}\right)\right| d\omega \leq$$

$$\leq \left(\frac{1}{(\alpha p+1)(\alpha p+2)}\right)^{\frac{1}{p}}\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{6e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{3e^{\lambda\frac{\xi_k}{m_2}}}\right)$$
$$+\left(\frac{1}{\alpha p+2}\right)^{\frac{1}{p}}\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{3e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{6e^{\lambda\frac{\xi_k}{m_2}}}\right).$$

Take of account of notations for the right-side of (13), we get

$$\sum_{k=1}^{n}\left\{\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}\left[\left(\frac{1}{(\alpha p+1)(\alpha p+2)}\right)^{\frac{1}{p}}\mathbf{A}_{1k}+\left(\frac{1}{\alpha p+2}\right)^{\frac{1}{p}}\mathbf{A}_{2k}\right]\right.$$
$$\left.+\frac{(\xi_k-x_k)^{\alpha+1}}{h}\left[\left(\frac{1}{(\alpha p+1)(\alpha p+2)}\right)^{\frac{1}{p}}\mathbf{A}_{3k}+\left(\frac{1}{\alpha p+2}\right)^{\frac{1}{p}}\mathbf{A}_{4k}\right]\right\}. \tag{20}$$

From (13) taken into account (20), we obtain (19). Hence proved. □

Theorem 2.17. *Let the considerations as defined in Theorem 2.7 are valid. If* $g'\in L[\frac{a_1}{m_1},\frac{a_2}{m_2}]$ *and* $|g'|^q\in K_{m_1,m_2}[\frac{a_1}{m_1},\frac{a_2}{m_2}]$, *then the following inequality*

$$\left|\sum_{k=1}^{n}\frac{(x_k-\xi_{k-1})^{\alpha}+(\xi_k-x_k)^{\alpha}}{h}g(x_k)-\frac{\Gamma(\alpha+1)}{h}\sum_{k=1}^{n}\left[J^{\alpha}_{x_k-}g(\xi_{k-1})+J^{\alpha}_{x_k+}g(\xi_k)\right]\right|$$
$$\leq\sum_{k=1}^{n}\left[\frac{(x_k-\xi_{k-1})^{\alpha+1}}{h}(\mathbf{a}_1\cdot\mathbf{P}_{1k}+\mathbf{a}_2\cdot\mathbf{P}_{2k})+\frac{(\xi_k-x_k)^{\alpha+1}}{h}(\mathbf{a}_1\cdot\mathbf{P}_{3k}+\mathbf{a}_2\cdot\mathbf{P}_{4k})\right] \tag{21}$$

holds for $\omega\in[0,1]$ *and* $m_1,m_2\in(0,1]$, p *and* q *are conjugate exponents,* $q\geq 1$,

$$\mathbf{a}_1=\frac{(\alpha+1)^{\frac{1}{q}-1}}{(\alpha+2)(\alpha+3)^{\frac{1}{q}}},\quad \mathbf{a}_2=\frac{(\alpha+2)^{\frac{1}{q}-1}}{(\alpha+3)^{\frac{1}{q}}}.$$

$$\mathbf{P}_{1k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{2m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{(\alpha+1)e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}},$$

$$\mathbf{P}_{2k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{(\alpha+2)e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}},$$

$$\mathbf{P}_{3k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{2m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{(\alpha+1)e^{\lambda\frac{\xi_k}{m_2}}}\right)^{\frac{1}{q}},$$

$$\mathbf{P}_{4k}=\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{(\alpha+2)e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right)^{\frac{1}{q}}.$$

Proof. By using the improved power-mean integral inequality and since exponentially (m_1, m_2)-convexity of $|g'|^q$, for the first integral on right-side of (13), we obtain

$$\int_0^1 \omega^\alpha \left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_{k-1}}{m_2}\right)\right|d\omega \leq \left(\int_0^1 (1-\omega)\,\omega^\alpha d\omega\right)^{1-\frac{1}{q}}$$
$$\times\left(m_1\frac{\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}\int_0^1 (1-\omega)\,\omega^{\alpha+1}d\omega + m_2\frac{\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{e^{\lambda\frac{\xi_{k-1}}{m_2}}}\int_0^1 (1-\omega)^2\,\omega^\alpha d\omega\right)^{\frac{1}{q}}$$
$$+\left(\int_0^1 \omega^{\alpha+1}d\omega\right)^{1-\frac{1}{q}}\left(\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}\int_0^1 \omega^{\alpha+2}d\omega + \frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{e^{\lambda\frac{\xi_{k-1}}{m_2}}}\int_0^1 \omega^{\alpha+1}(1-\omega)\,d\omega\right)^{\frac{1}{q}}$$

By computing the integrals, we get

$$\leq \frac{(\alpha+1)^{\frac{1}{q}-1}}{(\alpha+2)(\alpha+3)^{\frac{1}{q}}}\left[\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{2m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{(\alpha+1)\,e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right]^{\frac{1}{q}}$$
$$+\frac{(\alpha+2)^{\frac{1}{q}-1}}{(\alpha+3)^{\frac{1}{q}}}\left[\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_{k-1}}{m_2}\right)\right|^q}{(\alpha+2)\,e^{\lambda\frac{\xi_{k-1}}{m_2}}}\right]^{\frac{1}{q}}.$$

And for the second integral from (13), we have

$$\int_0^1 \omega^\alpha \left|g'\left(m_1\omega\frac{x_k}{m_1}+m_2(1-\omega)\frac{\xi_k}{m_2}\right)\right|d\omega$$
$$\leq \frac{(\alpha+1)^{\frac{1}{q}-1}}{(\alpha+2)(\alpha+3)^{\frac{1}{q}}}\left[\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{2m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{(\alpha+1)\,e^{\lambda\frac{\xi_k}{m_2}}}\right]^{\frac{1}{q}}$$
$$+\frac{(\alpha+2)^{\frac{1}{q}-1}}{(\alpha+3)^{\frac{1}{q}}}\left[\frac{m_1\left|g'\left(\frac{x_k}{m_1}\right)\right|^q}{e^{\lambda\frac{x_k}{m_1}}}+\frac{m_2\left|g'\left(\frac{\xi_k}{m_2}\right)\right|^q}{(\alpha+2)\,e^{\lambda\frac{\xi_k}{m_2}}}\right]^{\frac{1}{q}}.$$

Taking into account the notations, we obtain (21). □

3. Applications

For arbitrary numbers α and β we consider the following the means:

1. *Arithmetic mean* : $A(\alpha,\beta)=\dfrac{\alpha+\beta}{2}, \qquad \alpha,\beta\in\mathbb{R};$
2. *Logaritmic mean* : $L(\alpha,\beta)=\dfrac{\alpha-\beta}{\log\alpha-\log\beta}, \qquad \alpha,\beta>0;$
3. *The generalized logarithmic mean* : $L_r(\alpha,\beta)=\left[\dfrac{\alpha^{r+1}-\beta^{r+1}}{(r+1)(\alpha-\beta)}\right]^{\frac{1}{r}},$
   $r\in\mathbb{R}\setminus\{-1,0\}, \alpha,\beta\in\mathbb{R}.$

Proposition 3.1. *Suppose* $r \in Z \backslash \{-1, 0\}$ *and* $\xi_{k-1}, \xi_k \in \mathbb{R}$ *such that* $0 < \xi_{k-1} < \xi_k$, *then the following inequality*

$$\left| \sum_{k=1}^{n} \left[A^r (\xi_{k-1}, \xi_k) - L_r^r (\xi_{k-1}, \xi_k) \right] \right| \leq \frac{rh}{\alpha + 2} \sum_{k=1}^{n} \left[\frac{|\xi_{k-1}|^{r-1} + |\xi_k|^{r-1}}{\alpha + 1} + 2 \left| A^{r-1} (\xi_{k-1}, \xi_k) \right| \right]$$

holds true.

Proof. It is claimed from Corollary 2.2. for $g(x) = x^r$. ☐

Proposition 3.2. *Suppose* $q \geq 1$ *and* $\xi_{k-1}, \xi_k \in \mathbb{R}$ *such that* $0 < \xi_{k-1} < \xi_k$, *then the following inequality*

$$\left| \sum_{k=1}^{n} \left[A^{-1} (\xi_{k-1}, \xi_k) - L^{-1} (\xi_{k-1}, \xi_k) \right] \right| \leq \left(\frac{1}{2} \right)^{1-\frac{1}{q}} h \sum_{k=1}^{n} \left[\left(\frac{1}{3} \left| A^{-2} (\xi_{k-1}, \xi_k) \right|^q + \frac{1}{6} \left| \xi_{k-1}^{-2} \right|^q \right)^{\frac{1}{q}} + \left(\frac{1}{3} \left| A^{-2} (\xi_{k-1}, \xi_k) \right|^q + \frac{1}{6} \left| \xi_k^{-2} \right|^q \right)^{\frac{1}{q}} \right]$$

holds true.

Proof. It is claimed from Corollary 2.15 for $g(x) = \frac{1}{x}$. ☐

3.1. q-digamma function.

The **q**-digamma function is a special function in mathematics that arises in the study of q-series and related topics. It is a generalization of the classical digamma function, which is a well-known special function in its own right. The **q**-digamma function has many applications in various branches of mathematics and physics, including number theory, combinatorics, statistical mechanics, and quantum field theory. For example, it appears in the computation of partition functions in statistical mechanics, and in the calculation of Feynman diagrams in quantum field theory. Moreover, it has connections to other special functions, such as the **q**-gamma function and the **q**-beta function. Due to its rich properties and broad range of applications, the q-digamma function is an important tool in the mathematical sciences.

Suppose $0 < \mathbf{q} < 1$, the $\mathbf{q}-$digamma function $\varphi_{\mathbf{q}}$ is the $\mathbf{q}-$analogue of the digamma function φ defined by (see[27]).

$$\varphi_{\mathbf{q}} = -\ln(1 - \mathbf{q}) + \ln \mathbf{q} \sum_{j=0}^{\infty} \frac{\mathbf{q}^{j+x}}{1 - \mathbf{q}^{j+x}} = -\ln(1 - \mathbf{q}) + \ln \mathbf{q} \sum_{j=0}^{\infty} \frac{\mathbf{q}^{jx}}{1 - \mathbf{q}^{jx}}.$$

For $\mathbf{q} > 1$ and $x > 0$, $\mathbf{q}-$digamma function $\varphi_{\mathbf{q}}$ is defined by

$$\varphi_{\mathbf{q}} = -\ln(\mathbf{q} - 1) + \ln \mathbf{q} \left[x - \frac{1}{2} - \sum_{j=0}^{\infty} \frac{\mathbf{q}^{-(j+x)}}{1 - \mathbf{q}^{-(j+x)}} \right]$$

$$= -\ln(\mathbf{q}-1) + \ln\mathbf{q}\left[x - \frac{1}{2} - \sum_{j=0}^{\infty}\frac{\mathbf{q}^{-jx}}{1-\mathbf{q}^{-jx}}\right].$$

Proposition 3.3. *Suppose* $0 < \xi_{k-1} < \xi_k$, $q > 1$ *and* $\xi_{k-1}, \xi_k \in \mathbb{R}$ *and* $q^{-1} = 1 - p^{-1}$, *for* $k = 1, 2, 3, ..., n$, *then the following inequality*

$$\left|\sum_{k=1}^{n}\left[\varphi_{\mathbf{q}}\left(\frac{\xi_{k-1}+\xi_k}{2}\right) - \frac{1}{\xi_k - \xi_{k-1}}\int_{\xi_{k-1}}^{\xi_k}\varphi_{\mathbf{q}}(u)\,du\right]\right|$$
$$\leq \frac{1}{2^{\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} h\sum_{k=1}^{n}\left[\left(\left|\varphi'_{\mathbf{q}}\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right|^q + \left|\varphi'_{\mathbf{q}}(\xi_{k-1})\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\left|\varphi'_{\mathbf{q}}\left(\frac{\xi_{k-1}+\xi_k}{2}\right)\right|^q + \left|\varphi'_{\mathbf{q}}(\xi_k)\right|^q\right)^{\frac{1}{q}}\right]$$

holds true.

Proof. It is claimed from Corollary 2.3. for $g(x) = \varphi_{\mathbf{q}}(x)$, $x > 0$, since $g'(x) = \varphi'_{\mathbf{q}}(x)$. □

3.2. Conclusion. In this article, we have defined exponentially $(m_1, m_2)-$convex functions and demonstrated some algebraic properties of this class. A number of new integral inequalities were obtained. All these results are related and are new in the literature. Since the results of convex analysis are the basis and argument in favor of many inequalities in pure and applied sciences. We believe that our new results on relatively exponentially (m_1, m_2)-convex functions will open new directions for research in the field of inequalities, as well as in pure and applied sciences. In the future, our goal is to continue our research in this direction by using strongly convex functions theory.

References

[1] M. Alomari, M. Darus, S. S. Dragomir, P. Cerone, Ostrowski type inequalities for functions whose derivatives are $s-$convex in the second sense, *Applied Mathematics Letters* **23** (2010), no. 9, 1071-1076. DOI:10.1016/j.aml.2010.04.038

[2] M. Alomari, M. Darus, Some Ostrowski type inequalities for quasi-convex functions with applications to special means, *RGMIA Res. Rep. Coll.* **13** (2010), no. 2.

[3] M.U. Awan, M.A. Noor, K.I. Noor, Hermite-Hadamard inequalities for exponentially convex functions, *Appl. Math. Inf. Sci.* **12** (2018), no. 2, 405-409. DOI: 10.12785/amis/120215

[4] B. Bayraktar, S. Butt, S. Shaukat, J. Nápoles, New Hadamard Type Inequalities Via $(s, m_1, m_2)-$ *Convex Functions*, *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki* **31** (2021), no. 4, 597-612. DOI:10.35634/Vm210405

[5] B. Bayraktar, J.E. Nápoles Valdes, Integral inequalities for mappings whose derivatives are $(h; m; s)$convex modfied of second type via Katugampola integrals, *Annals of the University of Craiova, Mathematics and Computer Science Series* **49**, (2022), no 2, 371-383. DOI: 10.52846/ami.v49i2.1596

[6] S.I. Butt, J. Nasir, M.A. Dokuyucu, A.O. Akdemir, E. Set, Some Ostrowski-Mercer type inequalities for differentiable convex functions via fractional integral operators with strong kernels, *Applied and computational mathematics* **21** (2022), no. 3, 329–348.

[7] S.I. Butt, S. Yousaf, A.O. Akdemir, M.A. Dokuyucu, New Hadamard-type integral inequalities via a general form of fractional integral operators, *Chaos, Solitons and Fractals* **148** (2021), 111025. DOI: 10.1016/j.chaos.2021.111025

[8] P. Cerone, S.S. Dragomir, Ostrowski type inequalities for functions whose derivatives satisfy certain convexity assumptions, *Demonstratio Math.* **XXXVII** (2004), no. 2, 299–308.

[9] S.S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **1** (1998), no. 2, 59–66.

[10] İ. İşcan, New refinements for integral and sum forms of Hölder inequality, *Journal of Inequalities and Applications* **2019** (2019), no. 1, 1-11.

[11] H. Kadakal, (α, m_1, m_2)-convexity and some inequalities of Hermite-Hadamard type, *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.* **68** (2019), no. 2, 2128–2142. DOI: 10.31801/cfsuasmas.511184

[12] M. Kadakal, I. Işcan, H. Kadakal, K. Bekar, On improvements of some integral inequalities, *Researchgate*, Preprint, January, (2019). DOI: 10.13140/RG.2.2. 15052.46724.

[13] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[14] T. Lara, E. Rosales, J.L. Sánchez, New properties of $m-$convex functions, *International Journal of Mathematical Analysis* **9** (2015), no. 15, 735-742. DOI: 10.12988/ijma.2015.412389

[15] D.S. Mitrinovič, J. Pečarič, A. M. Fink, Inequalities involving functions and their integrals and derivatives, *Springer Science and Business Media* **53**, (1991).

[16] J.E. Nápoles, B. Bayraktar, On The Generalized Inequalities Of The Hermite – Hadamard Type, *FILOMAT* **35** (2021), no. 14, 4917–4924. https://doi.org/10.2298/FIL2114917N

[17] J.E. Nápoles, B. Bayraktar, S.I. Butt, New integral inequalities of Hermite–Hadamard type in a generalized context, *Punjab University Journal Of Mathematics*, **53** (2021), no. 11, 765–777. Doi:10.52280/Pujm.2021.531101

[18] J.E. Nápoles, F. Rabossi, A.D. Samaniego, CONVEX FUNCTIONS: ARIADNE'S THREAD OR CHARLOTTE'S SPIDERWEB?, *Advanced Mathematical Models and Applications* **5** (2020), no. 2, 176–191.

[19] M.A. Noor, K.I. Noor, M.U. Awan, Fractional Ostrowski inequalities for $s-$Godunova-Levin functions, *International Journal of Analysis and Applications* **5** (2014), no. 2, 167–173.

[20] M.E. Özdemir, S.I. Butt, B. Bayraktar, J. Nasir, Several integral inequalities for $(\alpha, s, m)-$convex functions, *AIMS Mathematics* **5** (2020), no. 4, 3906–3921. DOI: 10.3934/math.202053

[21] B.G. Pachpatte, On an inequality of Ostrowski type in three independent variables, *J. Math. Anal. Appl.* **249** (2000), 583–591. DOI:10.1006/jmaa.2000.6913

[22] A. Rafiq, N.A. Mir, F. Ahmad, Weighted Chebysev–Ostrowski type inequalities, *Applied Math. Mechanics (English Edition)* **28** (2007), no. 7, 901–906.

[23] S. Rashid, M.A. Noor, K.I. Noor, Fractional exponentially $m-$convex functions and inequalities, *International Journal of Analysis and Applications* **17** (2019), no. 3, 464–478.

[24] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are $s-$convex in the second sense via fractional integrals, *Computers and Mathematics with Applications* **63** (2012), no. 7, 1147–1154. DOI:10.1016/j.camwa.2011.12.023

[25] M.E. Özdemir, H. Kavurmaci, E. Set, Ostrowski's type inequalities for $(\alpha, m)-$*convex function*, *Kyungpook Mathematical Journal* **50** (2010), no. 3, 371–378. DOI: 10.5666/KMJ.2010.50.3.371

[26] G. Toader, Some generalizations of the convexity, *Proceedings of The Colloquium On Approximation and Optimization*, Univ. Cluj-Napoca, Cluj-Napoca, 1984, 329-338.

[27] G.N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1995.

[28] J. Xie, M.A. Ali, H. Budak, M. Fekan, T. Sitthiwirattham, Fractional Hermite-Hadamard inequality, Simpson's and Ostrowski's type inequalities for convex functions with respect to a pair of functions, *Rocky Mountain J. Math.* **53** (2023), no. 2, 611-628.

(Saad Ihsan Butt) COMSATS University Islamabad, Lahore Campus, Pakistan
E-mail address: saadihsanbutt@gmail.com

(Bahtiyar Bayraktar) Bursa Uludag University, Faculty of Education, Gorukle Capus, Bursa, Turkey
E-mail address: bbayraktar@uludag.edu.tr, bahtiyar.bayraktar@gmail.com

(Jamshed Nasir) Virtual University of Pakistan, Department of Mathematics, Pakistan
E-mail address: jnasir143@gmail.com