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On strong ergodicity of iterated function systems, with applications to time series models

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Abstract. We consider so-called iterated function systems, mainly as a mathematical model of (non-linear) autoregressive time series. We apply recent results on Markov chains to iterated function systems and such time series. The goal is to ensure the aperiodic strong ergodicity of Markov chains generated by iterated function systems, respectively of the time series.

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1. Introduction

In the last two decades the so-called "iterated function systems" (IFSs for short) or "iterated random functions" have attracted much attention. Researchers from quite different branches of applied mathematics, as engineers, physicists, mathematicians who study the construction of fractals or analyze certain autoregressive time series models, dealt with such systems.

From a mere theoretical standpoint, IFSs are a mathematical model consisting of two stochastic processes in discrete time, a state and an event process. The event process $(X_n, n \geq 0)$ is a sequence of independent identically distributed (i. i. d. for short) random variables with values in a measurable space $(X, \mathcal{X})$ distributed according to a probability measure p on $\mathcal{X}$. The state process $(W_n, n \geq 0)$ with values in the measurable space $(W, \mathcal{W})$ starts with a given random variable W_0 distributed according to a probability measure μ on $\mathcal{W}$ and is generated recursively by means of a measurable transition function $u : (W \times X, \mathcal{W} \otimes \mathcal{X}) \to (W, \mathcal{W})$ such that $W_{n+1} = u(W_n, X_n)$ for all $n \geq 0$. Obviously, the state process $(W_n, n \geq 0)$ is a Markov chain on $(W, \mathcal{W})$ with transition probability Q, where Q is determined by p and u as

$$Q(w, A) = p(u_w^{-1}(A)),\ w \in W, A \in \mathcal{W}.$$

Here $u_w : X \to W$ stands for the section of u defined by $u_w(x) = u(w, x), w \in W, x \in X$. We shall also use the section $u_x : W \to W$ of u defined by $u_x(w) = u(w, x)$, $x \in X$, $w \in W$.

The components of an IFS are collected in

Definition 1.1. *An IFS is a quadruple* $\{(W, \mathcal{W}), (X, \mathcal{X}), u, p\}$*, where*
- $(W, \mathcal{W})$ *and* $(X, \mathcal{X})$ *are measurable spaces,*
- $u : (W \times X, \mathcal{W} \otimes \mathcal{X}) \to (W, \mathcal{W})$ *is a measurable function,*
- p *is a probability measure on* $\mathcal{X}$*.*

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According to the Ionescu Tulcea theorem, for each probability measure μ on $\mathcal{W}$ there exist a probability space $(\Omega, \mathcal{K}, \mathbb{P}_\mu) = ((W \times X)^{\mathbb{N}}, (\mathcal{W} \otimes \mathcal{X})^{\mathbb{N}}, \mathbb{P}_\mu)$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and two sequences $(W_n, n \geq 0)$ und $(X_n, n \geq 0)$ with values in W respectively X such that $\mathbb{P}_\mu(W_0 \in A) = \mu(A), A \in \mathcal{W}$, $(X_n, n \geq 0)$ is i. i. d. with common probability distribution p, $W_{n+1} = u(W_n, X_n)$, $n \geq 0$.

Hence the random variable W_n may be represented as the n th iterate of u to mean that

$$W_n = u(u(\ldots u(u(W_0, X_0), X_1), \ldots), X_{n-1}).$$

When constructing fractals, see e. g. [3], X is a finite set $X = \{1, \ldots, N\}$, W a real interval endowed with the Borel σ-algebra $\mathcal{W} = \mathcal{B}_W$ and the mappings $u_x : W \to W$, $x \in X$, which are iterated, are chosen at random by means of the distribution p.

In the context of (non-linear) autoregressive time series models, W and X are (subsets of) Euclidean spaces, p is the noise-distribution and the process $(W_n, n \geq 0)$ is the time series which is recursively generated by the transition function u and the noise-terms, see e. g. [2] and [4].

In any case of an IFS the focus of attention is to ensure desirable properties for the Markov chain $(W_n, n \geq 0)$. Here we concentrate our interest to the (aperiodic) strong ergodicity of the Markov chain.

Definition 1.2. *Let $(W_n, n \geq 0)$ be a Markov chain on $(W, \mathcal{W})$ with transition probability Q. Let Q^n be the corresponding n-step transition probability and $Q^{(n)}$ the average or Cesaro sum $Q^{(n)} = \frac{1}{n} \sum_{j=0}^{n-1} Q^j(w, A)$, $w \in W$, $A \in \mathcal{W}$, $n = 1, 2, \ldots$.*

The Markov chain $(W_n, n \geq 0)$ is called strongly ergodic if and only if there exists a probability measure π on $\mathcal{W}$ such that

$$\forall w \in W \parallel \ Q^{(n)}(w, .) - \pi(.) \parallel \to 0$$

as $n \to \infty$, where $\parallel . \parallel$ denotes the norm of total variation on the space of signed measures on $\mathcal{W}$.

Moreover, the Markov chain $(W_n, n \geq 0)$ is called aperiodic strongly ergodic if and only if there exists a probability measure π on $(W, \mathcal{W})$ such that

$$\forall w \in W \parallel Q^n(w, .) - \pi(.) \parallel \to 0$$

as $n \to \infty$.

As for these definitions and their consequences, we refer to [5] and [7]. The probability measure π in Definition 1.2 is the unique invariant probability measure for Q, as easily can be checked. Therefore according to Proposition 9.2.2 in [5] strong ergodicity as defined above is equivalent to the so-called "positive Harris recurrence" of the Markov chain $(W_n, n \geq 0)$. As well the aperiodic strong ergodicity defined above is equivalent to the "aperiodic positive Harris recurrence" of the Markov chain $(W_n, n \geq 0)$ according to Theorems 4.3.3 and 4.3.4 in [5]. For a discussion of the property of "(aperiodic) positive Harris recurrence" we refer again to the books [5] and [7]. Of course, aperiodic strong ergodicity implies strong ergodicity and the latter property implies e. g. that the Markov chain $(W_n, n \geq 0)$ with π as initial distribution, i. e. W_0 is distributed according to π, is an ergodic stationary process (see Proposition 2.4.3 in [5]).

Next, we present a sufficient condition for a strongly ergodic Markov chain to be aperiodic.

Lemma 1.1. *Let the Markov chain $(W_n, n \geq 0)$ be strongly ergodic and its transition probability Q satisfy $\forall w \in W, A \in \mathcal{W} : Q(w, A) = \int_A q(w, w')\phi(dw')$ with $q(w, w') > 0$ for a nontrivial σ-finite measure ϕ on $\mathcal{W}$, $w, w' \in W$. Then $(W_n, n \geq 0)$ is aperiodic.*

Proof. If π is the unique invariant probability measure for Q given by the assumption, $(W_n, n \geq 0)$ is π-irreducible, since $\pi(A) > 0$ implies $\sum_{j=1}^{\infty} Q^j(w, A) > 0$ for all $w \in W$.

According to Proposition 5.2.4 in [7], there is a countable collection of "small sets" $C_i \in \mathcal{W}$ such that $W = \bigcup_{i=1}^{\infty} C_i$. As $\pi(W) = 1$, there exists at least one small set C_{i_0} with $\pi(C_{i_0}) > 0$. Proposition 4.2.7 in [5] ensures the existence of a so-called d-cycle $\{D_1, \ldots, D_d\}$ of disjoint sets $D_i \in \mathcal{W}$ such that $\pi(\bigcup_{i=1}^{d} D_i) = 1$.

The equality

$$\pi(A) = \int_W Q(w, A)\pi(dw) = \int_W \int_A q(w, w')\phi(dw')\pi(dw)$$
$$= \int_A \int_W q(w, w')\pi(dw)\phi(dw')$$

shows the equivalence of π and ϕ.

Now, if $d = 1$, then $\pi(D_1) = 1$. If $d > 1$ then

$$\pi(D_2) = \int_W Q(w, D_2)\pi(dw)$$
$$= \int_{D_1} Q(w, D_2)\pi(dw) + \int_{W \setminus \cup_{i=2}^{d} D_i} Q(w, D_2)\pi(dw) = \pi(D_1),$$

since $\{D_1, \ldots, D_d\}$ is a d-cycle. By symmetry, $\pi(D_i) = \frac{1}{d}$, $i = 1, \ldots, d$.
So, $\phi(D_i) > 0$, $i = 1, \ldots, d$. The assumption "$d > 1$" leads to a contradiction, since then $Q(w, D_1) = 0$ for all $w \in D_1$, which would imply $\phi(D_1) = 0$. Therefore "$d = 1$" is valid, i. e. $(W_n, n \geq 0)$ aperiodic.

By Theorems 4.3.2 and 4.3.4 in [5], $(W_n, n \geq 0)$ is aperiodic strongly ergodic.

□

Definition 1.3. *An IFS $\{(W, \mathcal{W}), (X, \mathcal{X}), u, p\}$ is called (aperiodic) strongly ergodic if and only if the Markov chain $(W_n, n \geq 0)$ generated by this IFS is (aperiodic) strongly ergodic.*

The presentation of our results is organized as follows: In Section 2 we give two sufficient conditions for the (aperiodic) strong ergodicity of an IFS. These are applied in Section 3 to some (non-linear) autoregressive time series models, i. e. for those models sufficient conditions are given for the (aperiodic) strong ergodicity of the corresponding time series. These applications exhibit the powerful potential of the available theory on Markov chains, respectively IFSs.

2. Two Theorems on IFSs

In [5] several theorems are presented in which the existence of a unique invariant probability measure π for a Markov chain is taken as an assumption or is proved in a first step on the way to derive the (aperiodic) positive Harris recurrence of the chain, respectively the (aperiodic) strong ergodicity (see the remarks following Definition 1.2). Following this approach we shall prove

Theorem 2.1. *Consider an IFS* $\{(W,\mathcal{W}),(X,\mathcal{X}),u,p\} = \{(\mathbb{R},\mathcal{B}),(\mathbb{R},\mathcal{B}),u,p\}$*, where* $\mathcal{B}$ *denotes the Borel* σ*-algebra. Assume that*

(1) $\int_X l(x)p(dx) < 1$*, where*

$$l(x) = \sup_{w' \neq w''} \frac{| u_x(w') - u_x(w'') |}{| w' - w'' |};$$

(2) *there exists* $w_0 \in W$ *such that* $\int_X | w_0 - u_x(w_0) | p(dx) < \infty$*;*

(3) *for all* $w \in W$ *the sections* $u_w : X \to W$ *are surjective and continuously differentiable with*

$$0 < \frac{d}{dx}u_w(x) < \infty;$$

(4) p *has a strictly positive density* $\widetilde{p}$ *with respect to the Lebesgue measure* λ *on* $\mathbb{R}$*.*

Then the IFS, i. e. the Markov chain $(W_n, n \geq 0)$ *generated by this IFS, is aperiodic strongly ergodic.*

Proof. Theorem 4 in [6] shows that (1) and (2) ensure the existence of a unique invariant probability measure π for $(W_n, n \geq 0)$, respectively its transition probability Q.

Now, according to the theorem on the transformation of densities each probability measure $Q(w,.)$ has a strictly positive Lebesgue density

$$q(w,w') = \widetilde{p}(u_w^{-1}(w')) \left(\frac{d}{dx}u_w(u_w^{-1}(w'))\right)^{-1}, w, w' \in W.$$

The Lebesgue measure λ is in turn absolutely continuous with respect to π.

If $A \in \mathcal{W}$ has positive Lebesgue measure $\lambda(A)$, then $Q(w,A) > 0$ for all $w \in W$ and therefore $\pi(A) = \int_W Q(w,A)\pi(dw) > 0$. As a consequence we get $Q(w,.) \ll \lambda \ll \pi$ for every $w \in W$. Theorem 4.4.1 in [5] ensures the positive Harris recurrence of the chain $(W_n, n \geq 0)$, in other words its strong ergodicity, while Lemma 1.1 then ensures its aperiodicity.

□

Remark 2.1. *Assumption (1) means that the mappings* u_x *are contractions in the mean, assumption (2) that they are in some sense non-explosive. Assumption (4) is satisfied in many cases, e. g. in the context of time series models for the normal distribution as noise-distribution.*

Another possibility to ensure strong ergodicity and then, by some additional assumption, even aperiodicity consists in assuming a so-called drift-condition for the Markov chain, see again [7] and [5]. We namely have

Theorem 2.2. *Consider an IFS* $\{(W,\mathcal{W}),(X,\mathcal{X}),u,p\}$*, where* $(W,\mathcal{W})$ *is a locally compact separable metric space endowed with the Borel* σ*-algebra* $\mathcal{W}$*. Moreover assume that*

(1) *the Markov chain* $(W_n, n \geq 0)$ *is* ϕ*-irreducible for a nontrivial* σ*-finite measure* ϕ *on* $\mathcal{W}$*, i. e.* $\phi(A) > 0$ *for* $A \in \mathcal{W}$ *implies* $\sum_{n=1}^{\infty} Q^n(w,A) > 0$ *for all* $w \in W$*;*

(2) *for all* $x \in X$ *the sections* u_x *are continuous;*

(3) *there exist a measurable function $V : W \to \mathbb{R}_+$ and a so-called moment function $h : W \to \mathbb{R}_+$ such that*

$$\int_X V(u(w,x))p(dx) \leq V(w) - h(w) + 1.$$

(A measurable function $h : W \to \mathbb{R}_+$ is called a moment function if and only if there exists a sequence of compact sets $K_n \uparrow W$ such that $\inf_{w \in K_n^c} h(w) \to \infty$ with $K_n^c = W \backslash K_n$.)

Under the above assumptions, the IFS, i. e. the Markov chain $(W_n, n \geq 0)$ generated by this IFS, is strongly ergodic.

The *proof* follows directly from that of Proposition 4.4.3 in [5], if that Proposition is applied to the Markov chain $(W_n, n \geq 0)$ respectively the transition probability Q of the IFS: Assumption (2) implies that the transition probability Q of $(W_n, n \geq 0)$ is weak-Feller, i. e. for each sequence $(w_n, n \geq 0) \to w$, $w_n, w \in W$, the corresponding sequence $Q(w_n, .)$ converges weakly to $Q(w, .)$. Moreover the Markov operator gives $QV(w) = \int_W V(w')Q(w, dw') = \int_X V(u(w,x))p(dx)$ according to the transformation theorem. □

In the next section we present examples for which the above theorems apply.

3. Applications to time series models

In this section we apply the above theorems to some time series models which are frequently studied in the context of financial markets, see e. g. [4], [1].

Example 3.1. *Consider an IFS $\{(W, \mathcal{W}), (X, \mathcal{X}), u, p\} = \{(\mathbb{R}, \mathcal{B}), (\mathbb{R}, \mathcal{B}), u, p\}$, where $u(w,x) = f(w) + g(w)x$ for measurable functions $f, g : W \to W$.*

Case (1): If the random variables X_n have common probability distribution p, let us assume that

(1a) $E[| X_n |] < \infty$;

(1b) *$g(.) > 0$, f and g Lipschitz-continuous such that*

$$l(f) + l(g)E[| X_n |] < 1,$$

where $l(f) = \sup_{w' \neq w''} \frac{| f(w') - f(w'') |}{| w' - w'' |}$, $l(g)$;

(1c) *p has a strictly positive Lebesgue density $\widetilde{p}$.*

Case (2): Assume that

(2a) *p is the standard normal distribution, i. e. $X_n \sim N(0,1)$, $n \geq 0$;*

(2b) *$g(.) > 0$, f, g continuous;*

(2c) *there exists a compact set $K \subset W$ such that $\sup_{w \in K} [f(w)^2 + g(w)^2] \leq 1$ and $f(w)^2 + g(w)^2 \leq 1 + (1-\gamma)w^2$ for all $w \notin K$ and for some $0 < \gamma < 1$.*

For this example we shall prove

Theorem 3.1. *In both Cases (1) and (2) the time series $(W_n, n \geq 0)$ generated by the IFS of Example 3.1, i. e. the recursively defined process*

$$W_{n+1} = f(W_n) + g(W_n)X_n, \; n \geq 0,$$

is an aperiodic strongly ergodic Markov chain.

Proof. In Case (1) all assumptions of Theorem 2.1 are satisfied:

(1a) implies (2), since for each $w_0 \in W$

$$\int_X \mid w_0 - u_x(w_0) \mid p(dx) \leq \mid w_0 - f(w_0) \mid + g(w_0) E[\mid X_n \mid],$$

(1b) implies (3), since $\frac{d}{dx} u_w(x) = g(w) > 0$,

(1c) is (4) and finally (1b) implies (1), since $l(x) \leq l(f) + l(g) \mid x \mid$, $x \in X$.

We show that in Case (2) all assumptions of Theorem 2.2 are satisfied. Since $Q(w,.)$ is the $N(f(w), g(w)^2)$-distribution for all $w \in W$, Assumption (1) is valid with $\phi =$ Lebesgue measure λ. Obviously (2b) implies (2). Next, (2c) ensures that Assumption (3) is satisfied with $V(w) = w^2$ and

$$h(w) = \begin{cases} 0 & , w \in K \\ \gamma w^2 & , w \notin K \end{cases},$$

since

$$\int_X V(u(w,x)) p(dx) = \int_X [f(w) + g(w)x]^2 p(dx)$$

$$= f(w)^2 + g(w)^2$$

due to the fact that $E[X_n] = 0$ and $E[X_n^2] = 1$.

Moreover, the Markov chain $(W_n, n \geq 0)$ is aperiodic by Lemma 1.1, since all $Q(w,.)$ are normally distributed.

□

A well-known special case of Example 3.1 is

Example 3.2. *The AR(1)-time series model described by the recursion*

$$W_{n+1} = \alpha W_n + X_n,\ n \geq 0$$

with i. i. d. X_n, $n \geq 0$, *with common distribution* $N(0, \sigma^2)$, $\sigma > 0$, *is a special case of Example 3.1 with* $f(w) = \alpha w$, $g(w) = \sigma$, p *the* $N(0,1)$*-distribution.*

The aperiodic strong ergodicity of the time series $(W_n, n \geq 0)$ *under the condition* $\mid \alpha \mid < 1$ *holds by Theorem 3.1, since* $l(f) = \mid \alpha \mid$ *and* $l(g) = 0$.

Another special case of Example 3.1 is

Example 3.3. *Consider the IFS of Example 3.1 with u given by*

$$u(w,x) = \alpha w + \sqrt{\alpha_0 + \alpha_1 w^2} x$$

with $\alpha \geq 0$, $\alpha_0 > 0$, $\alpha_1 \geq 0$, *i. e.* $f(w) = \alpha w$ *and* $g(w) = \sqrt{\alpha_0 + \alpha_1 w^2}$, *and with p the standard normal distribution* $N(0,1)$.

This IFS induces an ARCH(1)-time series model

$$W_{n+1} = \alpha W_n + \sqrt{\alpha_0 + \alpha_1 W_n^2} X_n$$

for which

$$\begin{aligned} E[W_{n+1} \mid W_0, \ldots, W_n] &= \alpha W_n \\ Var[W_{n+1} \mid W_0, \ldots, W_n] &= \alpha_0 + \alpha_1 W_n^2, n \geq 0. \end{aligned}$$

Theorem 3.1 applied to this special time series model yields two sets of sufficient conditions for the aperiodic strong ergodicity of the time series $(W_n, n \geq 0)$.

Theorem 3.2. *If the parameters of the Markov chain $(W_n, n \geq 0)$ generated in Example 3.3 satisfy one of the conditions*

$$\alpha + \sqrt{2\frac{\alpha_1}{\pi}} < 1 \tag{1}$$

or

$$\alpha_0 \leq 1 \text{ and } \alpha^2 + \alpha_1 < 1, \tag{2}$$

then $(W_n, n \geq 0)$ is aperiodic strongly ergodic.

Proof. Under Condition (1) we have Case (1) of Example 3.1. Since $E[\mid X_n \mid] = \sqrt{\frac{2}{\pi}}$, $l(f) = \alpha$, $l(g) = \sqrt{\alpha_1}$, the inequality (1) is nothing else but Assumption (1b) of Case (1). Under Condition (2) we are in Case (2) of Example 3.1: Choose $K = \{0\}$ and $\gamma = 1 - (\alpha^2 + \alpha_1) > 0$ to ensure Assumption (2c). □

Remark 3.1. *Conditions (1) and (2) in Theorem 3.2 are different in the sense that neither implies the other one. First, note that under both conditions we should have $\alpha < 1$. Under (2) the restrictions $\alpha_0 \leq 1$ and $\alpha_1 < 1$ should hold, whereas under (1) there is no restriction on α_0, while α_1 may moderately exceed 1. On the other hand, (2) allows e. g. a combination like $\alpha = 0.9$ and $\alpha_1 = 0.18$ which is not allowed under (1).*

As another interesting example to which Theorem 2.2 can be applied is a so-called autoregressive time series model with random coefficients (an RCA-model for short), see [8].

Example 3.4. *Consider an IFS $\{(W, \mathcal{W}), (X, \mathcal{X}), u, p\} = \{(\mathbb{R}, \mathcal{B}), (\mathbb{R}^2, \mathcal{B}^2), u, q \otimes r\}$, where*

- *p is the product of the measures q and r on $\mathcal{B}$;*
- *u has the form $u(w, x) = u(w, (\xi, \epsilon)) = (\alpha + \xi)w + \epsilon$ for $x = (\xi, \epsilon) \in \mathbb{R}^2$ and some $\alpha \in \mathbb{R}$.*

The measure $p = q \otimes r$ generates two independent sequences of i. i. d. random variables $(\xi_n, n \geq 0)$ and $(\epsilon_n, n \geq 0)$, where ξ_n is distributed according to q and ϵ_n according to r.

Let us assume that ϵ_n is standard-normally distributed, $\epsilon_n \sim N(0, 1)$, and $E[\xi] = 0$, $Var[\xi] = \sigma_\xi^2 < \infty$.

This IFS generates a time series $(W_n, n \geq 0)$ by the recursion

$$W_{n+1} = (\alpha + \xi_n)W_n + \epsilon_n, \ n \geq 0.$$

We can prove

Theorem 3.3. *Under the additional assumption*

$$\alpha^2 + \sigma_\xi^2 < 1,$$

the time series $(W_n, n \geq 0)$ of Example 3.4 is an aperiodic strongly ergodic Markov chain.

Proof. We show that all assumptions in Theorem 2.2 and Lemma 1.1 are satisfied.

If f denotes the density of the $N(0,1)$-distribution, we can represent the transition probability Q of $(W_n, n \geq 0)$ as

$$Q(w,A) = \int_X \mathbb{P}_\mu(W_{n+1} \in A \mid W_n = w, \xi_n = x) q(dx)$$

$$= \int_X \int_A f(\epsilon - \alpha w - xw) d\epsilon \; q(dx)$$

$$= \int_A \int_X f(\epsilon - \alpha w - xw) q(dx) d\epsilon, \;\; w \in W, A \in \mathcal{W}.$$

Therefore, each $Q(w,.)$ has a strictly positive Lebesgue density. Now, Assumption (1) of Theorem 2.2 is fulfilled. As (2) is obviously valid, (3) remains to be checked. With $V(w) = w^2$ we have

$$\int_X V(u(w,x)) p(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} [(\alpha + x)w + \epsilon]^2 q(dx) f(\epsilon) d\epsilon$$

$$= (\alpha w)^2 + w^2 E[\xi^2] + E[\epsilon^2]$$

$$= w^2[\alpha^2 + \sigma_\xi^2] + 1 = V(w)[\alpha^2 + \sigma_\xi^2] + 1$$

$$= V(w) - h(w) + 1$$

with $h(w) = [1 - \alpha^2 - \sigma_\xi^2] w^2$. By the additional assumption, h is a moment function.

Since Lemma 1.1 applies with $\phi =$ Lebesgue measure, the proof is complete.

□

The way we took to ensure the (aperiodic) strong ergodicity for certain autoregressive time series models may be modified by using different conditions to ensure the existence of a unique invariant probability measure, respectively different drift conditions in combination with additional assumptions. Various variants of such conditions and additional assumptions are dealt with in the literature cited above.

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