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Descriptive Analysis of New Model of Unbounded 3×3 Operator Matrix with Application

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Abstract. In the present paper, we are interested to develop some spectral properties of new model of unbounded block 3×3 of operator matrix with non diagonal domain, named one sided block 3×3 of operator matrix. Some new hypotheses are invested assuring a new approach to find a fine description of the spectrum, the resolvent expression as well as some essential spectra of such new matrix model independently of the knowledge of the union of some essential spectra of its diagonal operators entries. Our developed results extending some known results from I. Marzouk et al. (from Georgian Math. J., https://doi.org/10.1515/gmj-2023-2071 (2023)) to the case of one sided block 3×3 of operator matrix form. Physical model of neutron transport equation with one partly elastic diagonal collision operator is stated to exam the validity of our theoretical framework.

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1. Introduction

During the past decades the spectral analysis of the problem of operator matrices with mixed forms is one of the most famous topics in the studies of the spectral analysis problem of unbounded block operator matrix. A strong renewal of interest in the study of $n \times n$ block of operator matrix form was emerged in the literature in more than one occasion [2, 3, 4, 10, 12, 13, 19, 20, 22, 23, 30, 32], where $(n \geq 2)$. Unfortunately, one of the critical topics in the studies of such kind of problem is the case where n over strictly 2. In this prospect, many mathematicians tackled such fields by decomposing $n \times n$ block of operator matrix with the partitioned of 2×2 block operator matrix with entries blocks of the partition. Mainly, such concept of study fail not be interesting. So, it is interesting to prove the generality of such study.

Even in approach physics, many evolution equations may be modeled by the form of 3×3 block of unbounded operator matrix with mixed domains as mentioned by: J. A. Burns et al. in [3, 4] for wave equation with Viscoelastic Damping, G. Leugering in [19], T. Nishida et al. in [23] and G. Strohmer in [31] for equations describing the flow of viscous, compressible and heat conducting fluids in $\mathbb{R}^3$, D.L.Russel in [27, 28] for Euler Bernoulli Beam, and among others. Nevertheless, the above examples seem to indicate that one of the most complicated problems arises in the situation where the domain fail not be maximal or diagonal.

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Based upon, the literature of mathematics built a generalized study for the model of unbounded operator matrix with non maximal domain coined by the name of the one sided coupled operator matrix. Such model of operator was originate informed and invested in the serials of work by Klauss J. Engel in [5, 9, 11, 12], to offer to the reader the impact of the use of such model in the spectral analysis properties of 2×2 block of operator matrix case. So, attracted by the specificity of this nice notion of operator matrix form our current research prompts us to lead who this model can be tackled for the case of 3×3 operator matrix without using any partitioned form of 2×2 block of operator matrix with block operators entries. Mainly, we start with a general setting which could be useful to draw an interesting results intervening in the theory of operators matrices.

Therefore, our interest in this paper is to find some new criteria on the operators entries of our matrix form allowing to show their compatibility in the formulation of a new factorization form different then the most known in the theory of operator matrices by Frobenuis-Schur factorization form. The most relevance of such new obtained factorization form seems to be remarked to derive new argumentations and criterions which could be useful in spectral description of the spectrum, the resolvent form of our studied matrix form as well as in proving the interaction of the Fredholmness properties between our matrix and in terms of their diagonal entries. The illustration of the validity of our development sketched to an example of an integral differential equation. Precisely, trough some arguments of Abdul-Majeed Al-Izeri et al. developed in [1], which works straightforwardly in neutron transport equation with partly elastic collision operator, we iterate our approach in order to adopt our main investigations of our matrix framework to the neutron transport equation with one partly elastic diagonal collision operator and with specific boundary condition. Specifically, our idea in this paper aims not only to enlarge the spectral analysis developed by Klauss J. Engel in [5] but also to introduce a new model of unbounded block 3×3 of operator matrix allowing to develop with a new concept an appropriate criteria to ameliorate and extend some works done by I. Walha et al. in [2, 20].

The outline of our paper is organized as follows: Section 2 is devoted to recall some basic notations and definitions from the theory of operators and presents their fundamental properties. In Section 3, we introduce a new model of unbounded block 3×3 operator matrix $\mathbb{M}$. Sufficient criteria are introduced to guarantee our interest and to prove an improvement in the theory of operators matrices and an amelioration to many earlier works. In the last section, generic example of neutron transport equation with one partly elastic diagonal collision operator is introduced to exam the validity and the accuracy ideas developed in the theoretical part of this paper.

2. Framework and basic definitions

In this section, we introduce the framework and we prove some preliminary results such as the analysis of perturbed Fredholm operators via Fredholm perturbation and relations between those operators and their corresponding essential spectra.

Let X and Y be two Banach spaces. We denote by:

- $\mathcal{L}(X,Y)$: the set of bounded linear operators from X to Y.
- $\mathcal{C}(X,Y)$: the set of densely defined closed linear operators from X to Y.
- $\mathcal{K}(X,Y)$: the set of compact operators from X to Y.

- $\mathcal{PK}(X)$: the set of polynomially compact operators which defined as:

$$\mathcal{PK}(X) :=\{\ A \in \mathcal{L}(X) :\ \text{there exists a nonzero complex polynomial}$$
$$P(z) = \sum_{k=0}^{n} a_k z^k \text{ satisfying } P(A) \in \mathcal{K}(X)\}.$$

- ${}^{C}\Omega$: the complement of a subset $\Omega \subset \mathbb{C}$. Before moving to define sets of Fredholm operators, we need to clarify to the readers the meaning of the following notations for $T \in \mathcal{C}(X,Y)$ as: $\mathcal{D}(T)$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset Y$ for the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $N(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $R(T)$.

Sets of Fredholm operators of unbounded linear operator were defined as follows:

Definition 2.1. **(i)** Set of upper semi-Fredholm operators from X into Y is defined by:

$$\Phi_+(X,Y) := \{T \in \mathcal{C}(X,Y)\ :\ \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}.$$

(ii) Set of lower semi-Fredholm operators from X into Y is defined by:

$$\Phi_-(X,Y) := \{T \in \mathcal{C}(X,Y)\ :\ \beta(T) < \infty \text{ and } R(T) \text{is closed in } Y\}.$$

(iii) Set of Fredholm operator from X into Y is defined as:

$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y).$$

For $T \in \Phi(X,Y)$, the index of T is defined by the number $i(T) := \alpha(T) - \beta(T)$.
The set of upper (resp. lower) Weyl operators from X into Y is defined by:

$$\mathcal{W}_+(X,Y) := \{T \in \mathcal{C}(X,Y) : T \in \Phi_+(X,Y) \text{ with } i(T) \leq 0\}$$
$$\big(\text{resp. } \mathcal{W}_-(X,Y) := \{T \in \mathcal{C}(X,Y) : T \in \Phi_-(X,Y) \text{ with } i(T) \geq 0\}\big).$$

More general them the sets of upper (resp. lower) semi-Fredholm and upper (resp. lower) Weyl operators, we introduce the following definition.

Definition 2.2. Let $T \in \mathcal{C}(X,Y)$.
(i) Sets of left and right Fredholm operators from X into Y are defined respectively as:

$$\Phi_l(X,Y) := \{T \in \Phi_+(X,Y) :\ R(T) \text{ is complemented subspace of } Y\}$$

and

$$\Phi_r(X) := \{T \in \Phi_-(X,Y) :\ N(T) \text{ is complemented subspace of } X\}.$$

(ii) The sets of left and right Weyl operators from X into Y are defined respectively by:

$$\mathcal{W}_l(X,Y) := \{T \in \Phi_l(X,Y) :\ i(T) \leq 0\}$$

and

$$\mathcal{W}_r(X,Y) := \{T \in \Phi_r(X,Y) :\ i(T) \geq 0\}.$$

Consequently from Definitions 2.1 and 2.2, we introduce the set of Weyl operators which is defined as:

$$\begin{aligned}\mathcal{W}(X,Y) &:= \mathcal{W}_+(X,Y) \cap \mathcal{W}_-(X,Y)\\ &= \mathcal{W}_l(X,Y) \cap \mathcal{W}_r(X,Y) = \{T \in \Phi(X,Y) :\ i(T) = 0\}.\end{aligned}$$

If $X = Y$, the sets $\mathcal{L}(X,X)$, $\mathcal{C}(X,X)$, $\mathcal{K}(X,X)$, $\Phi_+(X,X)$, $\Phi_-(X,X)$, $\Phi(X,X)$, $\mathcal{W}_+(X,X)$, $\mathcal{W}_-(X,X)$, $\Phi_l(X,X)$, $\Phi_r(X,X)$, $\mathcal{W}_l(X,X)$, $\mathcal{W}_r(X,X)$ and $\mathcal{W}(X,X)$ will be replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi(X)$, $\mathcal{W}_+(X)$, $\mathcal{W}_-(X)$, $\Phi_l(X)$, $\Phi_r(X)$, $\mathcal{W}_l(X)$, $\mathcal{W}_r(X)$ and $\mathcal{W}(X)$, respectively.

For $\mathbf{H} \in \{\Phi_+, \Phi_-, \Phi_l, \Phi_r, \Phi, \mathcal{W}_+, \mathcal{W}_-, \mathcal{W}_l, \mathcal{W}_r, \mathcal{W}\}$, the following notations will be required to clarified the analysis results developed along this paper:

- $P(\mathbf{H}(X,Y))$: defines the set of Fredholm perturbation from X into Y as:

$$P \in P(\mathbf{H}(X,Y)) \quad \text{if and only if} \quad T + P \in \mathbf{H}(X,Y), \quad \text{for all} \quad T \in \mathbf{H}(X,Y).$$

- $\sigma_{\mathbf{H}}(T)$: defines the essential spectrum of $T \in C(X)$ as:

$$\sigma_{\mathbf{H}}(T) := \{\mu \in \mathbb{C} : \ \mu - T \notin \mathbf{H}(X)\}.$$

In the following, we list some basic notations about some classes of Fredholm perturbations those are used in our formulation:

Definition 2.3. Let $T \in \mathcal{L}(X,Y)$.
(i) T is said a weakly compact operator from X into Y, if $T(M)$ is relatively weakly compact on Y, for every bounded subset $M \subset X$.

The set of weakly compact operators from X into Y will be denoted by $\mathcal{WC}(X,Y)$.
(ii) T is called a strictly singular operator if, for every infinite-dimensional subspace M, the restriction of T to M is not a homeomorphism, that is $m(TJ_M) = 0$, where J_M is the natural inclusion of M into Y.

The set of strictly singular operators from X into Y will be denoted by $\mathcal{SS}(X,Y)$.

If $X = Y$, the family of weakly compact (resp. strictly singular) operators on X, $\mathcal{WC}(X) := \mathcal{WC}(X,X)$ (resp. $\mathcal{SS}(X) := \mathcal{SS}(X,X)$) is a closed two-sided ideal of $\mathcal{L}(X)$.

Remark 2.1. Let (Ω, Σ, μ) stands for a positive measure space. X_p denotes the space $L_p(\Omega, d\mu)$ $(1 \le p \le \infty)$, where (Ω, Σ, μ) stands for a positive measure space.

Following Theorem 1 in [24, 25], in a special case for $L_1(\Omega, d\mu)$-space (respectively $C(\Omega)$-spaces, with Ω is a compact Hausdorff space), we obtain:

$$\mathcal{WC}(L_1(\Omega, d\mu)) = \mathcal{SS}(X_1)L_1(\Omega, d\mu)).$$

However, when dealing with reflexive space $L_p(\Omega, d\mu)$, $1 < p < \infty$, we have $\mathcal{L}(L_p(\Omega, d\mu)) = \mathcal{WC}(L_p(\Omega, d\mu))$. On the other hand, it follows from Theorem 5.2 in [14] that:

$$\mathcal{K}(L_p(\Omega, d\mu)) \subset_{\neq} \mathcal{SS}(L_p(\Omega, d\mu)) \subset_{\neq} \mathcal{WC}(L_p(\Omega, d\mu))$$

with $p \neq 2$. In particular case for $p = 2$ we get:

$$\mathcal{K}(L_p(\Omega, d\mu)) = \mathcal{SS}(L_p(\Omega, d\mu)) = \mathcal{WC}(L_p(\Omega, d\mu)).$$

In the theory of operators matrices, the analysis of the invertibility problem of operator matrix block 3×3 become an attractive ways to keep some spectral properties of bounded operator matrix.

Proposition 2.1. *Let the Banach spaces $\boldsymbol{U}_i$, for $1 \le i \le 3$, $(A, E, K) \in \mathcal{L}(\boldsymbol{U}_1) \times \mathcal{L}(\boldsymbol{U}_2) \times \mathcal{L}(\boldsymbol{U}_3)$, $B \in \mathcal{L}(\boldsymbol{U}_2, \boldsymbol{U}_1)$, $C \in \mathcal{L}(\boldsymbol{U}_3, \boldsymbol{U}_1)$, $D \in \mathcal{L}(\boldsymbol{U}_1, \boldsymbol{U}_2)$, $F \in \mathcal{L}(\boldsymbol{U}_3, \boldsymbol{U}_2)$, $G \in \mathcal{L}(\boldsymbol{U}_3, \boldsymbol{U}_1)$, $H \in \mathcal{L}(\boldsymbol{U}_3, \boldsymbol{U}_2)$, and consider a bounded operator matrix on $\prod_{i=1}^{3} \boldsymbol{U}_i$ as:*

$$\mathbb{M} := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}.$$

Assume that A and $\Delta_1 := E - DA^{-1}B$ are two invertible operators, then the following items are equivalent:
(i) $\mathbb{M}$ *is invertible in* $\mathcal{L}(\boldsymbol{U}_1 \times \boldsymbol{U}_2 \times \boldsymbol{U}_3)$.
(ii) $\Delta_2 := K - GA^{-1}C - [H - GA^{-1}B]\Delta_1^{-1}[F - DA^{-1}C]$ *is invertible in* $\boldsymbol{U}_3$.

Proof. We begin to proof **(ii)** $\Longrightarrow$ **(i)**.

Based on the Frobenius-Schur factorization of the operator $\mathbb{M}$, for invertible operators A and Δ_1, we may rewritten then as:

$$\mathbb{M} := \mathbb{P}_r \text{diag}(A, \Delta_1, \Delta_2)\mathbb{P}_l$$

where:

$$\mathbb{P}_r := \begin{pmatrix} I & 0 & 0 \\ \mathbb{P}_{r1} & I & 0 \\ \mathbb{P}_{r2} & \mathbb{P}_{r3} & I \end{pmatrix} \qquad \begin{array}{c} \mathbb{P}_{r1} := DA^{-1} \\ \mathbb{P}_{r2} := GA^{-1} \\ \mathbb{P}_{r3} := [H - GA^{-1}B]\Delta_1^{-1} \end{array}$$

and

$$\mathbb{P}_l := \begin{pmatrix} I & \mathbb{P}_{l1} & \mathbb{P}_{l2} \\ 0 & I & \mathbb{P}_{l3} \\ 0 & 0 & I \end{pmatrix} \qquad \begin{array}{c} \mathbb{P}_{l1} := A^{-1}B \\ \mathbb{P}_{l2} := A^{-1}C \\ \mathbb{P}_{l3} := \Delta_1^{-1}[F - DA^{-1}C] \end{array}$$

Therefore, it is seen to observe that the following matrix form $\mathbb{P}_l^{-1}\text{diag}(A^{-1}, \Delta_1^1, \Delta_2^{-1})\mathbb{P}_r^{-1}$ is the inverse of $\mathbb{M}$, for:

$$\mathbb{P}_l^{-1} := \begin{pmatrix} I & -\mathbb{P}_{l1} & -\mathbb{P}_{l2} + \mathbb{P}_{l1}\mathbb{P}_{l3} \\ 0 & I & -\mathbb{P}_{l3} \\ 0 & 0 & I \end{pmatrix} \text{ and } \mathbb{P}_r^{-1} := \begin{pmatrix} I & 0 & 0 \\ -\mathbb{P}_{r1} & I & 0 \\ -\mathbb{P}_{r2} + \mathbb{P}_{r3}\mathbb{P}_{r1} & -\mathbb{P}_{r3} & I \end{pmatrix}.$$

We proof **(i)** $\Longrightarrow$ **(ii)**.

Indeed, assume that $\mathbb{M}$, A and Δ_1 are invertible.

A short computation leads the following expression:

$$\mathbb{P}_r^{-1}\mathbb{M}\mathbb{P}_l^{-1} := \text{diag}(A, \Delta_1, \Delta_2).$$

Which amounts that the operator Δ_2 is invertible. □

3. Principal and main results

We state the Banach spaces $\mathbf{U}_i$, for $1 \leq i \leq 4$, and we consider:

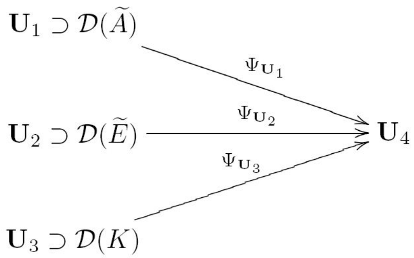
• the maximal operator $\widetilde{A}$ (resp. $\widetilde{E}$ and K) defined with maximal domain $\mathcal{D}(\widetilde{A})$ (resp. $\mathcal{D}(\widetilde{E})$ and $\mathcal{D}(K)$) in U_1 (resp. in U_2 and U_3) as:

$$\widetilde{A} : \mathcal{D}(\widetilde{A}) \subset \mathbf{U}_1 \longrightarrow \mathbf{U}_1 \ \text{ (resp. } \widetilde{E} : \mathcal{D}(\widetilde{E}) \subset \mathbf{U}_2 \longrightarrow \mathbf{U}_2 \text{ and } \ K : \mathcal{D}(K) \subset \mathbf{U}_3 \to \mathbf{U}_3),$$

• the linear operators:

$$\begin{array}{ll} A : \mathcal{D}(A) \subset \mathbf{U}_1 \to \mathbf{U}_1 & E : \mathcal{D}(E) \subset \mathbf{U}_2 \to \mathbf{U}_2 \\ B : \mathcal{D}(B) \subset \mathbf{U}_2 \to \mathbf{U}_1 & C : \mathcal{D}(C) \subset \mathbf{U}_3 \to \mathbf{U}_1 \\ D : \mathcal{D}(D) \subset \mathbf{U}_1 \to \mathbf{U}_2 & F : \mathcal{D}(F) \subset \mathbf{U}_3 \to \mathbf{U}_2 \\ G : \mathcal{D}(G) \subset \mathbf{U}_1 \to \mathbf{U}_3 & H : \mathcal{D}(H) \subset \mathbf{U}_2 \to \mathbf{U}_3 \end{array}$$

• the space of boundaries condition $\mathbf{U}_4$ for the graph norm on $\mathcal{D}(\widetilde{A}), \mathcal{D}(\widetilde{E}), \mathcal{D}(K)$, defines the boundary operators $\Psi_{\mathbf{U}_i}, 1 \leq i \leq 3$, as in the following diagram which is needed to define our abstract framework of operator matrix with non maximal domain as described in the next definition.



Definition 3.1. In the product of the Banach spaces $\mathbf{U} := \prod_{k=1}^{3} \mathbf{U}_k$, we define the operator matrix $\mathbb{M}$ on its non diagonal domain

$$\mathcal{D}(\mathbb{M}) := \left\{ \begin{pmatrix} f \\ g \\ h \end{pmatrix} : \begin{array}{l} f \in \mathcal{D}(\widetilde{A}) \\ g \in \mathcal{D}(\widetilde{E}) \\ h \in \mathcal{D}(K) \end{array} \quad \text{and} \quad \Psi_{\mathbf{U}_1}(f) = \Psi_{\mathbf{U}_2}(g) = \Psi_{\mathbf{U}_3}(h) \right\}$$

as:

$$\mathbb{M}\begin{pmatrix} f \\ g \\ h \end{pmatrix} := \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & K \end{pmatrix}\begin{pmatrix} f \\ g \\ h \end{pmatrix} := \begin{pmatrix} \widetilde{A}f + Bg + Ch \\ Df + \widetilde{E}g + Fh \\ Gf + Hg + Kh \end{pmatrix}, \ \forall \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}).$$

The idea of this work consists in introducing an abstract setting allowing to analysis some spectral properties of the above kind matrix form.

Especially, we provide a new arguments which ensure the computation of some essential spectra of the matrix form introducing in Definition 3.1 independently of their two Schur complements but in relation with their diagonal operators entries.

Such treatment are not artificial but these are meaningful as an easier manner in the expression of the eigenvalues of some physical problems.

To explain such interest, we define some other operators

$$\widetilde{A}_{|} = \widetilde{A}\,|_{\ker \Psi_{\mathbf{U}_1}} \quad \text{and} \quad \widetilde{E}_{|} = \widetilde{E}\,|_{\ker \Psi_{\mathbf{U}_2}},$$

due to fact that we deal with unbounded model of 3×3 block operator matrix defined with non diagonal domain. The specificity of such operators is offered to provide some spectral interactions between the matrix $\mathbb{M}$ and another matrix form $\mathbb{M}_0 := diag(\widetilde{A}_{|}, \widetilde{E}_{|}, K)$ defined with diagonal domain $\mathcal{D}(\mathbb{M}_0) := \mathcal{D}(\widetilde{A}_{|}) \times \mathcal{D}(\widetilde{E}_{|}) \times \mathcal{D}(K)$.

We assume the following hypothesis on the entries of the operator matrix $\mathbb{M}$.

(H1) The operators $\widetilde{A}$, $\widetilde{E}$ and K are densely defined and closed linear operators.

(H2) The operators $\Psi_{\mathbf{U}_i}$ are surjective, for $i = \{1, 2\}$.

(H3) $\begin{cases} \mathcal{D}(\widetilde{A}) \supset \mathcal{D}(D) \cap \mathcal{D}(G), \\ D \text{ and } G \text{ are bounded from } \mathcal{D}(\widetilde{A}) \text{ into } \mathbf{U}_2 \text{ and } \mathbf{U}_3, \text{ respectively.} \end{cases}$

(H4) $\begin{cases} \mathcal{D}(\widetilde{E}) \supset \mathcal{D}(B) \cap \mathcal{D}(H), \\ B \text{ and } H \text{ are bounded from } \mathcal{D}(\widetilde{E}) \text{ into } \mathbf{U}_1 \text{ and } \mathbf{U}_3, \text{ respectively.} \end{cases}$

(H5) $\begin{cases} \mathcal{D}(K) \supset \mathcal{D}(C) \cap \mathcal{D}(F), \\ C \text{ and } F \text{ are bounded from } \mathcal{D}(K) \text{ into } \mathbf{U}_1 \text{ and } \mathbf{U}_2, \text{ respectively.} \end{cases}$

We list some results whose will be essential to formulate our goal.

Remark 3.1. Assume that the assumptions (H1) − (H2) are satisfied. Then, we have:
(i) $\Psi_{\mathbf{U}_1}(\mathcal{D}(\widetilde{A})) = \{0\}$, resp. $\Psi_{\mathbf{U}_2}(\mathcal{D}(\widetilde{E})) = \{0\}$, the operator $\widetilde{A}_| := \widetilde{A}_{|\ker \Psi_{\mathbf{U}_i}}$, resp. $\widetilde{E}_| := \widetilde{E}_{|\ker \Psi_{\mathbf{U}_2}}$ is closed and therefore, $\mathcal{D}(\widetilde{A}_|)$, resp. $\mathcal{D}(\widetilde{E}_|)$, is a closed subset of $\mathbf{U}_1$, resp. $\mathbf{U}_2$. Moreover, for $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|)$, one has

$$\mathcal{D}(\widetilde{A}) = \mathcal{D}(\widetilde{A}_|) \oplus \ker(\mu - \widetilde{A}) \ \text{ and } \ \mathcal{D}(\widetilde{E}) = \mathcal{D}(\widetilde{E}_|) \oplus \ker(\mu - \widetilde{E}).$$

(ii) Following Lemma 1.2 in [15] in view of the last item, we deduce the following continuous bijections:

$$\Psi_{1\mu} := \Psi_{\mathbf{U}_1 |\ker(\mu - \widetilde{A})} \quad \text{and} \quad \Psi_{2\mu} := \Psi_{\mathbf{U}_2 |\ker(\mu - \widetilde{E})},$$

from $\ker(\mu - \widetilde{A})$ and $\ker(\mu - \widetilde{E})$ into $\mathbf{U}_1$ and $\mathbf{U}_2$, respectively, for $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|)$.
(iii) According assumption $(H2)$ with the item **(ii)**, one has two isomorphisms $\Psi_{2\mu}$ and $\Psi_{3\mu}$ which having continuous inverses.

It follows from the last remark that for $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$, the inverse of $\Psi_{1\mu}$ and $\Psi_{2\mu}$ will be essential to define the bounded operators $L_{1\mu}$, $L_{2\mu}$ and $L_{3\mu}$ as stated in the following Lemma:

Lemma 3.1. *Let $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$, then we define the bounded operators:*

(i) $$\begin{cases} L_{1\mu}: & \mathcal{D}(\widetilde{E}) \longrightarrow \mathcal{D}(\widetilde{A}) \\ & g \longmapsto L_{1\mu}(g) = \Psi_{1\mu}^{-1} \circ \Psi_{\boldsymbol{U}_2}(g), \end{cases}$$

(ii) $$\begin{cases} L_{2\mu}: & \mathcal{D}(K) \longrightarrow \mathcal{D}(\widetilde{A}) \\ & h \longmapsto L_{2\mu}(h) = \Psi_{1\mu}^{-1} \circ \Psi_{\boldsymbol{U}_3}(h), \end{cases}$$

(iii) $$\begin{cases} L_{3\mu}: & \mathcal{D}(K) \longrightarrow \mathcal{D}(\widetilde{E}) \\ & h \longmapsto L_{3\mu}(h) = \Psi_{2\mu}^{-1} \circ \Psi_{\boldsymbol{U}_3}(h). \end{cases}$$

Furthermore, the linear operators $\Psi_{\boldsymbol{U}_1}$, $\Psi_{\boldsymbol{U}_2}$ and $\Psi_{\boldsymbol{U}_3}$ obey to the following relations:

$$\Psi_{\boldsymbol{U}_1}(L_{1\mu}g) = \Psi_{\boldsymbol{U}_2}(g), \quad \Psi_{\boldsymbol{U}_1}(L_{2\mu}h) = \Psi_{\boldsymbol{U}_3}(h), \quad \Psi_{\boldsymbol{U}_2}(L_{3\mu}h) = \Psi_{\boldsymbol{U}_3}(h) \tag{1}$$

for all $g \in \mathcal{D}(\widetilde{E})$ and $h \in \mathcal{D}(K)$.

Proof. The results follows immediately from Lemma 3.1. □

Remark 3.2. We should be observe from Lemma 3.1, that, for $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$, the unique operator $L_{1\mu}$ (resp. $L_{2\mu}$ and $L_{3\mu}$) obey to Eq. (1) with $Im(L_{1\mu}) \subset Ker(\mu - \widetilde{A})$ (resp. $Im(L_{2\mu}) \subset Ker(\mu - \widetilde{A})$ and $Im(L_{3\mu}) \subset Ker(\mu - \widetilde{E})$).

For $\mu \in \rho(\widetilde{A}_|) \cap \rho(\widetilde{E}_|) \cap \rho(K)$, we state the following bounded operator, in view of the hypotheses (H1) − (H5), as:

$$\begin{aligned} &\mathbb{U}_1(\mu) := -L_{1\mu} + (\mu - \widetilde{A}_|)^{-1}B && \mathbb{V}_1(\mu) := (\mu - \widetilde{E}_|)^{-1}D \\ &\mathbb{U}_2(\mu) := L_{1\mu}L_{3\mu} - L_{2\mu} + (\mu - \widetilde{A}_|)^{-1}C && \mathbb{V}_2(\mu) := (\mu - K)^{-1}G \\ &\mathbb{U}_3(\mu) := -L_{3\mu} + (\mu - \widetilde{E}_|)^{-1}F && \mathbb{V}_3(\mu) := (\mu - K)^{-1}H. \end{aligned}$$

$$\mathbb{W}_\mu := \begin{pmatrix} I & \mathbb{U}_1(\mu) & \mathbb{U}_2(\mu) \\ \mathbb{V}_1(\mu) & I & \mathbb{U}_3(\mu) \\ \mathbb{V}_2(\mu) & \mathbb{V}_3(\mu) & I \end{pmatrix} \in \mathcal{L}(\mathcal{D}(\widetilde{A}) \times \mathcal{D}(\widetilde{E}) \times \mathcal{D}(K)), \tag{2}$$

which are powerful tools to reach a formulation of a fine decomposition of our model of operator matrix.

Lemma 3.2. *Let* $\mu \in \varrho(\widetilde{A}_{|}) \cap \varrho(\widetilde{E}_{|}) \cap \varrho(K)$.
Suppose that the hypotheses (H1) $-$ (H5) *hold true.*
Then, the operator matrix $\mu - \mathbb{M}$ *obey to the following decomposition on* $\mathcal{D}(\mathbb{M})$*:*

$$\mu - \mathbb{M} = (\mu - \mathbb{M}_0)\mathbb{W}_{\mu}. \tag{3}$$

Proof. Let $\mu \in \varrho(\widetilde{A}_{|}) \cap \varrho(\widetilde{E}_{|}) \cap \varrho(K)$.
Firstly, we shall show that:

$$\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}) \quad \Longleftrightarrow \quad \mathbb{W}_{\mu}\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}_0).$$

To get this equivalent, let consider $\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\widetilde{A}) \times \mathcal{D}(\widetilde{E}) \times \mathcal{D}(K)$.

A short computation, reveals that:

$$\mathbb{W}_{\mu}\begin{pmatrix} f \\ g \\ h \end{pmatrix} = \begin{pmatrix} f - L_{1\mu}g + \underbrace{L_{1\mu}L_{3\mu}h - L_{2\mu}h}_{=0} + (\mu - \widetilde{A}_{|})^{-1}Bg + (\mu - \widetilde{A}_{|})^{-1}Ch \\ (\mu - \widetilde{E}_{|})^{-1}Df + g - L_{3\mu}h + (\mu - \widetilde{E}_{|})^{-1}Fh \\ (\mu - K)^{-1}Gf + (\mu - K)^{-1}Hg + h \end{pmatrix}.$$

Hence,

$$\begin{aligned} \mathbb{W}_{\mu}\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}_0) &= \mathcal{D}(\widetilde{A}_{|}) \times \mathcal{D}(\widetilde{E}_{|}) \times \mathcal{D}(K) \\ &= (\mathcal{D}(\widetilde{A}) \cap \ker \Psi_{\mathbf{U}_1}) \times (\mathcal{D}(\widetilde{E}) \cap \ker \Psi_{\mathbf{U}_2}) \times \mathcal{D}(K). \end{aligned}$$

That is,

$$\mathbb{W}_{\mu}\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}_0) \quad \Longleftrightarrow \quad \begin{cases} f - L_{1\mu}g + (\mu - \widetilde{A}_{|})^{-1}Bg + (\mu - \widetilde{A}_{|})^{-1}Ch \in \ker \Psi_{\mathbf{U}_1} \\ (\mu - \widetilde{E}_{|})^{-1}Df + g - L_{3\mu}h + (\mu - \widetilde{E}_{|})^{-1}Fh \in \ker \Psi_{\mathbf{U}_2} \\ (\mu - K)^{-1}Gf + (\mu - K)^{-1}Hg + h \in \mathcal{D}(K). \end{cases}$$

Using the linearity of the operators $\Psi_{\mathbf{U}_i}$ with the definition of $L_{i\mu}, 1 \leq i \leq 3$, one has:

$$\begin{aligned} &\Psi_{\mathbf{U}_1}(f - L_{1\mu}g + (\mu - \widetilde{A}_{|})^{-1}Bg + (\mu - \widetilde{A}_{|})^{-1}Ch) = \\ &\quad = \Psi_{\mathbf{U}_1}(f) - \Psi_{\mathbf{U}_1}(L_{1\mu}g) + \Psi_{\mathbf{U}_1}((\mu - \widetilde{A}_{|})^{-1}Bg) + \Psi_{\mathbf{U}_1}((\mu - \widetilde{A}_{|})^{-1}Ch) \\ &\quad = \Psi_{\mathbf{U}_1}(f) - \Psi_{\mathbf{U}_2}(g). \end{aligned}$$

and

$$\begin{aligned} &\Psi_{\mathbf{U}_2}((\mu - \widetilde{E}_{|})^{-1}Df + g - L_{3\mu}h + (\mu - \widetilde{E}_{|})^{-1}Fh) = \\ &\quad = \Psi_{\mathbf{U}_2}((\mu - \widetilde{E}_{|})^{-1}Df) + \Psi_{\mathbf{U}_2}(g) - \Psi_{\mathbf{U}_2}(L_{3\lambda}h) + \Psi_{\mathbf{U}_2}((\mu - \widetilde{E}_{|})^{-1}Fh) \\ &\quad = \Psi_{\mathbf{U}_2}(g) - \Psi_{\mathbf{U}_3}(h). \end{aligned}$$

Therefore,

$$\mathbb{W}_\mu \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}_0) \quad \text{if and only if} \quad \begin{cases} \Psi_{\mathbf{U}_1}(f) - \Psi_{\mathbf{U}_2}(g) = 0, f \in \mathcal{D}(\widetilde{A}) \text{ and } g \in \mathcal{D}(\widetilde{E}) \\ \text{and} \\ \Psi_{\mathbf{U}_2}(g) - \Psi_{\mathbf{U}_3}(h) = 0, g \in \mathcal{D}(\widetilde{E}) \text{ and } h \in \mathcal{D}(K) \end{cases}$$

$$\text{if and only if} \quad \begin{cases} f \in \mathcal{D}(\widetilde{A}), g \in \mathcal{D}(\widetilde{E}), h \in \mathcal{D}(K) \\ \text{and} \\ \Psi_{\mathbf{U}_1}(f) = \Psi_{\mathbf{U}_2}(g) = \Psi_{\mathbf{U}_3}(h) \end{cases}$$

$$\text{if and only if} \quad \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}).$$

Secondly, we will proof that:

$$(\mu - \mathbb{M}) \begin{pmatrix} f \\ g \\ h \end{pmatrix} = (\mu - \mathbb{M}_0)\mathbb{W}_\mu \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad \text{for all} \quad \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M}).$$

In fact, for $\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{D}(\mathbb{M})$, we obtain formally

$$(\mu - \mathbb{M}_0)\mathbb{W}_\mu \begin{pmatrix} f \\ g \\ h \end{pmatrix} := \begin{pmatrix} (\mu - \widetilde{A}_|)\left(f - L_{1\mu}g + (\mu - \widetilde{A}_|)^{-1}Bg + (\mu - \widetilde{A}_|)^{-1}Ch\right) \\ (\mu - \widetilde{E}_|)\left((\mu - \widetilde{E}_|)^{-1}Df + g - L_{3\mu}h + (\mu - \widetilde{E}_|)^{-1}Fh\right) \\ (\mu - K)\left((\mu - K)^{-1}Gf + (\mu - K)^{-1}Hg + h\right) \end{pmatrix}$$

$$:= \begin{pmatrix} (\mu - \widetilde{A})f + Bg + Ch \\ Df + (\mu - \widetilde{E})g + Fh \\ Gf + Hg + (\mu - K)h \end{pmatrix} = (\mu - \mathbb{M}) \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

while $L_{1\mu} \in \ker(\mu - \widetilde{A})$ and $L_{3\mu} \in \ker(\mu - \widetilde{E})$ (see Remark 3.2 and Lemma 3.1 for more details). □

As a first towards a new chap of some essential spectra of one sided coupled block 3×3 operator matrix $\mathbb{M}$ will be invested as below.

So, the following proposition will be essential to present the key tool for our investigations.

Proposition 3.3. *Assume, for* $\mu \in \mathbb{C} \backslash \sigma(\widetilde{A}_|) \cup \sigma(\widetilde{E}_|) \cup \sigma(K)$, *that:*
(i) *the hypotheses* (H1)-(H5) *are fulfilled.*
(ii) $0 \notin \sigma(\Delta_1(\mu))$, *for* $\Delta_1(\mu) := I - \mathbb{V}_1(\mu)\mathbb{U}_1(\mu)$.

Then, the following assertions are equivalents:
(i) $0 \in \sigma(\mathbb{W}_\mu)$.
(ii) $0 \in \sigma(\Delta_2(\mu))$, *where* $\Delta_2(\mu)$ *is given by:*

$$\Delta_2(\mu) := I - \mathbb{V}_2(\mu)\mathbb{U}_2(\mu) - [\mathbb{V}_3(\mu) - \mathbb{V}_2(\mu)\mathbb{U}_1(\mu)]\Delta_1(\mu)^{-1}[\mathbb{U}_3(\mu) - \mathbb{V}_1(\mu)\mathbb{U}_2(\mu)].$$

Moreover, in this case, the resolvent formula of $\mathbb{W}_\mu$ *is given by the following matrix:*

$$\mathbb{W}_\mu^{-1} := diag(I, \Delta_1(\mu)^{-1}, \Delta_2(\mu)^{-1}) + (W_{ij})_{1\leq i,j\leq 3}, \quad (4)$$

where:

$$\begin{aligned}
W_{11} &:= \mathbb{U}_1(\mu)\Delta_1(\mu)^{-1}\mathbb{V}_1(\mu) + \Big[\mathbb{U}_1\Delta_1(\mu)^{-1}\left[\mathbb{U}_3(\mu) - \mathbb{V}_1(\mu)\mathbb{U}_2(\mu)\right] - \mathbb{U}_2(\mu)\Big]\Delta_2(\mu)^{-1} \\
&\quad \times \Big[\left[\mathbb{V}_3(\mu) - \mathbb{V}_2(\mu)\mathbb{U}_1(\mu)\right]\Delta_1(\mu)^{-1}\mathbb{V}_1(\mu) - \mathbb{V}_2(\mu)\Big], \\
W_{12} &:= -\mathbb{U}_1(\mu)\Delta_1(\mu)^{-1} - \Big[\mathbb{U}_1(\mu)\Delta_1(\mu)^{-1}\left[\mathbb{U}_3(\mu) - \mathbb{V}_1(\mu)\mathbb{U}_2(\mu)\right] - \mathbb{U}_2(\mu)\Big]\Delta_2(\mu)^{-1} \\
&\quad \times \left[\mathbb{V}_3(\mu) - \mathbb{V}_2(\mu)\mathbb{U}_1(\mu)\right]\Delta_1(\mu)^{-1}, \\
W_{13} &:= \Big[\mathbb{U}_1(\mu)\Delta_1(\mu)^{-1}\left[\mathbb{U}_3(\mu) - \mathbb{V}_1(\mu)\mathbb{U}_2(\mu)\right] - \mathbb{U}_2(\mu)\Big]\Delta_2(\mu)^{-1}, \\
W_{21} &:= -\Delta_1(\mu)^{-1}\mathbb{V}_1(\mu) - \Delta_1(\mu)^{-1}\left[\mathbb{U}_3(\mu) - \mathbb{V}_1(\mu)\mathbb{U}_2(\mu)\right], \\
&\quad \times \Delta_2(\mu)^{-1}\Big[\left[\mathbb{V}_3(\mu) - \mathbb{V}_2(\mu)\mathbb{U}_1(\mu)\right]\Delta_1(\mu)^{-1}\mathbb{V}_1(\mu) - \mathbb{V}_2(\mu)\Big], \\
W_{22} &:= \triangle_1(\mu)^{-1}\left[\mathbb{U}_3(\mu) - \mathbb{V}_1(\mu)\mathbb{U}_2(\mu)\right]\triangle_2(\mu)^{-1}\left[\mathbb{V}_3(\mu) - \mathbb{V}_2(\mu)\mathbb{U}_1(\mu)\right]\triangle_1(\mu)^{-1}, \\
W_{23} &:= -\triangle_1(\mu)^{-1}\left[\mathbb{U}_3(\mu) - \mathbb{V}_1(\mu)\mathbb{U}_2(\mu)\right]\triangle_2(\mu)^{-1}, \\
W_{31} &:= \triangle_2(\mu)^{-1}\Big[\left[\mathbb{V}_3(\mu) - \mathbb{V}_2(\mu)\mathbb{U}_1(\mu)\right]\triangle_1(\mu)^{-1}\mathbb{V}_1(\mu) - \mathbb{V}_2(\mu)\Big], \\
W_{32} &:= \triangle_2(\mu)^{-1}\left[\mathbb{V}_3(\mu) - \mathbb{V}_2(\mu)\mathbb{U}_1(\mu)\right]\triangle_1(\mu)^{-1}, \\
W_{33} &:= 0.
\end{aligned}$$

Proof. The results may be checked directly from the use of Proposition 2.1 and Eq. (2). □

The main advantage of Proposition 3.3 with the factorization 3 makes the computation of some essential spectra of $\mathbb{M}$ easy and in a fast manner as well as to characterize its invertibility. Thus, we summarize such arguments in the following theorem.

Theorem 3.4. *For the considered operator* $\mathbb{M}$ *defined as in Definition 3.1 under the hypotheses* (H1)-(H5)*. The following assertions hold true for* $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$ *such that* $0 \notin \sigma(\Delta_1(\mu))$*:*

(i) $\mu \in \sigma(\mathbb{M})$ *implies that* $0 \in \sigma(\Delta_2(\mu))$.

(ii) *If* $I - \Delta_1(\mu) \in \mathcal{PK}(\mathcal{D}(\widetilde{E}))$ *and* $I - \Delta_2(\mu) \in \mathcal{PK}(\mathcal{D}(K))$*, then we get:*

$$\mu \in \sigma(\mathbb{M}) \iff 0 \in \sigma(\Delta_2(\mu)).$$

Moreover, in this case the resolvent expression of $\mathbb{M}$ *is formally given by:*

$$\begin{aligned}
(\mu - \mathbb{M})^{-1} &:= (\mu - \mathbb{M}_0)^{-1} + (W_{ij})_{1\leq i,j\leq 3}(\mu - \mathbb{M}_0)^{-1} \\
&\quad + diag\left(0, \left(\Delta_1(\mu)^{-1} - I\right)(\mu - \widetilde{E}_|)^{-1}, \left(\Delta_2(\mu)^{-1} - I\right)(\mu - K)^{-1}\right). \quad (5)
\end{aligned}$$

Proof. Let $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$ such that $0 \in \varrho(\Delta_1(\mu))$.

(i) While $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$, one has $\mu - \mathbb{M}_0$ is invertible with bounded inverse on $\mathcal{D}(\widetilde{A}_|) \times \mathcal{D}(\widetilde{E}_|) \times \mathcal{D}(K)$.

This yields the required result by using Lemma 3.2 and Proposition 3.3 in view of the criterions posed on this item.

(ii) We will proof only the reverse implication, while the direct seems from item **(i)**.

Indeed, in view of the polynomially compactness arguments of the operators that $I - \Delta_1(\mu)$ and $I - \Delta_2(\mu)$ with the fact that $\mu - \mathbb{M}_0$ is invertible with bound inverse, we deduce from Proposition 3.3 that $\Delta_2(\mu)$ is invertible, therefore it is injective.

Hence, the desired result holds.

Therefore, in what follows, the expression of the resolvent of $\mathbb{M}$ may be obvious from the use of Eqs. (3) and (4). □

Remark 3.3. (i) Let (Ω, λ) be a σ-finite measure space. Thus, consider $\mathcal{X} = L_1(\Omega, d\lambda)$ (respectively $\mathcal{X} = C(\Omega)$-spaces with Ω is a compact Hausdorff space) for which $I - \Delta_1(\mu) \in \mathcal{WC}(\mathcal{X})$ (resp. $I - \Delta_2(\mu) \in \mathcal{WC}(\mathcal{X})$) satisfies for $P(z) = z^2$. Therefore, in such spaces, we obtain $P(I - \Delta_1(\mu)) = (I - \Delta_1(\mu))^2 \in \mathcal{K}(\mathcal{X})$ (resp. $P(I - \Delta_1(\mu)) = (I - \Delta_1(\mu))^2 \in \mathcal{K}(\mathcal{X})$) (as the product of two weakly compact linear operators in $L_1(\Omega, d\lambda)$ (respectively $C(\Omega)$, where Ω is a compact Hausdorff space) is compact from [26] (resp. [16])). So, we conclude that $P(I - \Delta_1(\mu)) \in \mathcal{K}(\mathcal{X})$ (resp. $P(I - \Delta_2(\mu)) \in \mathcal{K}(\mathcal{X})$) and therefore, $I - \Delta_1(\mu) \in \mathcal{PK}(\mathcal{X})$ (resp. $I - \Delta_2(\mu) \in \mathcal{PK}(\mathcal{X})$). Consequently, we may the required results of Theorem 3.4 under weakly compact assumptions in $L_1(\Omega, d\lambda)$ (respectively $C(\Omega)$, where Ω is a compact Hausdorff space).

(ii) Obviously, for compact operators (or strictly singular operators) $I - \Delta_1(\mu)$ and $I - \Delta_2(\mu)$ in $L_2(\Omega, d\lambda)$ and for particular polynomial $P(z) = z$, we deduce that $P(I - \Delta_1(\mu)) = I - \Delta_1(\mu) \in \mathcal{K}(L_2(\Omega, d\lambda))$ and $P(I - \Delta_2(\mu)) = I - \Delta_2(\mu) \in \mathcal{K}(L_2(\Omega, d\lambda))$. In this fact, we conclude the validity of the results of Theorem 3.4 in such particular spaces and for such particular polynomial case.

New we are in the position to formulate our interest.

Theorem 3.5. *For $\mu \in \rho(\widetilde{A}_|) \cap \rho(\widetilde{E}_|) \cap \rho(K)$, suppose that the following items hold true:*

(i) *the hypotheses* (H1)-(H5) *associated to the operators $\mathbb{M}$ are satisfied.*

(ii) $0 \notin \sigma(\Delta_1(\mu)) \cup \sigma(\Delta_2(\mu))$.

(iii) $I - \Delta_1(\mu) \in \mathcal{PK}(\mathcal{D}(\widetilde{E}))$ *and* $I - \Delta_2(\mu) \in \mathcal{PK}(\mathcal{D}(K))$.

Then,

$$(\mu - \mathbb{M})^{-1} - (\mu - \mathbb{M}_0)^{-1} \in P(\mathbf{H}(\boldsymbol{U}))$$

implies that

$$\sigma_{\mathbf{H}}(\mathbb{M}) = \sigma_{\mathbf{H}}(\widetilde{A}_|) \cup \sigma_{\mathbf{H}}(\widetilde{E}_|) \cup \sigma_{\mathbf{H}}(K), \qquad \text{for } \sigma_{\mathbf{H}}(.) \in \{\sigma_{\Phi_l}(.), \sigma_{\Phi_r}(.), \sigma_{\Phi}(.)\}$$

and

$$\sigma_{\mathbf{H}}(\mathbb{M}) \subseteq \sigma_{\mathbf{H}}(\widetilde{A}_|) \cup \sigma_{\mathbf{H}}(\widetilde{E}_|) \cup \sigma_{\mathbf{H}}(K), \qquad \text{for } \sigma_H(.) \in \{\sigma_{\mathcal{W}_l}(.), \sigma_{\mathcal{W}_r}(.), \sigma_{\mathcal{W}(.)}\}.$$

Assume further that ${}^C\sigma_{\Phi_*}(\widetilde{A}_|)$, ${}^C\sigma_{\Phi_*}(\widetilde{E}_|)$ and ${}^C\sigma_{\Phi_*}(K)$ are connected, then we get:

$$\gamma \in \sigma_{\mathcal{W}_*}(\mathbb{M}) \iff \gamma \in \sigma_{\mathcal{W}_*}(\widetilde{A}_|) \cup \sigma_{\mathcal{W}_*}(\widetilde{E}_|) \cup \sigma_{\mathcal{W}_*}(K),$$

for $\Big(\sigma_{\Phi_*}(.), \sigma_{\mathcal{W}_*}(.)\Big) \in \Big\{\Big(\sigma_{\Phi_l}(.), \sigma_{\mathcal{W}_l}(.)\Big), \Big(\sigma_{\Phi_r}(.), \sigma_{\mathcal{W}_r}(.)\Big)\Big\}$.

Proof. The required estimations follow immediately from Theorem 3.4, Theorem 2.3 in [6] and Lemma 4.1 in [7]. □

Remark 3.4. (i) The results of Theorem 3.5 remain true for some criterions of weakly compactness in $L_1(\Omega, d\lambda)$.

Precisely, if we assume, for $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$, that $\mathbb{U}_k(\mu)$ and $\mathbb{V}_k(\mu)$, $1 \leq k \leq 3$, are weakly compact operators in $L_1(\Omega, d\lambda)$, where $(\Omega, d\lambda)$ is a positive measure space, we deduce that:

$$(\mu - \mathbb{M})^{-1} - (\mu - \mathbb{M}_0)^{-1} \in \mathcal{WC}(L_1(\Omega, d\lambda) \times L_1(\Omega, d\lambda) \times L_1(\Omega, d\lambda)) \subset P(\mathbf{H}(\mathbf{U})).$$

(ii) For compact operators $\mathbb{U}_k(\mu)$ and $\mathbb{V}_k(\mu), 1 \leq k \leq 3$, in $L_p -$ spaces, for $1 \leq p \leq +\infty$ and $\mu \in \varrho(\widetilde{A}_|) \cap \varrho(\widetilde{E}_|) \cap \varrho(K)$, we infer that:

$$(\mu - \mathbb{M})^{-1} - (\mu - \mathbb{M}_0)^{-1} \in \mathcal{K}(L_p(\Omega, d\lambda) \times L_p(\Omega, d\lambda) \times L_p(\Omega, d\lambda)) \subset P(\mathbf{H}(\mathbf{U})).$$

Therefore, we illustrate the validity of the obtained results of Theorem 3.5 in such particular case.

Now, we deal in the last section with a generic physical example of integro differential equation named neutron transport equation with one partly elastic diagonal collision operator. This subsequent physical model is intended to illustrate the validity of our general framework.

4. Example of neutron transport equation with one partly elastic diagonal collision operator

In this section, we state an example of integro-differential equation to fit the importance of our theoretical results modeled as the following form for $(x, \xi, t) \in \Omega \times V \times \mathbb{R}^+$:

$$(\mathbf{T.E}) \qquad \frac{\partial u_i(x,\xi,t)}{\partial t} + \xi \frac{\partial u_i(x,\xi,t)}{\partial x} - \sigma_i(\xi) u_i(x,\xi,t) = \\ = \sum_{j=1, i\neq j}^{3} \int_V \kappa_{c_{ij}}(x,\xi,\xi')\, u_j(x,\xi',t)\, d\xi' + \widetilde{K}_i, \;\; 1 \leq i \leq 3,$$

with initial condition

$$(\mathbf{I.C}) \qquad u_i(x,\xi,0) = u_i^0(x,\xi)$$

where:

. $\widetilde{K}_i := \begin{cases} \displaystyle\int_{\mathbb{S}^{N-1}} \kappa_e(x,\rho,\omega,\omega')\, u_i(x,\rho\omega',t)\, d\omega' + \sum_{m=1}^{l} K_d^{(m)} u_i(x,\xi,t), & i = 3 \\ 0, & i \neq 3. \end{cases}$

. $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$: is an open and bounded set of $\mathbb{R}^N$ $(N \geq 3)$ endowed with the Lebesgue measure dx,

. the space of admissible velocities is defined by:

$$V := \{\xi = \rho\omega,\ \omega \in \mathbb{S}^{N-1}, 0 \leq \rho_{min} \leq \rho \leq \rho_{max} < \infty\} := I \times \mathbb{S}^{N-1}$$

and endowed with Lebesgue measure $d\xi = \rho^{N-1} d\rho d\omega$, ($d\omega$ denotes the Lebesgue measure on the unit sphere $\mathbb{S}^{N-1}$),

. $\mu()$: is a positive Radon measurement on $\mathbb{R}^N$ with $\mu(0) = 0$,

. $\sigma_i(.) \in L^\infty(V, d\mu(\xi))$: denotes the collision frequency,

. $u_i(x, \xi, t)$: represents the number density of particle having the position x and the velocity ξ.

Our interest consists at showing what are this kind of (**T.E**) fit into framework of 3×3 groups of transport equations with one partly elastic diagonal collision operator modeled as follows:

$$\mathbb{M}_{\mathcal{T}.\mathcal{E}} := \begin{pmatrix} T_1 & K_{c_{12}} & K_{c_{13}} \\ K_{c_{21}} & T_2 & K_{c_{23}} \\ K_{c_{31}} & K_{c_{32}} & T_{\mathbb{H}} + K_{c_{33}} + K_e + K_d \end{pmatrix}$$

with specific boundary condition given by:

(**B.C**) $$u^- := \mathcal{H} u^+$$

. Γ_- : denotes the incoming part of the boundary of the phase space $\Omega \times V$ and defined as:

$$\Gamma_- := \Big\{ (x, \xi) \in \partial\Omega \times V,\ \xi.\eta(x) \le 0 \Big\},$$

where $\eta(x)$ stands for the outward normal unit at $x \in \partial\Omega$.

. Γ_+ : denotes the outgoing part of the boundary of the phase space $\Omega \times V$ and defined as:

$$\Gamma_+ := \Big\{ (x, \xi) \in \partial\Omega \times V,\ \xi.\eta(x) \ge 0 \Big\}.$$

. $\mathcal{X}^{p,-}$ and $\mathcal{X}^{p,+}$: denotes the boundary spaces which defined as:

$$\mathcal{X}^{p,-} := L^p(\Gamma_-, |\ v.\nu\nu_x\ |\ d\gamma_x d\mu(\xi)) \text{ and } \mathcal{X}^{p,+} := L^p(\Gamma_+, |\ v.\nu\nu_x\ |\ d\gamma_x d\mu(\xi)).$$

. $\mathcal{H} \in \mathcal{L}(\mathcal{X}^{p,+}, \mathcal{X}^{p,-})$: is an abstract bounded linear operator defined on suitable boundary spaces relating the traces of u, for $u \in \mathcal{W}_p := \{u \in \mathcal{X}^p :\ \ \xi\frac{\partial u}{\partial x} \in \mathcal{X}^p\}$, i.e., u^- on Γ_- with the range of u^+ on Γ_+ by $\mathcal{H}$, as follows:

$$\mathcal{H} := \begin{pmatrix} 0 & 0 & \mathbb{H} \\ 0 & 0 & \mathbb{H} \\ 0 & 0 & \mathbb{H} \end{pmatrix}.$$

While:

. each operator $T_j,\ j = \{1, 2\}$, is defined by:

$$\begin{cases} T_j : \mathcal{D}(T_j) \subset \mathcal{X}^p \longrightarrow \mathcal{X}^p, \quad \text{for } \mathcal{X}^p := L^p(\Omega \times V, dx d\mu(\xi)), \quad \text{with } \ 1 < p < \infty \\ u_j \longmapsto T_j u_j, \ \ (T_j u_j)(x, \xi) := -\xi \dfrac{\partial u_j(x, \xi)}{\partial x} - \sigma_j(\xi) u_j(x, \xi),\ \forall u_j \in \mathcal{D}(T_j) = \mathcal{W}_p \end{cases}$$

. the streaming operator $T_{\mathbb{H}}$ is defined by:

$$\begin{cases} T_{\mathbb{H}} : \mathcal{D}(T_{\mathbb{H}}) \subset \mathcal{X}^p \longrightarrow \mathcal{X}^p, \quad \mathcal{D}(T_{\mathbb{H}}) = \{h \in \mathcal{W}_p :\ h^- = \mathbb{H}\ h^+\} \\ \qquad h \longmapsto T_{\mathbb{H}}\ h, \ \ (T_{\mathbb{H}} h)(x, \xi) = -\xi \dfrac{\partial h(x, \xi)}{\partial x} - \sigma_3(\xi) h(x, \xi), \end{cases}$$

. the classical collision operator

$$\begin{aligned} K_{c_{ij}} \in \mathcal{L}(\mathcal{X}^p) : \mathcal{X}^p \ni u \quad &\longmapsto \quad K_{c_{ij}} u, \\ (x, \xi) \quad &\longmapsto \quad K_{c_{ij}} u(x, \xi) := \int_V \kappa_{c_{ij}}(x, \xi, \xi')\ u(x, \xi', t)\ d\xi', \end{aligned}$$

corresponds physically to fission, high energy elastic slowing down and thermal inelastic scattering.

. the elastic operator K_e is described by an integral operator of the form:

$$K_e u(x, \xi) := \int_{\mathbb{S}^{N-1}} \kappa_e(x, \rho, \omega, \omega')\ u(x, \rho\omega')\ d\omega',$$

for low energy neutrons describing microscopic events in which the kinetic energy is conserved and velocities are changed only in their direction.

. the high energy inelastic scattering K_d is described by a downshift operator of the form:

$$K_d u(x,\xi) := \sum_{m=1}^{l} K_d^{(m)} u(x,\xi) = \sum_{m=1}^{l} \int_{\mathbb{S}^{N-1}} \kappa_d^m(x, e_m(\rho), \omega, \omega')\; u(x, e_m(\rho)\omega', t)\; d\omega',$$

where $K_d^{(m)}, (1 \leq m \leq l)$ describes an event in which a discrete energy E_m is lost by a neutron at position x with initial speed $e_m(\rho)$ and final speed ρ.

Keeping with the theoretical part, we point that our above described physical model of neutron transport equation may be translated in our matrix terminology described in section 3, by taking $U_k = \mathcal{X}^p$, $1 \leq k \leq 3$, the closed operator $\widetilde{A} := T_1$ (resp. $\widetilde{E} := T_2$ and $K := T_{\mathbb{H}} + K_{c_{33}} + K_e + K_d$) with $\mathcal{D}(\widetilde{A}) := \mathcal{W}_p$ (resp. $\mathcal{D}(\widetilde{E}) := \mathcal{W}_p$ and $\mathcal{D}(K) := \mathcal{D}(T_{\mathbb{H}})$, while the off-diagonal operators entries B, C, D, F, G and H correspond to the collision operators $K_{c_{ij}}$, $i \neq j$, that is, $B = K_{c_{12}}, C = K_{c_{13}}, D = K_{c_{21}}, F = K_{c_{23}}, G = K_{c_{31}}$ and $H = K_{c_{32}}$.

According to the boundaries condition (**B.C**) of this kind of physical model, we express the domain $\mathcal{D}(\mathbb{M}_{\mathcal{T}.\mathcal{E}})$ in the same notation of Definition 3.1 as follows:

$$\mathcal{D}(\mathbb{M}_{\mathcal{T}.\mathcal{E}}) := \left\{ \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{W}_p \times \mathcal{W}_p \times \mathcal{D}(T_{\mathbb{H}}) : \quad \Psi_{\mathbf{U}_1}(f) = \Psi_{\mathbf{U}_2}(g) = \Psi_{\mathbf{U}_3}(h) \right\},$$

where the functions $\Psi_{\mathbf{U}_i}$, for $1 \leq i \leq 3$, are identified as well:

$$\left\{ \begin{array}{lcl} \Psi_{\mathbf{U}_i} & : & \mathcal{W}_p \longrightarrow \mathcal{X}^{p,-}, \ \text{ for } 1 \leq i \leq 2 \\ & & u \longmapsto \Psi_{\mathbf{U}_i}(u) = u^- \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{lcl} \Psi_{\mathbf{U}_3} & : & \mathcal{W}_p \longrightarrow \mathcal{X}^{p,-} \\ & & h \longmapsto \Psi_{\mathbf{U}_3}(h) = \mathbb{H}h^+. \end{array} \right.$$

Therefore, the associated operators $\widetilde{A}_|$ and $\widetilde{E}_|$ appears in this physical example as the model of an streaming operators with vacuum boundaries condition modeled as follows:

$$\widetilde{A}_| \ := \ T_{1|} = T_1|_{\ker \Psi_{\mathbf{U}_1}}, \quad \mathcal{D}(\widetilde{A}_|) := \left\{ f \in \mathcal{W}_p : \ f^- = 0 \right\}$$

and

$$\widetilde{E}_| \ := \ T_{2|} = T_2|_{\ker \Psi_{\mathbf{U}_2}}, \quad \mathcal{D}(\widetilde{E}_|) := \left\{ g \in \mathcal{W}_p : \ g^- = 0 \right\}.$$

Remark 4.1. **(i)** Following Definition 2.2 in [1], one has $t^{\pm}(x,\xi) = 0, t \mp (x,\xi) > 0$, for $(x,\xi) \in \Gamma_{\pm}$.

Therefore, we deduce that in all cases $x - t^-(x,\xi)\xi \in \Gamma_-$.

(ii) In view of the above description and keeping into account from the arguments that the operators T_k, $k = 1, 2$ and $T_{\mathbb{H}}$ are closed and densely defined, we conclude the validity of the hypothesis (H1), introduced in Section 3, in this physical example.

(iii) Due to Theorem 1 p. 252 in [8], the hypothesis (H2) is fulfilled, while the traces mappings $\Psi_{\mathbf{U}_i}$, for $1 \leq i \leq 3$, are continuous and subjective.

(iv) The physical problem studied in this section presents a perturbation of the operator

$$\mathbb{T} := \operatorname{diag}(T_1, T_2, T_{\mathbb{H}})$$

by the bounded collision operator

$$\mathbb{K} := \begin{pmatrix} 0 & K_{c_{12}} & K_{c_{13}} \\ K_{c_{21}} & 0 & K_{c_{23}} \\ K_{c_{31}} & K_{c_{32}} & K_{c_{33}} + K_e + K_d \end{pmatrix}.$$

So, the assumptions (H3) – (H5) are satisfied with our present situation.

Consider the real number

$$\tilde{\mu}_j := ess - inf\,\{\sigma_j(\xi),\ \xi \in V\},\ \ 1 \leq j \leq 3.$$

Now, we are in position to express the bounded operators $L_{1\mu}$, $L_{2\mu}$ and $L_{3\mu}$ corresponding to the theoretical part of this paper.

For this interest, we will use the following terminology.

Lemma 4.1. *Let $\mu \in \varrho(T_{1|}) \cap \varrho(T_{2|}) \cap \varrho(T_{\mathbb{H}} + K_e + K_d + K_{c33})$.*
$L_{1\mu}$, $L_{2\mu}$ and $L_{3\mu}$ are bounded operators which are expressed as the mapping:

$$\begin{aligned} L_{1\mu}\ :\ \mathcal{W}_p &\longrightarrow \mathcal{W}_p \\ g &\longmapsto L_{1\mu}g \\ (x,\xi) &\longmapsto (L_{1\mu}g)(x,\xi) := g(x - t^-(x,\xi)\xi,\xi)e^{-(\sigma_1(\xi)+\mu)t^-(x,\xi)},\ \ \textit{for } Re\mu > -\tilde{\mu}_1 \\ &\qquad = \mathbb{H}\ h(x - \tau(x,\xi)\xi,\xi)e^{-(\sigma_1(\xi)+\mu)t^-(x,\xi)},\ \ (x,\xi) \in \Gamma_+, \end{aligned}$$

$$\begin{aligned} L_{2\mu}\ :\ \mathcal{D}(T_{\mathbb{H}}) &\longrightarrow \mathcal{W}_p \\ h &\longmapsto L_{2\mu}h \\ (x,\xi) &\longmapsto (L_{2\mu}h)(x,\xi) := h(x - t^-(x,\xi)\xi,\xi)e^{-(\sigma_2(\xi)+\mu)t^-(x,\xi)},\ \ \textit{for } Re\mu > -\tilde{\mu}_2 \\ &\qquad = \mathbb{H}\ h(x - \tau(x,\xi)\xi,\xi)e^{-(\sigma_2(\xi)+\mu)t^-(x,\xi)},\ \ (x,\xi) \in \Gamma_+, \end{aligned}$$

$$\begin{aligned} L_{3\mu}\ :\ \mathcal{D}(T_{\mathbb{H}}) &\longrightarrow \mathcal{W}_p \\ h &\longmapsto L_{2\mu}h \\ (x,\xi) &\longmapsto (L_{3\mu}h)(x,\xi) := h(x - t^-(x,\xi)\xi,\xi)e^{-(\sigma_3(\xi)+\mu)t^-(x,\xi)},\ \ \textit{for } Re\mu > -\tilde{\mu}_3 \\ &\qquad = \mathbb{H}\ h(x - \tau(x,\xi)\xi,\xi)e^{-(\sigma_3(\xi)+\mu)t^-(x,\xi)},\ \ (x,\xi) \in \Gamma_+, \end{aligned}$$

where $\tau(x,\xi) := t^+(x,\xi) + t^-(x,\xi)$, for any $(x,\xi) \in \overline{\Omega} \times V$.

Proof. Let $\mu \in \varrho(T_{1|}) \cap \varrho(T_{2|}) \cap \varrho(T_{\mathbb{H}} + K_e + K_d + K_{c33})$.

Before moving to find an expression of the operator $L_{1\mu}$, we will proceed with two steps:

* Step1: we will express $\ker(\mu - T_1)$ as well:

$$\begin{aligned} \ker(\mu - T_1) &:= \{f \in \mathcal{D}(T_1) :\ (\mu - T_1)f = 0\} \\ := \Big\{ f \in \mathcal{D}(T_1) : (x,\xi) &\longmapsto f(x,\xi) := f(x - t^-(x,\xi)\xi,\xi)e^{-(\sigma_1(\xi)+\mu)t^-(x,\xi)},\ \text{for } \mathrm{Re}\mu > -\tilde{\mu}_1 \Big\}. \end{aligned}$$

* Step2: we will solve the equation

$$\Psi_{\mathbf{U}_1}(L_{1\mu}g) = \Psi_{\mathbf{U}_2}(g),\ \ \text{for } (f,g) \in \mathcal{D}(T_1) \times \mathcal{D}(T_2).$$

Since $L_{1\mu}g \in \ker(\mu - T_1)$, for $g \in \mathcal{D}(T_2)$, then the following statement holds:

$$\begin{aligned}\Psi_{\mathbf{U}_1}(L_{1\mu}g) = \Psi_{\mathbf{U}_2}(g) \quad &\Longleftrightarrow \quad \Psi_{\mathbf{U}_1}(f) = \Psi_{\mathbf{U}_2}(g), \quad f \in \mathcal{D}(T_1)\\ &\Longleftrightarrow \quad f^- := f|_{\Gamma_-} = g^- = \mathbb{H}\ h^+ := \mathbb{H}\ h|_{\Gamma_+}, h \in \mathcal{D}(T_{\mathbb{H}})\\ &\Longleftrightarrow \quad f(x - t^-(x,\xi)\xi, \xi) = g(x - t^-(x,\xi)\xi, \xi)\\ &\qquad\qquad = \mathbb{H}h(x - \tau(x,\xi)\xi, \xi), \quad (x,\xi) \in \Gamma_+.\end{aligned}$$

Therefore, the required statement holds.

We can adopt the same reasoning to check the explicit expressions of $L_{2\mu}$ and $L_{3\mu}$. □

The perturbed arguments used on the theoretical part will be verified for such physical model via some criterions involving the regularity definition of the collision operator invested by M. Mokhtar-Kharroubi in [21] as follows:

Definition 4.1. A classical collision operator $K_{cij}, 1 \le i,j \le 3$, is said to be regular if the following assertions are fulfilled.

($\mathbf{a}_1$) $\{K_{cij}(x) : x \in \Omega\}$ is a set of collectively compact operators on $L^p(V, d\mu(\xi))$, that is, $\{K_{cij}(x)u : x \in \Omega, \|u\|_{L^p(V,d\mu(\xi))} \le 1\}$ is relatively compact in $L^p(V, d\mu(\xi))$.

($\mathbf{a}_2$) For each $u' \in L^q(V, d\mu(\xi)), \{K_{cij}(x)u' : x \in \Omega, \|u'\|_{L^q(V,d\mu(\xi))} \le 1\}$ is relatively compact in $L^q(V, d\mu(\xi))$, where $L^q(V, d\mu(\xi))$ is the dual space of $L^p(V, d\mu(\xi))$ and $q = \frac{p}{p-1}$.

We assume that the measure $\mu(\cdot)$ satisfies the hypothesis:

$$(\mathbf{a}_3) \qquad \left\{\begin{array}{l}\text{The hyper planes have zero } \upsilon - measure, i.e.\\ \text{for each } e \in \mathbb{S}^{N-1},\ \upsilon\{\xi \in \mathbb{R}^N,\ \xi.e = 0\} = 0\\ \text{where } \mathbb{S}^{N-1} \text{ denotes the unit sphere of } \mathbb{R}^N.\end{array}\right.$$

Lemma 4.2. *Assume that the class of elastic collision operators K_e satisfies the following assertions:*

($\mathbf{a}_4$) *For every $u \in L^q(\mathbb{S}^{N-1}, d\omega)$, the subset*

$$\left\{\int_{\mathbb{S}^{N-1}} \kappa_e(x,\rho,\omega,\omega')\ u(\omega')\ d\omega' : (x,\rho) \in \Omega \times I,\ \|u\|_{L^p(\mathbb{S}^{N-1},d\omega)} \le 1\right\}$$

is relatively compact in $L^p(\mathbb{S}^{N-1}, d\omega)$.

($\mathbf{a}_5$) *For every $u' \in L^q(\mathbb{S}^{N-1}, d\omega)$, the subset*

$$\left\{\int_{\mathbb{S}^{N-1}} \kappa_e(x,\rho,\omega,.)\ u'(\omega)\ d\omega : (x,\rho) \in \Omega \times I\right\}$$

is relatively compact in $L^q(\mathbb{S}^{N-1}, d\omega)$.

Thus, K_{eii} may be regard as a bounded elastic collision operator, namely,

$$\|K_e\| := ess \sup_{(x,\rho)\in\Omega\times I} \|K_e(x,\rho)\|_{\mathcal{L}(L^p(\mathbb{S}^{N-1},d\omega))}.$$

Finally, following [29], we suppose for $m \in \{1, \cdots, \nu\}$, that: ($\mathbf{a}_6$) kernels κ_d^m are assumed to be bounded.

Before further proceed, we introduce

$$\mathcal{S} := \left\{\mu \in \mathbb{C} :\ Re\mu > -\tilde{\mu}_3 + \parallel \widetilde{K}_3 \parallel\right\}.$$

Lemma 4.3. *We fix $\mu \in \varrho(T_{1|}) \cap \varrho(T_{2|}) \cap \mathcal{S}$ for which $r_\sigma((\mu - T_{\mathbb{H}} - \widetilde{K}_3)^{-1}K_{c33}) < 1$ and we assume that:*

(i) *the assumptions $(\boldsymbol{a}_1) - (\boldsymbol{a}_6)$ are fulfilled.*
(ii) *Ω is a convex and bounded subset of $\mathbb{R}^N$.*
(iii) *$\mathbb{H}$ is a compact operator on $\mathcal{X}^p$, for $p > 1$.*

Then, we obtain the compactness criterion of the operators $\mathbb{U}_k(\mu)$ and $\mathbb{V}_k(\mu)$ in $\mathcal{X}^p$, for $p > 1$ and $1 \le k \le 3$.

Proof. The required statements may be obvious from the use of the compactness arguments of the boundaries operator $\mathbb{H}$, of the operators $(\mu - T_{1|})^{-1}K_{c12}$, $(\mu - T_{1|})^{-1}K_{c13}$, $(\mu - T_{2|})^{-1}K_{c23}$, $(\mu - T_{2|})^{-1}K_{c21}$, derived by K. Latrach in [18], and of the operators $(\mu - T_{\mathbb{H}} - \widetilde{K}_3)^{-1}K_{c33}$, $(\mu - T_{\mathbb{H}} - \widetilde{K}_3 - K_{c33})^{-1}K_{c31}$, $(\mu - T_{\mathbb{H}} - \widetilde{K}_3 - K_{c33})^{-1}K_{c32}$ established in Proposition 3.5 in [1], for $\mu \in \varrho(T_{1|}) \cap \varrho(T_{2|}) \cap \mathcal{S}$ such that $r_\sigma((\mu - T_{\mathbb{H}} - \widetilde{K}_3)^{-1}K_{c33}) < 1$ and from the property of the set $\mathcal{K}(\mathcal{X}^p)$, $p > 1$. □

The effectiveness and the applicability of the items **(ii)** and **(iii)** of Theorem 3.5 will be verified for such physical model in the required lemma.

Lemma 4.4. *Let $\mu \in \varrho(T_{1|}) \cap \varrho(T_{2|}) \cap \mathcal{S}$, for which $r_\sigma((\mu - T_{\mathbb{H}} - \widetilde{K}_3)^{-1}K_{c33}) < 1$ and we assume that:*

(i) *the assumptions $(\boldsymbol{a}_1) - (\boldsymbol{a}_6)$ are fulfilled.*
(ii) *Ω is a convex and bounded subset of $\mathbb{R}^N$.*
(iii) *$\mathbb{H}$ is a compact operator on $\mathcal{X}^p$, for $p > 1$.*

Then, $I - \Delta_1(\mu)$ and $I - \Delta_2(\mu)$ are two polynomially compact operators on $\mathcal{X}^p$, $p > 1$, moreover, in such case, the following statements hold:

$$\begin{aligned} 0 \in \varrho(\Delta_1(\mu)) \cap \varrho(\Delta_2(\mu)) &\iff \Delta_1(\mu) \text{ and } \Delta_2(\mu) \text{ are two invertible operators} \\ &\iff \Delta_1(\mu) \text{ and } \Delta_2(\mu) \text{ are two injective operators.} \end{aligned}$$

Proof. Let $\mu \in \varrho(T_{1|}) \cap \varrho(T_{2|}) \cap \mathcal{S}$, for which $r_\sigma((\mu - T_{\mathbb{H}} - \widetilde{K}_3)^{-1}K_{c33}) < 1$.

The criterions of polynomially compactness properties of the operators $I - \Delta_1(\mu)$ and $I - \Delta_2(\mu)$ required directly from the use of Lemma 4.3.

Now, to rich the equivalence statement, we will proceed by steps only for the reverse implication while the direct implication seems to be trivial from the definition.

* Step 1: We proof that $0 \notin \sigma_p(\Delta_1(\mu))$, where $\sigma_p(.)$ denotes the punctual spectrum.

We will find the unknown function u solution of the equation $(\Delta_1(\mu))u = 0$.

Taking into account from Remark 3.2, we conclude that:

$$\begin{aligned} \Delta_1(\mu)u = 0 \quad &\iff \quad (\mu - \widetilde{E}_|)^{-1}K_{c_{21}}[-L_{1\mu} + (\mu - \widetilde{A}_|)^{-1}K_{c_{12}}]u = u \\ &\iff \quad K_{c_{21}}(\mu - \widetilde{A}_|)^{-1}[-(\mu - \widetilde{A}_|)L_{1\mu} + K_{c_{12}}]u = (\mu - \widetilde{E}_|)u \\ &\iff \quad \Big(\mu - \widetilde{E}_| - K_{c_{21}}(\mu - \widetilde{A}_|)^{-1}K_{c_{12}}\Big)u = 0 \\ &\iff \quad u = 0 \\ &\iff \quad \ker(\Delta_1(\mu)) = \{0\}. \end{aligned}$$

* Step2: We will proof that $\Delta_2(\mu)$ is invertible.

In fact, we rewrite the operator $I-\Delta_2(\mu)$ as well:

$$\begin{aligned} I-\Delta_2(\mu) \quad := \quad & (\mu-K)^{-1}K_{c31}(\mu-T_{1|})^{-1}\Big[(\mu-T_{1|})[L_{1\mu}L_{3\mu}-L_{2\mu}]+K_{c13}\Big] \\ & -\Big[(\mu-K)^{-1}\Big(K_{c32}-K_{c31}(\mu-T_{1|})^{-1}[-(\mu-T_{1|})L_{1\mu}+K_{c12}]\Big)\Big] \\ & \times\Delta_1(\mu)^{-1}\Big[(\mu-T_{2|})^{-1}\Big(-(\mu-T_{2|})L_{3\mu}+K_{c23}\Big) \\ & -(\mu-T_{2|})^{-1}K_{c21}(\mu-T_{1|})^{-1}\Big((\mu-T_{1|})[L_{1\mu}L_{3\mu}-L_{2\mu}]+K_{c13}\Big)\Big]. \end{aligned}$$

According to Remark 3.2, the last equality is written as:

$$\begin{aligned} I-\Delta_2(\mu) \quad := \quad & (\mu-K)^{-1}K_{c31}(\mu-T_{1|})^{-1}K_{c13} \\ & -(\mu-K)^{-1}K_{c32}\Delta_1(\mu)^{-1}(\mu-T_{2|})^{-1}\Big[K_{c23}-K_{c21}(\mu-T_{1|})^{-1}K_{c13}\Big] \\ & +(\mu-K)^{-1}K_{c31}(\mu-T_{1|})^{-1}K_{c12} \\ & \times\Delta_1(\mu)^{-1}(\mu-T_{2|})^{-1}\Big[K_{c23}-K_{c21}(\mu-T_{1|})^{-1}K_{c13}\Big]. \end{aligned} \tag{6}$$

This shows in view of Lemma 4.3 and Eq. (2.9) in [1], that (6) obeys to the following inequality for $\mu\in\varrho(T_{1|})\cap\varrho(T_{2|})\cap\mathcal{S}\subset\varrho(T_{1|})\cap\varrho(T_{2|})\cap\varrho(T_{\mathbb{H}}+\widetilde{K})$, such that $r_\sigma((\mu-T_{\mathbb{H}}-\widetilde{K})^{-1}K_{c33})<1$:

$$\|I-\Delta_2(\mu)\|\leq\frac{\|\ K_{c_{33}}\ \|}{\text{Re }\mu+\tilde{\mu}_3-\|\ \widetilde{K}\ \|}\left(\|\ K_{c_{31}}\ \|\ \mathbb{B}_1+\|\ K_{c_{32}}\ \|\ \mathbb{B}_2\right),$$

where $\mathbb{B}_k$ is a bounded operator for $k\in\{1,2\}$.

Moreover, $\lim\limits_{\text{Re }\mu\to+\infty}\|\ I-\Delta_2(\mu)\ \|=0$, and therefore, we get:

$$\lim_{\text{Re }\mu\to+\infty} r_\sigma(I-\Delta_2(\mu))=0,\ \text{ since } r_\sigma(I-\Delta_2(\mu))\leq\|\ I-\Delta_2(\mu)\ \|.$$

Consequently, we deduce that there exists $\mu_0\in\mathcal{S}$ large enough for which the operator $I-(I-\Delta_2(\mu_0)):=\Delta_2(\mu_0)$ is invertible.

Based on the theorem of Gohberg-Smulyan's in [17, Theorem 11.4], in view the compactness-valued assumption of the function $I-\Delta_2(\mu)$ on the connected set $\mathcal{S}$, we claim the invertibility of $\Delta_2(\mu)$ for all $\mu\in\mathcal{S}$ except for a countable subset contained in $\mathcal{S}$. Thus, $0\in\varrho(\Delta_2(\mu))$. □

The eigenvalues associated to the physical model of transport equation with one partly elastic diagonal collision operator modeled by (**T.E**) with specific boundary condition (**B.C**) are localized in the the half plane as follows.

Theorem 4.5. *Let $\mu\in\varrho(T_{1|})\cap\varrho(T_{2|})\cap\mathcal{S}$, for which $r_\sigma((\mu-T_{\mathbb{H}}-\widetilde{K})^{-1}K_{c33})<1$, $\mathbb{H}\in\mathcal{K}(\mathcal{X}^p)$, for $p>1$, and Ω is a convex and bounded subset of $\mathbb{R}^N$.*

Assume that the assumptions $(\boldsymbol{a}_1)-(\boldsymbol{a}_6)$ are fulfilled.

Then, we get:

$$\begin{aligned} \sigma_{\mathbf{H}}(\mathbb{M}_{\mathcal{T}.\mathcal{E}}) \quad & \subseteq \quad \bigcup_{i=1}^{2}\sigma_{\mathbf{H}}(T_{i|})\cup\sigma_{\mathbf{H}}(T_{\mathbb{H}}+\widetilde{K}) \\ & \subset \quad \Big\{\mu\in\mathbb{C}:\ Re\mu\leq-min\Big(\tilde{\mu}_1,\ \tilde{\mu}_2,\ \tilde{\mu}_3-\|\widetilde{K}\ \|\Big)\Big\}, \end{aligned}$$

for $\sigma_{\mathbf{H}}(.)\in\{\sigma_{\Phi_l}(.),\sigma_{\Phi_r}(.),\sigma_{\Phi}(.),\sigma_{\mathcal{W}_l}(.),\sigma_{\mathcal{W}_r}(.),\sigma_{\mathcal{W}}(.)\}$.

Proof. Taking into account from Remark 4.1, Lemmas 4.1, 4.3 and 4.4, one has the validity of the claim of Theorem 3.5 for this physical model. More generally, we obtain:

$$\sigma_{\mathbf{H}}(\mathbb{M}_{\mathcal{T}.\mathcal{E}}) = \sigma_{\mathbf{H}}(\mathbb{M}_{0\mathcal{T}.\mathcal{E}}),$$

where $\mathbb{M}_{0\mathcal{T}.\mathcal{E}} := \text{diag}\,(T_{1|}, T_{2|}, T_{\mathbb{H}} + \widetilde{K} + K_{c33})$.

Following [1], in view of the assumptions, we have:

$$\sigma(T_{\mathbb{H}} + \widetilde{K} + K_{c_{33}}) \cap \left\{\mu \in \mathbb{C} : \ \text{Re}\mu > -\tilde{\mu}_3 + \parallel \widetilde{K} \parallel\right\}$$

consisting of at most isolated eigenvalues with finite algebraic multiplicity on $\mathcal{X}^p$. Thus, the essential spectrum of $T_{\mathbb{H}} + \widetilde{K} + K_{c_{33}}$ is given as follows:

$$\sigma_{\mathcal{W}}(T_{\mathbb{H}} + \widetilde{K} + K_{c_{33}}) = \sigma_{\mathcal{W}}(T_{\mathbb{H}} + \widetilde{K}) \subset \left\{\mu \in \mathbb{C} : \ \text{Re}\mu \leq -\tilde{\mu}_3 + \parallel \widetilde{K} \parallel\right\}.$$

As $T_{k|}, k \in \{1, 2\}$, is the streaming operator with vacuum boundaries condition, we infer from that proceed that:

$$\sigma_{\mathcal{W}}(T_{k|}) = \{\mu \in \mathbb{C} : \ \text{Re}\mu \leq -\tilde{\mu}_k\}.$$

Therefore, the following statement hold from what proceed:

$$\begin{aligned}\sigma_{\mathbf{H}}(\mathbb{M}_{\mathcal{T}.\mathcal{E}}) &\subset \bigcup_{i=1}^{2} \sigma_{\mathcal{W}}(T_{i|}) \cup \sigma_{\mathcal{W}}(T_{\mathbb{H}} + \widetilde{K}) \\ &\subset \left\{\mu \in \mathbb{C} : \ \text{Re}\ \mu \leq -min(\widetilde{\mu}_1,\ \widetilde{\mu}_2,\ \widetilde{\mu}_3 + \|\widetilde{K}\,\|)\right\},\end{aligned}$$

where $\sigma_{\mathbf{H}}(.) \in \{\sigma_{\Phi_l}(.), \sigma_{\Phi_r}(.), \sigma_{\Phi}(.), \sigma_{\mathcal{W}_l}(.), \sigma_{\mathcal{W}_r}(.), \sigma_{\mathcal{W}}(.)\}$. □

5. Conclusion

We have introduced a new model of unbounded block 3 × 3 of operator matrix, named one sided block 3 × 3 of operator matrix. Some new hypotheses are invested to provide a fine decomposition of such model of operator matrix form. The relevance of such new obtained decomposition form of such kind of operator matrix seems to be remarked to derive sufficient criteria assuring the invertibility of such matrix form as well as a new technique to present a fine expression of its resolvent. Our analysis is not just a simple adaptation of the already handled unbounded block 3 × 3 operator matrix with non diagonal domain case. But, there is a new structure and argumentation of the analytical study of the accurate description of the eigenvalues of this operator matrix shape model. As well, a new generic example of neutron transport equation with one partly elastic diagonal collision operators is presented to clarify better contribution of the well-posed theoretical results. Our contribution provide an amelioration and an extension of the work done by I. Marzouk et al. in [20] to the case of unbounded model of full 3 × 3 block operator matrix defined with non diagonal domain.

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