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A New Analytical Solutions for Systems of Nonlinear Time-Fractional Partial Differential Equations using the Khalouta-Daftardar-Jafari Method

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Abstract. In this paper, we investigate the analytical solutions for systems of nonlinear time-fractional partial differential equations by using Khalouta-Daftardar-Jafari method (KHDJM). The fractional derivatives are described in Caputo sense. We discuss the method in general and provide examples for the illustration purpose. Also we provide some results for the convergence of Khalouta-Daftardar-Jafari method. The results in this work show that the proposed method is an effective tool for the solutions of systems of nonlinear time-fractional partial differential equations.

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1. Introduction

Over the past decades, many problems in mathematical physics and engineering such as plasma physics, fluid dynamics, hydrodynamics, nanotechnology, and electromagnetic waves have been successfully formulated through systems of nonlinear fractional partial differential equations. Therefore, it is very important to find efficient methods for solving systems of nonlinear partial differential equations. Many researchers have introduced new methods in the literature where several transforms coupled with semi-analytical techniques have been used to solve these systems. The coupled method of natural transform and Adomian decomposition method called the natural decomposition method (NDM) was introduced in [5] for solving time-fractional nonlinear system of KdV equation, and presents the approximate solution in the form of an infinite series. In [7], Ali Khalouta proposed two different methods based on coupling the new general transform with homotopy perturbation method and variational iteration method called homotopy perturbation transform method (HPTM) and variational iteration transform method (VITM), respectively, to resolve time-fractional system of nonlinear equations of unsteady flow of a polytropic gas in two dimensions, and the solution is presented in the form of the Mittag-Leffler function. In [11], Saadeh et al. used a coupled method of the Laplace transform and residual power series method called Laplace residual power series method (LRPSM) to expand the solution of the nonlinear time-fractional coupled Hirota–Satsuma and KdV equations in the form of a rapidly convergent series. In [1], Agarwal et al. applied a coupled method of

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the Elzaki transform and homotopy analysis method called homotopy analysis Elzaki transform method (HAETM) to find an approximate analytical solution of space-time fractional coupled Burger's equations. In [8], Ali Khalouta proposed a new hybrid method of Elzaki transform and differential transform method called Elzaki differential transform method (EDTM) to solve the fractional SIS epidemic model involving Caputo and Caputo-Fabrizio fractional derivatives.

Our objective of this work is to couple the Khalouta transform method and the Daftardar-Jafari method for solving general systems of nonlinear time-fractional partial differential equations. This method is called the Khalouta-Daftardar-Jafari method (KHDJM). The KHDJM has reduced the computational workload compared with other existing methods in the literature. The convergence and absolute truncation error of the proposed method also provided in this paper.

This paper is organized in six sections. In the second section, we present some preliminaries and basic definitions related to fractional calculus and Khalouta transform method. In the third section, we give the main principle of the KHDJM for solving general systems of nonlinear time-fractional partial differential equations. In the fourth section, we study the convergence analysis of KHDJM. To demonstrate the efficiency and accuracy of the current method, two numerical applications are presented in the fifth section with the use of the results of the third section. Finally, in the sixth section, we conclude this paper with some remarks.

2. Preliminaries and results

We give in this section some fundamental definitions and results about fractional calculus and Khalouta transform for use in this study.

Definition 2.1. [10] The Riemann-Liouville fractional integral of a function $\mathcal{X}(\vartheta,\xi)$ is defined as

$$\mathbb{I}_{\xi}^{\alpha}\mathcal{X}(\vartheta,\xi)=\frac{1}{\Gamma(\alpha)}\int_{0}^{\xi}(\xi-\epsilon)^{\alpha-1}\mathcal{X}(\vartheta,\epsilon)d\epsilon,\ \text{if}\ \alpha>0, \tag{1}$$

and

$$\mathbb{I}_{\xi}^{\alpha}\mathcal{X}(\vartheta,\xi)=\mathcal{X}(\vartheta,\xi),\ \text{if}\ \alpha=0, \tag{2}$$

where $\Gamma(.)$ represents the the gamma function.

Definition 2.2. [10] The Caputo time-fractional derivative of a function $\mathcal{X}(\vartheta,\xi)$ is defined as

$$\mathbb{D}_{\xi}^{\alpha}\mathcal{X}(\vartheta,\xi)=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{\xi}(\xi-\epsilon)^{n-\alpha-1}\mathcal{X}^{(n)}(\vartheta,\epsilon)d\epsilon,\ \text{if}\ n-1<\alpha<n, \tag{3}$$

and

$$\mathbb{D}_{\xi}^{\alpha}\mathcal{X}(\vartheta,\xi)=\mathcal{X}^{(n)}(\vartheta,\xi),\ \text{if}\ \alpha=n, \tag{4}$$

Definition 2.3. [10] The Mittag-Leffler function for one parameter is described as follows

$$E_{\alpha}(z)=\sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+1)},\alpha,z\in\mathbb{C},Re(\alpha)\geq 0. \tag{5}$$

Now, we show the main results related to the Khalouta transform of the Riemann-Liouville fractional integral and the Caputo fractional derivative.

Definition 2.4. [9] The Khalouta transform of a function $\mathcal{X}(\vartheta,\xi)$ of exponential order is defined by the following integral

$$\mathbb{KH}\left[\mathcal{X}(\vartheta,\xi)\right]=\mathcal{K}(\vartheta,s,\gamma,\eta)=\frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{s\xi}{\gamma\eta}\right)\mathcal{X}(\vartheta,\xi)d\xi, s,\gamma,\eta>0, \tag{6}$$

where the function $\mathcal{X}(\vartheta,\xi)$ is defined on the set

$$\mathcal{A}=\left\{\mathcal{X}(\vartheta,\xi):\exists M,\kappa_1,\kappa_2>0,|\mathcal{X}(\vartheta,\xi)|<M\exp\left(\kappa_j\left|\xi\right|\right),\text{ if } \xi\in(-1)^j\times[0,\infty)\right\}, \tag{7}$$

Theorem 2.1. [9] *The basic properties of the Khalouta transform are as follows.*

1) If $\mathcal{X}(\vartheta,\xi)$ and $\mathcal{Y}(\vartheta,\xi)$ are defined on the set $\mathcal{A}$, then for all constants a and b, we have

$$\mathbb{KH}\left[a\mathcal{X}(\vartheta,\xi)+b\mathcal{Y}(\vartheta,\xi)\right]=a\mathbb{KH}\left[\mathcal{X}(\vartheta,\xi)\right]+b\mathbb{KH}\left[\mathcal{Y}(\vartheta,\xi)\right]. \tag{8}$$

2) If the n^{th} derivative of $\mathcal{X}(\vartheta,\xi)$ with respect to ξ is $\mathcal{X}^{(n)}(\vartheta,\xi)$, then its Khalouta transform is given as

$$\mathbb{KH}\left[\mathcal{X}^{(n)}(\vartheta,\xi)\right]=\left(\frac{s}{\gamma\eta}\right)^n\mathcal{K}(\vartheta,s,\gamma,\eta)-\sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\mathcal{X}^{(n)}(\vartheta,0), n\geq 1. \tag{9}$$

3) If the Khalouta transform of $\mathcal{X}(\vartheta,\xi)$ and $\mathcal{Y}(\vartheta,\xi)$ are $\mathcal{K}(\vartheta,s,\gamma,\eta)$ and $\mathcal{H}(\vartheta,s,\gamma,\eta)$ respectively, defined on the set $\mathcal{S}$, then

$$\mathbb{KH}\left[\left(\mathcal{X}*\mathcal{Y}\right)(\vartheta,\xi))\right]=\int_0^\infty \mathcal{X}(\vartheta,\epsilon)\mathcal{Y}(\xi-\epsilon)d\epsilon=\frac{\gamma\eta}{s}\mathcal{K}(\vartheta,s,\gamma,\eta)\mathcal{H}(\vartheta,s,\gamma,\eta), \tag{10}$$

where $\mathbb{KH}\left[\left(\mathcal{X}*\mathcal{Y}\right)(\vartheta,\xi))\right]$ is the Khalouta convolution of the functions $\mathcal{X}(\vartheta,\xi)$ and $\mathcal{Y}(\vartheta,\xi)$.

4) The Khalouta transforms of some special functions are as follows

$$\begin{aligned}\mathbb{KH}\left[1\right] &= 1,\\ \mathbb{KH}\left[\xi\right] &= \frac{\gamma\eta}{s},\\ \mathbb{KH}\left[\frac{\xi^n}{n!}\right] &= \left(\frac{\gamma\eta}{s}\right)^n, n=0,1,2,...\\ \mathbb{KH}\left[\frac{\xi^\alpha}{\Gamma(\alpha+1)}\right] &= \left(\frac{\gamma\eta}{s}\right)^\alpha, \alpha>-1,\end{aligned} \tag{11}$$

Proof. The proof of this theorem is found in [9]. □

Theorem 2.2. *Let $\mathcal{K}(\vartheta,s,\gamma,\eta)$ be the Khalouta transform of the function $\mathcal{X}(\vartheta,\xi)$ with respect to ξ. Then the Khalouta transform of Riemann-Liouville fractional integral of order $\alpha>0$, is defined as*

$$\mathbb{KH}\left[\mathbb{I}_\xi^\alpha\mathcal{X}(\vartheta,\xi)\right]=\left(\frac{\gamma\eta}{s}\right)^\alpha\mathcal{K}(\vartheta,s,\gamma,\eta). \tag{12}$$

Proof. Applying the Khalouta transform to equation (1) gives

$$\begin{aligned}\mathbb{KH}\left[\mathbb{I}_\xi^\alpha\mathcal{X}(\vartheta,\xi)\right] &= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)}\int_0^\xi(\xi-\epsilon)^{\alpha-1}\mathcal{X}(\vartheta,\epsilon)d\epsilon\right]\\ &= \mathbb{KH}\left[\frac{1}{\Gamma(\alpha)}\xi^{\alpha-1}*\mathcal{X}(\vartheta,\xi)\right].\end{aligned} \tag{13}$$

Using Theorem 2.1 leads to

$$\begin{aligned}\mathbb{KH}\left[\mathbb{I}_\xi^\alpha \mathcal{X}(\vartheta,\xi)\right] &= \frac{\gamma\eta}{s}\mathbb{KH}\left[\frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\right]\mathbb{KH}\left[\mathcal{X}(\vartheta,\xi)\right] \\ &= \frac{\gamma\eta}{s}\left(\frac{\gamma\eta}{s}\right)^{\alpha-1}\mathcal{K}(\vartheta,s,\gamma,\eta) \\ &= \left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathcal{K}(\vartheta,s,\gamma,\eta). \qquad (14)\end{aligned}$$

Therefore, the proof is complete. □

Theorem 2.3. *Let $\mathcal{K}(\vartheta,s,\gamma,\eta)$ be the Khalouta transform of the function $\mathcal{X}(\vartheta,\xi)$ with respect to ξ. Then the Khalouta transform of the Caputo time-fractional derivative of order $n-1<\alpha\leq n, n\in\mathbb{Z}^+$, is defined as*

$$\mathbb{KH}\left[\mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi)\right] = \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathcal{K}(\vartheta,s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}\mathcal{X}^{(k)}(\vartheta,0). \qquad (15)$$

Proof. First, we put

$$\mathcal{Z}(\vartheta,\xi) = \mathcal{X}^{(n)}(\vartheta,\xi). \qquad (16)$$

Therefore, the Caputo time-fractional derivative (3) can be written as follows

$$\begin{aligned}\mathbb{KH}\left[\mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi)\right] &= \frac{1}{\Gamma(n-\alpha)}\int_0^\xi(\xi-\epsilon)^{n-\alpha-1}\mathcal{X}^{(n)}(\vartheta,\epsilon)d\epsilon \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^\xi(\xi-\epsilon)^{n-\alpha-1}\mathcal{Z}(\vartheta,\epsilon)d\epsilon \\ &= \mathbb{I}_\xi^{n-\alpha}\mathcal{Z}(\vartheta,\xi). \qquad (17)\end{aligned}$$

Applying the Khalouta transform to equation (17) and using Theorem 2.2, we get

$$\mathbb{KH}\left[\mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi)\right] = \mathbb{KH}\left[\mathbb{I}_\xi^{n-\alpha}\mathcal{Z}(\vartheta,\xi)\right] = \left(\frac{\gamma\eta}{s}\right)^{n-\alpha}\mathcal{H}(\vartheta,s,\gamma,\eta), \qquad (18)$$

where $\mathcal{H}(\vartheta,s,\gamma,\eta)$ is the Khalouta transform of the function $\mathcal{Z}(\vartheta,\xi)$ with respect to ξ.

Applying the Khalouta transform to equation (16) and using Theorem 2.1, we get

$$\begin{aligned}\mathbb{KH}\left[\mathcal{Z}(\vartheta,\xi)\right] &= \mathbb{KH}\left[\mathcal{X}^{(n)}(\vartheta,\xi)\right], \\ \mathcal{H}(\vartheta,s,\gamma,\eta) &= \left(\frac{s}{\gamma\eta}\right)^{n}\mathcal{K}(\vartheta,s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\mathcal{X}^{(k)}(\vartheta,0). \qquad (19)\end{aligned}$$

Substituting equation (19) into equation (18), leads to

$$\begin{aligned}\mathbb{KH}\left[\mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi)\right] &= \left(\frac{\gamma\eta}{s}\right)^{n-\alpha}\left(\left(\frac{s}{\gamma\eta}\right)^{n}\mathcal{K}(\vartheta,s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\mathcal{X}^{(k)}(\vartheta,0)\right) \\ &= \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathcal{K}(\vartheta,s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-k}\mathcal{X}^{(k)}(\vartheta,0). \qquad (20)\end{aligned}$$

Therefore, the proof is complete. □

3. Principle of the KHDJM

We present in this section the main principle of the KHDJM to solve systems of nonlinear time-fractional partial differential equations.

Theorem 3.1. *Consider the following general system of nonlinear time-fractional partial differential equations*

$$\begin{cases} \mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi) = \mathcal{L}(\mathcal{X}(\vartheta,\xi)) + \mathcal{N}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi)) + \mathcal{F}(\vartheta,\xi), n-1<\alpha\leq n \\ \mathbb{D}_\xi^\beta \mathcal{Y}(\vartheta,\xi) = \mathcal{P}(\mathcal{Y}(\vartheta,\xi)) + \mathcal{M}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi)) + \mathcal{G}(\vartheta,\xi), n-1<\beta\leq n \end{cases}, n\in\mathbb{N}^* \tag{21}$$

under the initial conditions

$$\begin{cases} \mathcal{X}^{(k)}(\vartheta,0) = \mathcal{X}_k(\vartheta) \\ \mathcal{Y}^{(k)}(\vartheta,0) = \mathcal{Y}_k(\vartheta) \end{cases}, k=0,1,2,...,n-1, \tag{22}$$

where $\mathbb{D}_\xi^\alpha$ *and* $\mathbb{D}_\xi^\beta$ *are the Caputo time-fractional derivative operators of the functions* $\mathcal{X}(\vartheta,\xi)$ *and* $\mathcal{Y}(\vartheta,\xi)$ *respectively,* $\mathcal{L},\mathcal{P}$ *are linear operators,* $\mathcal{N},\mathcal{M}$ *are nonlinear operators and* $\mathcal{F}(\vartheta,\xi),\mathcal{G}(\vartheta,\xi)$ *are known continuous functions.*

Using KHDJM, the solution of the system (21) can be expressed as an infinite series as follows

$$\begin{cases} \mathcal{X}(\vartheta,\xi) = \sum_{i=0}^{\infty} \mathcal{X}_i(\vartheta,\xi) \\ \mathcal{Y}(\vartheta,\xi) = \sum_{i=0}^{\infty} \mathcal{Y}_i(\vartheta,\xi) \end{cases}. \tag{23}$$

Proof. Applying the Khalouta transform on both sides of (21), we get

$$\begin{cases} \mathbb{KH}\left[\mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi)\right] = \mathbb{KH}\left[\mathcal{L}(\mathcal{X}(\vartheta,\xi)) + \mathcal{N}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi)) + \mathcal{F}(\vartheta,\xi)\right] \\ \mathbb{KH}\left[\mathbb{D}_\xi^\beta \mathcal{Y}(\vartheta,\xi)\right] = \mathbb{KH}\left[\mathcal{P}(\mathcal{Y}(\vartheta,\xi)) + \mathcal{M}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi)) + \mathcal{G}(\vartheta,\xi)\right] \end{cases}. \tag{24}$$

Using Theorem 2.3 and initial conditions (22), (24) becomes

$$\begin{cases} \mathbb{KH}\left[\mathcal{X}(\vartheta,\xi)\right] = & \sum_{k=0}^{n-1} \left(\frac{\gamma\eta}{s}\right)^k \mathcal{X}^{(k)}(\vartheta,0) + \left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}\left[\mathcal{F}(\vartheta,\xi)\right] \\ & + \left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}\left[\mathcal{L}(\mathcal{X}(\vartheta,\xi)) + \mathcal{N}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi))\right] \\ \mathbb{KH}\left[\mathcal{Y}(\vartheta,\xi)\right] = & \sum_{k=0}^{n-1} \left(\frac{\gamma\eta}{s}\right)^k \mathcal{Y}^{(k)}(\vartheta,0) + \left(\frac{\gamma\eta}{s}\right)^\beta \mathbb{KH}\left[\mathcal{G}(\vartheta,\xi)\right] \\ & + \left(\frac{\gamma\eta}{s}\right)^\beta \mathbb{KH}\left[\mathcal{P}(\mathcal{Y}(\vartheta,\xi)) + \mathcal{M}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi))\right] \end{cases}. \tag{25}$$

Taking the inverse Khalouta transform to both sides of (25) to get

$$\begin{cases} \mathcal{X}(\vartheta,\xi) = \mathcal{R}(\vartheta,\xi) + \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\alpha \mathbb{KH}\left[\mathcal{L}(\mathcal{X}(\vartheta,\xi)) + \mathcal{N}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi))\right]\right] \\ \mathcal{Y}(\vartheta,\xi) = \mathcal{T}(\vartheta,\xi) + \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^\beta \mathbb{KH}\left[\mathcal{P}(\mathcal{Y}(\vartheta,\xi)) + \mathcal{M}(\mathcal{X}(\vartheta,\xi),\mathcal{Y}(\vartheta,\xi))\right]\right] \end{cases}, \tag{26}$$

where $\mathcal{R}(\vartheta,\xi)$ and $\mathcal{T}(\vartheta,\xi)$ represents the terms arising from the source terms and the prescribed initial conditions.

Further, we apply the iterative method introduced by Daftardar-Gejji and Jafari [3], which represents solutions $\mathcal{X}(\vartheta,\xi)$ and $\mathcal{Y}(\vartheta,\xi)$ in infinite series of components

$$\begin{cases} \mathcal{X}(\vartheta,\xi) = \sum\limits_{i=0}^{\infty} \mathcal{X}_i(\vartheta,\xi), \\ \mathcal{Y}(\vartheta,\xi) = \sum\limits_{i=0}^{\infty} \mathcal{Y}_i(\vartheta,\xi), \end{cases} \tag{27}$$

Since $\mathcal{L}$ and $\mathcal{P}$ are linear operators, so we have

$$\mathcal{L}\left(\sum_{i=0}^{\infty} \mathcal{X}_i(\vartheta,\xi)\right) = \sum_{i=0}^{\infty} \mathcal{L}\left(\mathcal{X}_i(\vartheta,\xi)\right), \tag{28}$$

$$\mathcal{P}\left(\sum_{i=0}^{\infty} \mathcal{Y}_i(\vartheta,\xi)\right) = \sum_{i=0}^{\infty} \mathcal{P}\left(\mathcal{Y}_i(\vartheta,\xi)\right), \tag{29}$$

and the nonlinear operators $\mathcal{N}$ and $\mathcal{M}$ can be decomposed as

$$\begin{aligned} &\mathcal{N}\left(\sum_{i=0}^{\infty} \mathcal{X}_i(\vartheta,\xi), \mathcal{Y}_i(\vartheta,\xi)\right) = \mathcal{N}\left(\mathcal{X}_0(\vartheta,\xi), \mathcal{Y}_0(\vartheta,\xi)\right) + \\ &+ \sum_{i=1}^{\infty} \left\{ \mathcal{N}\left(\sum_{j=0}^{i} \mathcal{X}_j(\vartheta,\xi), \sum_{j=0}^{i} \mathcal{Y}_j(\vartheta,\xi)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} \mathcal{X}_j(\vartheta,\xi), \sum_{j=0}^{i-1} \mathcal{Y}_j(\vartheta,\xi))\right) \right\}, \end{aligned} \tag{30}$$

$$\begin{aligned} &\mathcal{M}\left(\sum_{i=0}^{\infty} \mathcal{X}_i(\vartheta,\xi), \mathcal{Y}_i(\vartheta,\xi)\right) = \mathcal{M}\left(\mathcal{X}_0(\vartheta,\xi), \mathcal{Y}_0(\vartheta,\xi)\right) + \\ &+ \sum_{i=1}^{\infty} \left\{ \mathcal{M}\left(\sum_{j=0}^{i} \mathcal{X}_j(\vartheta,\xi), \sum_{j=0}^{i} \mathcal{Y}_j(\vartheta,\xi)\right) - \mathcal{M}\left(\sum_{j=0}^{i-1} \mathcal{X}_j(\vartheta,\xi), \sum_{j=0}^{i-1} \mathcal{Y}_j(\vartheta,\xi))\right) \right\}. \end{aligned} \tag{31}$$

Substituting equations (27)–(31) into (26) , we obtain

$$\begin{aligned} &\sum_{i=0}^{\infty} \mathcal{X}_i(\vartheta,\xi) = \mathcal{R}(\vartheta,\xi) + \mathbb{KH}^{-1}\left(\left(\frac{\gamma\eta}{s}\right)^{\alpha} \mathbb{KH}\left[\sum_{i=0}^{\infty} \mathcal{L}\left(\mathcal{X}_i(\vartheta,\xi)\right) + \mathcal{N}\left(\mathcal{X}_0(\vartheta,\xi), \mathcal{Y}_0(\vartheta,\xi)\right)\right.\right. \\ &\left.\left.+ \sum_{i=1}^{\infty} \left\{ \mathcal{N}\left(\sum_{j=0}^{i} \mathcal{X}_j(\vartheta,\xi), \sum_{j=0}^{i} \mathcal{Y}_j(\vartheta,\xi)\right) - \mathcal{N}\left(\sum_{j=0}^{i-1} \mathcal{X}_j(\vartheta,\xi), \sum_{,j=0}^{i-1} \mathcal{Y}_j(\vartheta,\xi))\right) \right\}\right]\right), \end{aligned} \tag{32}$$

$$\begin{aligned} &\sum_{i=0}^{\infty} \mathcal{Y}_i(\vartheta,\xi) = \mathcal{T}(\vartheta,\xi) + \mathbb{KH}^{-1}\left(\left(\frac{\gamma\eta}{s}\right)^{\beta} \mathbb{KH}\left[\sum_{i=0}^{\infty} \mathcal{P}\left(\mathcal{Y}_i(\vartheta,\xi)\right) + \mathcal{M}\left(\mathcal{X}_0(\vartheta,\xi), \mathcal{Y}_0(\vartheta,\xi)\right)\right.\right. \\ &\left.\left.+ \sum_{i=1}^{\infty} \left\{ \mathcal{M}\left(\sum_{j=0}^{i} \mathcal{X}_j(\vartheta,\xi), \sum_{j=0}^{i} \mathcal{Y}_j(\vartheta,\xi)\right) - \mathcal{M}\left(\sum_{j=0}^{i-1} \mathcal{X}_j(\vartheta,\xi), \sum_{j=0}^{i-1} \mathcal{Y}_j(\vartheta,\xi))\right) \right\}\right]\right). \end{aligned} \tag{33}$$

Using equations (32) and (33), we define the following iterative formula

$$\begin{cases} \mathcal{X}_0(\vartheta,\xi) = \mathcal{R}(\vartheta,\xi) \\ \mathcal{Y}_0(\vartheta,\xi) = \mathcal{T}(\vartheta,\xi) \end{cases},$$
$$\begin{cases} \mathcal{X}_1(\vartheta,\xi) = \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\mathcal{L}\left(\mathcal{X}_0(\vartheta,\xi)\right)+\mathcal{N}\left(\mathcal{X}_0(\vartheta,\xi),\mathcal{Y}_0(\vartheta,\xi)\right)\right]\right] \\ \mathcal{Y}_1(\vartheta,\xi) = \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\mathcal{L}\left(\mathcal{Y}_0(\vartheta,\xi)\right)+\mathcal{N}\left(\mathcal{X}_0(\vartheta,\xi),\mathcal{Y}_0(\vartheta,\xi)\right)\right]\right] \end{cases}, \tag{34}$$
$$\begin{cases} \mathcal{X}_{m+1}(\vartheta,\xi) = \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\mathcal{L}\left(\mathcal{X}_m(\vartheta,\xi)\right)+\mathcal{N}\left(\sum_{j=0}^{m}\mathcal{X}_j(\vartheta,\xi),\sum_{j=0}^{m}\mathcal{Y}_j(\vartheta,\xi)\right) -\mathcal{N}\left(\sum_{j=0}^{m-1}\mathcal{X}_j(\vartheta,\xi),\sum_{j=0}^{m-1}\mathcal{Y}_j(\vartheta,\xi)\right)\right]\right] \\ \mathcal{Y}_{m+1}(\vartheta,\xi) = \mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\mathcal{P}\left(\mathcal{Y}_m(\vartheta,\xi)\right)+\mathcal{M}\left(\sum_{j=0}^{m}\mathcal{X}_j(\vartheta,\xi),\sum_{j=0}^{m}\mathcal{Y}_j(\vartheta,\xi)\right) -\mathcal{M}\left(\sum_{j=0}^{m-1}\mathcal{X}_j(\vartheta,\xi),\sum_{j=0}^{m-1}\mathcal{Y}_j(\vartheta,\xi)\right)\right]\right] \end{cases}.$$

Therefore, the approximate analytical solution of system (21), is given by

$$\begin{cases} \mathcal{X}(\vartheta,\xi) = \lim_{m\to\infty}\sum_{i=0}^{m}\mathcal{X}_i(\vartheta,\xi) = \sum_{i=0}^{\infty}\mathcal{X}_i(\vartheta,\xi) \\ \mathcal{Y}(\vartheta,\xi) = \lim_{m\to\infty}\sum_{i=0}^{m}\mathcal{Y}_i(\vartheta,\xi) = \sum_{i=0}^{\infty}\mathcal{Y}_i(\vartheta,\xi) \end{cases}.$$

Therefore, the proof is complete. □

4. Convergence analysis of the KHDJM

We study in this section the convergence of the KHDJM, when it is used in system (21).

Theorem 4.1. *Let* $\begin{cases} \mathcal{X}_i(\vartheta,\xi) \\ \mathcal{Y}_i(\vartheta,\xi) \end{cases}$ *and* $\begin{cases} \mathcal{X}(\vartheta,\xi) \\ \mathcal{Y}(\vartheta,\xi) \end{cases}$ *be in Banach space* $\mathcal{B}$*. Then the KHDJM series solution* $\begin{cases} \sum_{i=0}^{\infty}\mathcal{X}_i(\vartheta,\xi) \\ \sum_{i=0}^{\infty}\mathcal{Y}_i(\vartheta,\xi) \end{cases}$ *defined by (23) converges to the exact solution of system (21) provided that* $\begin{cases} 0<\varrho<1 \\ 0<\theta<1 \end{cases}$ *and* $\begin{cases} \mathcal{X}_0(\vartheta,\xi) \\ \mathcal{Y}_0(\vartheta,\xi) \end{cases} \in \mathcal{B}$ *are bounded.*

Proof. Considering the partial sum sequences $\{\mathcal{S}_n(\vartheta,\xi)\}_{n=0}^{\infty}$ and $\{\mathcal{S}'_n(\vartheta,\xi)\}_{n=0}^{\infty}$ of the form

$$\begin{cases} \mathcal{S}_0(\vartheta,\xi) = \mathcal{X}_0(\vartheta,\xi) \\ \mathcal{S}'_0(\vartheta,\xi) = \mathcal{Y}_0(\vartheta,\xi) \end{cases},$$
$$\begin{cases} \mathcal{S}_1(\vartheta,\xi) = \mathcal{X}_0(\vartheta,\xi)+\mathcal{X}_1(\vartheta,\xi) \\ \mathcal{S}'_1(\vartheta,\xi) = \mathcal{Y}_0(\vartheta,\xi)+\mathcal{Y}_1(\vartheta,\xi) \end{cases},$$
$$\begin{cases} \mathcal{S}_2(\vartheta,\xi) = \mathcal{X}_0(\vartheta,\xi)+\mathcal{X}_1(\vartheta,\xi)+\mathcal{X}_2(\vartheta,\xi) \\ \mathcal{S}'_2(\vartheta,\xi) = \mathcal{Y}_0(\vartheta,\xi)+\mathcal{Y}_1(\vartheta,\xi)+\mathcal{Y}_2(\vartheta,\xi) \end{cases},$$

$$\vdots$$

$$\left\{\begin{array}{l} \mathcal{S}_n(\vartheta,\xi) = \mathcal{X}_0(\vartheta,\xi) + \mathcal{X}_1(\vartheta,\xi) + \mathcal{X}_2(\vartheta,\xi) + ... + \mathcal{X}_n(\vartheta,\xi) \\ \mathcal{S}'_n(\vartheta,\xi) = \mathcal{Y}_0(\vartheta,\xi) + \mathcal{Y}_1(\vartheta,\xi) + \mathcal{Y}_2(\vartheta,\xi) + ... + \mathcal{Y}_n(\vartheta,\xi) \end{array}\right. .$$

To achieve the desired result, we will prove that $\{\mathcal{S}_n(\vartheta,\xi)\}_{n=0}^{\infty}$ and $\{\mathcal{S}'_n(\vartheta,\xi)\}_{n=0}^{\infty}$ be Cauchy sequences in $\mathcal{B}$.

From the last hypothesis of the theorem, we have $\left\{\begin{array}{l} \varrho \in (0,1) \\ \theta \in (0,1) \end{array}\right.$ then

$$\left\{\begin{array}{c} \|\mathcal{S}_{n+1}(\vartheta,\xi) - \mathcal{S}_n(\vartheta,\xi)\| \leq \|\mathcal{X}_{n+1}(\vartheta,\xi)\| \leq \varrho \|\mathcal{X}_n(\vartheta,\xi)\| \\ \leq \varrho^2 \|\mathcal{X}_{n-1}(\vartheta,\xi)\| \leq ... \leq \varrho^{n+1} \|\mathcal{X}_0(\vartheta,\xi)\| \\ \|\mathcal{S}'_{n+1}(\vartheta,\xi) - \mathcal{S}'_n(\vartheta,\xi)\| \leq \|\mathcal{Y}_{n+1}(\vartheta,\xi)\| \leq \theta \|\mathcal{Y}_n(\vartheta,\xi)\| \\ \leq \theta^2 \|\mathcal{Y}_{n-1}(\vartheta,\xi)\| \leq ... \leq \theta^{n+1} \|\mathcal{Y}_0(\vartheta,\xi)\| \end{array}\right. .$$

For any $n, m \in \mathbb{N}$ with $n \geq m$, we have

$$\left\{\begin{array}{c} \|\mathcal{S}_n(\vartheta,\xi) - \mathcal{S}_m(\vartheta,\xi)\| = \|\mathcal{S}_n(\vartheta,\xi) - \mathcal{S}_{n-1}(\vartheta,\xi) + \mathcal{S}_{n-1}(\vartheta,\xi) - \mathcal{S}_{n-2}(\vartheta,\xi) \\ +... + \mathcal{S}_{m+1}(\vartheta,\xi) - \mathcal{S}_m(\vartheta,\xi)\| \\ \leq \|\mathcal{S}_n(\vartheta,\xi) - \mathcal{S}_{n-1}(\vartheta,\xi)\| + \|\mathcal{S}_{n-1}(\vartheta,\xi) - \mathcal{S}_{n-2}(\vartheta,\xi)\| \\ +... + \|\mathcal{S}_{m+1}(\vartheta,\xi) - \mathcal{S}_m(\vartheta,\xi)\| \\ \leq \varrho^n \|\mathcal{X}_0(\vartheta,\xi)\| + \varrho^{n-1} \|\mathcal{X}_0(\vartheta,\xi)\| + ... + \varrho^{m+1} \|\mathcal{X}_0(\vartheta,\xi)\| \\ = \varrho^{m+1}\left(1 + \varrho + ... + \varrho^{n-m-1}\right) \|\mathcal{X}_0(\vartheta,\xi)\| \\ \leq \varrho^{m+1}\left(\frac{1-\varrho^{n-m}}{1-\varrho}\right) \|\mathcal{X}_0(\vartheta,\xi)\| \\ \|\mathcal{S}'_n(\vartheta,\xi) - \mathcal{S}'_m(\vartheta,\xi)\| = \|\mathcal{S}'_n(\vartheta,\xi) - \mathcal{S}'_{n-1}(\vartheta,\xi) + \mathcal{S}'_{n-1}(\vartheta,\xi) - \mathcal{S}'_{n-2}(\vartheta,\xi) \\ +... + \mathcal{S}'_{m+1}(\vartheta,\xi) - \mathcal{S}'_m(\vartheta,\xi)\| \\ \leq \|\mathcal{S}'_n(\vartheta,\xi) - \mathcal{S}'_{n-1}(\vartheta,\xi)\| + \|\mathcal{S}'_{n-1}(\vartheta,\xi) - \mathcal{S}'_{n-2}(\vartheta,\xi)\| \\ +... + \|\mathcal{S}'_{m+1}(\vartheta,\xi) - \mathcal{S}'_m(\vartheta,\xi)\| \\ \leq \theta^n \|\mathcal{Y}_0(\vartheta,\xi)\| + \theta^{n-1} \|\mathcal{Y}_0(\vartheta,\xi)\| + ... + \theta^{m+1} \|\mathcal{Y}_0(\vartheta,\xi)\| \\ = \theta^{m+1}\left(1 + \theta + ... + \theta^{n-m-1}\right) \|\mathcal{Y}_0(\vartheta,\xi)\| \\ \leq \theta^{m+1}\left(\frac{1-\theta^{n-m}}{1-\theta}\right) \|\mathcal{Y}_0(\vartheta,\xi)\| \end{array}\right. .$$

Since $\left\{\begin{array}{l} 0 < \varrho < 1 \\ 0 < \theta < 1 \end{array}\right.$, we have $\left\{\begin{array}{l} 1 - \varrho^{n-m} < 1 \\ 1 - \theta^{n-m} < 1 \end{array}\right.$, then

$$\left\{\begin{array}{l} \|\mathcal{S}_n(\vartheta,\xi) - \mathcal{S}_m(\vartheta,\xi)\| \leq \frac{\varrho^{m+1}}{1-\varrho} \|\mathcal{X}_0(\vartheta,\xi)\| \\ \|\mathcal{S}'_n(\vartheta,\xi) - \mathcal{S}'_m(\vartheta,\xi)\| \leq \frac{\theta^{m+1}}{1-\theta} \|\mathcal{Y}_0(\vartheta,\xi)\| \end{array}\right. .$$

Since $\left\{\begin{array}{l} \mathcal{X}_0(\vartheta,\xi) \\ \mathcal{Y}_0(\vartheta,\xi) \end{array}\right.$ is bounded, then $\left\{\begin{array}{l} \|\mathcal{X}_0(\vartheta,\xi)\| < \infty \\ \|\mathcal{Y}_0(\vartheta,\xi)\| < \infty \end{array}\right.$.

So

$$\left\{\begin{array}{l} \lim\limits_{n,m\to\infty} \|\mathcal{S}_n(\vartheta,\xi) - \mathcal{S}_m(\vartheta,\xi)\| = 0 \\ \lim\limits_{n,m\to\infty} \|\mathcal{S}'_n(\vartheta,\xi) - \mathcal{S}'_m(\vartheta,\xi)\| = 0 \end{array}\right. .$$

Hence, $\{\mathcal{S}_n(\vartheta,\xi)\}_{n=0}^{\infty}$ and $\{\mathcal{S}'_n(\vartheta,\xi)\}_{n=0}^{\infty}$ are Cauchy sequences in Banach space $\mathcal{B}$. It concludes that the solution of system (21) in series is convergent.

Therefore, the proof is complete. □

Theorem 4.2. *The maximum absolute truncation error of the series solution (23) for system (21) is estimated to be*

$$\begin{cases} \left\| \mathcal{X}(\vartheta,\xi) - \sum\limits_{i=0}^{j} \mathcal{X}_i(\vartheta,\xi) \right\| \leq \frac{\varrho^{j+1}}{1-\varrho} \left\| \mathcal{X}_0(\vartheta,\xi) \right\| \\ \left\| \mathcal{Y}(\vartheta,\xi) - \sum\limits_{i=0}^{j} \mathcal{Y}_i(\vartheta,\xi) \right\| \leq \frac{\theta^{j+1}}{1-\theta} \left\| \mathcal{Y}_0(\vartheta,\xi) \right\| \end{cases}.$$

Proof. Let the series $\begin{cases} \sum\limits_{i=0}^{j} \mathcal{X}_i(\vartheta,\xi) \\ \sum\limits_{i=0}^{j} \mathcal{Y}_i(\vartheta,\xi) \end{cases}$ be finite. Then

$$\begin{cases} \left\| \mathcal{X}(\vartheta,\xi) - \sum\limits_{i=0}^{j} \mathcal{X}_i(\vartheta,\xi) \right\| & \leq \left\| \sum\limits_{i=j+1}^{\infty} \mathcal{X}_i(\vartheta,\xi) \right\| \\ & \leq \sum\limits_{i=j+1}^{\infty} \left\| \mathcal{X}_i(\vartheta,\xi) \right\| \leq \sum\limits_{i=j+1}^{\infty} \varrho^j \left\| \mathcal{X}_0(\vartheta,\xi) \right\| \\ & \leq \varrho^{j+1} \left(1 + \varrho + \varrho^2 + ...\right) \left\| \mathcal{X}_i(\vartheta,\xi) \right\| \\ & \leq \frac{\varrho^{j+1}}{1-\varrho} \left\| \mathcal{X}_0(\vartheta,\xi) \right\| \\ \left\| \mathcal{Y}(\vartheta,\xi) - \sum\limits_{i=0}^{j} \mathcal{Y}_i(\vartheta,\xi) \right\| & \leq \left\| \sum\limits_{i=j+1}^{\infty} \mathcal{Y}_i(\vartheta,\xi) \right\| \\ & \leq \sum\limits_{i=j+1}^{\infty} \left\| \mathcal{Y}_i(\vartheta,\xi) \right\| \leq \sum\limits_{i=j+1}^{\infty} \varrho^j \left\| \mathcal{Y}_0(\vartheta,\xi) \right\| \\ & \leq \varrho^{j+1} \left(1 + \varrho + \varrho^2 + ...\right) \left\| \mathcal{Y}_0(\vartheta,\xi) \right\| \\ & \leq \frac{\theta^{j+1}}{1-\theta} \left\| \mathcal{Y}_0(\vartheta,\xi) \right\| \end{cases}.$$

Therefore, the proof is complete. □

5. Numerical applications

We demonstrate in this section, the validity and effectiveness of the results obtained using KHDJM to find the exact solution of some special cases of systems of nonlinear time-fractional partial differential equations (21).

Example 5.1. Consider the following system of nonlinear time-fractional partial differential equations

$$\begin{cases} \mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi) = \mathcal{X}_{\vartheta\vartheta}(\vartheta,\xi) + 2\mathcal{X}(\vartheta,\xi)\mathcal{X}_\vartheta(\vartheta,\xi) - (\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_\vartheta \\ \mathbb{D}_\xi^\beta \mathcal{Y}(\vartheta,\xi) = \mathcal{Y}_{\vartheta\vartheta}(\vartheta,\xi) + 2\mathcal{Y}(\vartheta,\xi)\mathcal{Y}_\vartheta(\vartheta,\xi) - (\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_\vartheta \end{cases}, \tag{35}$$

under the initial conditions

$$\begin{cases} \mathcal{X}(\vartheta,0) = \sin(\vartheta) \\ \mathcal{Y}(\vartheta,0) = \sin(\vartheta) \end{cases}, \tag{36}$$

where $\mathbb{D}_\xi^\alpha \mathcal{X}(\vartheta,\xi)$ and $\mathbb{D}_\xi^\beta \mathcal{Y}(\vartheta,\xi)$ are the Caputo time-fractional derivatives of order $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ respectively.

For $\alpha = \beta = 1$ the exact solution of system (35) is given by [2]

$$\begin{cases} \mathcal{X}(\vartheta,\xi) = e^{-\xi}\sin(\vartheta) \\ \mathcal{Y}(\vartheta,\xi) = e^{-\xi}\sin(\vartheta) \end{cases}. \tag{37}$$

Applying the Khalouta transform to system (35) and use Theorem 2.3 and initial conditions (36) to get

$$\left\{\begin{array}{l} \mathbb{KH}\left[\mathcal{X}(\vartheta,\xi)\right]=\sin(\vartheta)+\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\begin{array}{c}\mathcal{X}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{X}(\vartheta,\xi)\mathcal{X}_{\vartheta}(\vartheta,\xi)\\ -(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right]\\ \mathbb{KH}\left[\mathcal{Y}(\vartheta,\xi)\right]=\sin(\vartheta)+\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\begin{array}{c}\mathcal{Y}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{Y}(\vartheta,\xi)\mathcal{Y}_{\vartheta}(\vartheta,\xi)\\ -(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right]\end{array}\right. . \tag{38}$$

Taking the inverse Khalouta transform of (38), this yields

$$\left\{\begin{array}{l} \mathcal{X}(\vartheta,\xi)=\sin(\vartheta)+\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\begin{array}{c}\mathcal{X}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{X}(\vartheta,\xi)\mathcal{X}_{\vartheta}(\vartheta,\xi)\\ -(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right]\right]\\ \mathcal{Y}(\vartheta,\xi)=\sin(\vartheta)+\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\begin{array}{c}\mathcal{Y}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{Y}(\vartheta,\xi)\mathcal{Y}_{\vartheta}(\vartheta,\xi)\\ -(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right]\right]\end{array}\right. .$$

The KHDJM then implies that we have the following iteration

$$\left\{\begin{array}{l}\mathcal{X}_0(\vartheta,\xi)=\sin(\vartheta)\\ \mathcal{Y}_0(\vartheta,\xi)=\sin(\vartheta)\end{array}\right. ,$$

$$\left\{\begin{array}{l}\mathcal{X}_1(\vartheta,\xi)=-\sin(\vartheta)\frac{\xi^{\alpha}}{\Gamma(\alpha+1)}\\ \mathcal{Y}_1(\vartheta,\xi)=-\sin(\vartheta)\frac{\xi^{\beta}}{\Gamma(\beta+1)}\end{array}\right. ,$$

$$\left\{\begin{array}{l}\mathcal{X}_2(\vartheta,\xi)=\sin(\vartheta)\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}+2\sin(\vartheta)\cos(\vartheta)\left(-\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{\xi^{3\alpha}}{\Gamma(3\alpha+1)}\right.\\ \qquad\left.+\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{\xi^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\right)\\ \mathcal{Y}_2(\vartheta,\xi)=\sin(\vartheta)\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}+2\sin(\vartheta)\cos(\vartheta)\left(-\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}+\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)}\frac{\xi^{3\beta}}{\Gamma(3\beta+1)}\right.\\ \qquad\left.+\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{\xi^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)}\right)\end{array}\right. ,$$

$$\vdots$$

Thus, the series solution of system (35) is written as

$$\left\{\begin{array}{ll}\mathcal{X}(\vartheta,\xi)= & \sin(\vartheta)\left(1-\frac{\xi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}+...\right)+2\sin(\vartheta)\cos(\vartheta)\left(-\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right.\\ & \left.+\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{\xi^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{\Gamma(\beta+\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{\xi^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+...\right)\\ \mathcal{Y}(\vartheta,\xi)= & \sin(\vartheta)\left(1-\frac{\xi^{\beta}}{\Gamma(\beta+1)}+\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}+...\right)+2\sin(\vartheta)\cos(\vartheta)\left(-\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}\right.\\ & \left.+\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)}\frac{\xi^{3\beta}}{\Gamma(3\beta+1)}+\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{\xi^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)}+...\right)\end{array}\right. . \tag{39}$$

If $\alpha=\beta$ in (39), then the series solution of system (35) is given by

$$\left\{\begin{array}{l}\mathcal{X}(\vartheta,\xi)=\sin(\vartheta)\left(1-\frac{\xi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}+...\right)=\sin(\vartheta)E_{\alpha}(-\xi^{\alpha})\\ \mathcal{Y}(\vartheta,\xi)=\sin(\vartheta)\left(1-\frac{\xi^{\beta}}{\Gamma(\beta+1)}+\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}+...\right)=\sin(\vartheta)E_{\beta}(-\xi^{\beta})\end{array}\right. , \tag{40}$$

where $E_{\alpha}(-\xi^{\alpha})$ and $E_{\beta}(-\xi^{\beta})$ are the Mittag Leffler functions defined by (5).

When $\alpha=\beta=1$ in (40), we obtain

$$\left\{\begin{array}{l}\mathcal{X}(\vartheta,\xi)=e^{-\xi}\sin(\vartheta)\\ \mathcal{Y}(\vartheta,\xi)=e^{-\xi}\sin(\vartheta)\end{array}\right. ,$$

which is the exact solution given in (37).

The two-dimensional plots of the approximate and exact solutions of $\mathcal{X}(\vartheta,\xi) = \mathcal{Y}(\vartheta,\xi)$ using the KHDJM for system (35) with different values of $\alpha = \beta$ is given in Figure 1. Furthermore, the values of approximate solutions obtained by KHDJM, exact solutions, and absolute errors of $\mathcal{X}(\vartheta,\xi) = \mathcal{Y}(\vartheta,\xi)$ for system (35) with different values of $\alpha = \beta$ are provided in Table 1.

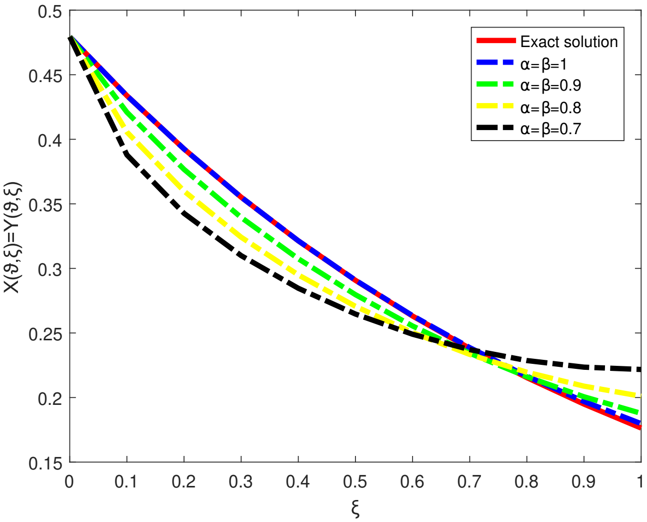

FIGURE 1. Two-dimensional plots representation of the approximate and exact solutions $\mathcal{X}(\vartheta,\xi) = \mathcal{Y}(\vartheta,\xi)$ of system (35) at $\vartheta = 1$.

ξ	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	Exact solution	$\lvert\mathcal{X}_{Exact} - \mathcal{X}_{KHDJM}\rvert$
0.01	0.45902	0.46670	0.47160	0.45604	0.45604	3.9886×10^{-13}
0.03	0.43682	0.44948	0.45870	0.46526	0.46526	9.6600×10^{-11}
0.05	0.42006	0.43523	0.44706	0.45604	0.45604	1.2382×10^{-9}
0.07	0.40598	0.42260	0.43620	0.44701	0.44701	6.6372×10^{-9}
0.09	0.39362	0.41109	0.42593	0.43816	0.43816	2.3242×10^{-8}

TABLE 1. Comparison of numerical values of the approximate and exact solutions $\mathcal{X}(\vartheta,\xi) = \mathcal{Y}(\vartheta,\xi)$ of system (35) for distinct values of fractional parameters $\alpha = \beta$.

Example 5.2. Consider the following system of nonlinear time-fractional partial differential equations

$$\begin{cases} \mathbb{D}_{\xi}^{\alpha}\mathcal{X}(\vartheta,\xi) = -\mathcal{X}_{\vartheta\vartheta}(\vartheta,\xi) - 2\mathcal{X}(\vartheta,\xi)\mathcal{X}_{\vartheta}(\vartheta,\xi) - (\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta} \\ \mathbb{D}_{\xi}^{\beta}\mathcal{Y}(\vartheta,\xi) = -\mathcal{Y}_{\vartheta\vartheta}(\vartheta,\xi) - 2\mathcal{Y}(\vartheta,\xi)\mathcal{Y}_{\vartheta}(\vartheta,\xi) - (\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta} \end{cases}, \tag{41}$$

under the initial conditions

$$\begin{cases} \mathcal{X}(\vartheta,0) = e^{\vartheta} \\ \mathcal{Y}(\vartheta,0) = -e^{\vartheta} \end{cases}, \tag{42}$$

where $\mathbb{D}_{\xi}^{\alpha}\ \mathcal{X}(\vartheta,\xi)$ and $\mathbb{D}_{\xi}^{\beta}\mathcal{Y}(\vartheta,\xi)$ are the Caputo time-fractional derivatives of order $0 < \alpha \leq 1$ and $0 < \beta \leq 1$ respectively.

For $\alpha=\beta=1$ the exact solution of system (41) is given by [4]

$$\begin{cases} \mathcal{X}(\vartheta,\xi)=e^{\vartheta-\xi} \\ \mathcal{Y}(\vartheta,\xi)=-e^{\vartheta-\xi} \end{cases}. \tag{43}$$

Applying the Khalouta transform to system (41) and use Theorem 2.3 and initial conditions (42) to get

$$\begin{cases} \mathbb{KH}\left[\mathcal{X}(\vartheta,\xi)\right]=e^{\vartheta}-\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\begin{array}{c}\mathcal{X}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{X}(\vartheta,\xi)\mathcal{X}_{\vartheta}(\vartheta,\xi)\\ +(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right] \\ \mathbb{KH}\left[\mathcal{Y}(\vartheta,\xi)\right]=-e^{\vartheta}-\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\begin{array}{c}\mathcal{Y}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{Y}(\vartheta,\xi)\mathcal{Y}_{\vartheta}(\vartheta,\xi)\\ +(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right] \end{cases}. \tag{44}$$

Taking the inverse Khalouta transform of (44), this yields

$$\begin{cases} \mathcal{X}(\vartheta,\xi)=e^{\vartheta}-\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathbb{KH}\left[\begin{array}{c}\mathcal{X}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{X}(\vartheta,\xi)\mathcal{X}_{\vartheta}(\vartheta,\xi)\\ +(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right]\right] \\ \mathcal{Y}(\vartheta,\xi)=-e^{\vartheta}-\mathbb{KH}^{-1}\left[\left(\frac{\gamma\eta}{s}\right)^{\beta}\mathbb{KH}\left[\begin{array}{c}\mathcal{Y}_{\vartheta\vartheta}(\vartheta,\xi)+2\mathcal{Y}(\vartheta,\xi)\mathcal{Y}_{\vartheta}(\vartheta,\xi)\\ +(\mathcal{X}(\vartheta,\xi)\mathcal{Y}(\vartheta,\xi))_{\vartheta}\end{array}\right]\right] \end{cases},$$

The KHDJM then implies that we have the following iteration

$$\begin{cases} \mathcal{X}_0(\vartheta,\xi)=e^{\vartheta} \\ \mathcal{Y}_0(\vartheta,\xi)=-e^{\vartheta} \end{cases}, \qquad \begin{cases} \mathcal{X}_1(\vartheta,\xi)=-e^{\vartheta}\frac{\xi^{\alpha}}{\Gamma(\alpha+1)} \\ \mathcal{Y}_1(\vartheta,\xi)=e^{\vartheta}\frac{\xi^{\beta}}{\Gamma(\beta+1)} \end{cases},$$

$$\begin{cases} \mathcal{X}_2(\vartheta,\xi)=e^{\vartheta}\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}+2e^{2\vartheta}\left(\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{\xi^{3\alpha}}{\Gamma(+\alpha+1)}\right. \\ \qquad\left.-\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{\xi^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}\right) \\ \mathcal{Y}_2(\vartheta,\xi)=-e^{\vartheta}\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}+2e^{2\vartheta}\left(\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}-\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)}\frac{\xi^{3\beta}}{\Gamma(3\beta+1)}\right. \\ \qquad\left.-\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{\xi^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)}\right) \end{cases},$$

Thus, the series solution of system (41) is written as

$$\begin{cases} \mathcal{X}(\vartheta,\xi)= & e^{\vartheta}\left(1-\frac{\xi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}+...\right)+2e^{2\vartheta}\left(\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}\right. \\ & \left.-\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)\Gamma(\alpha+1)}\frac{\xi^{3\alpha}}{\Gamma(+\alpha+1)}-\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)}\frac{\xi^{2\alpha+\beta}}{\Gamma(2\alpha+\beta+1)}+...\right) \\ \mathcal{Y}(\vartheta,\xi)= & -e^{\vartheta}\left(1-\frac{\xi^{\beta}}{\Gamma(\beta+1)}+\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}+...\right)+2e^{2\vartheta}\left(\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}\right. \\ & \left.-\frac{\Gamma(2\beta+1)}{\Gamma(\beta+1)\Gamma(\beta+1)}\frac{\xi^{3\beta}}{\Gamma(3\beta+1)}-\frac{\xi^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(\alpha+1)}\frac{\xi^{2\beta+\alpha}}{\Gamma(2\beta+\alpha+1)}+...\right) \end{cases}. \tag{45}$$

If $\alpha=\beta$ in (45), then the series solution of system (41) is given by

$$\begin{cases} \mathcal{X}(\vartheta,\xi)=e^{\vartheta}\left(1-\frac{\xi^{\alpha}}{\Gamma(\alpha+1)}+\frac{\xi^{2\alpha}}{\Gamma(2\alpha+1)}+...\right)=e^{\vartheta}E_{\alpha}(-\xi^{\alpha}) \\ \mathcal{Y}(\vartheta,\xi)=-e^{\vartheta}\left(1-\frac{\xi^{\beta}}{\Gamma(\beta+1)}+\frac{\xi^{2\beta}}{\Gamma(2\beta+1)}+...\right)=-e^{\vartheta}E_{\beta}(-\xi^{\beta}) \end{cases}, \tag{46}$$

where $E_{\alpha}(-\xi^{\alpha})$ and $E_{\beta}(-\xi^{\beta})$ are the Mittag Leffler functions defined by (5).

When $\alpha=\beta=1$ in (46), we obtain

$$\begin{cases} \mathcal{X}(\vartheta,\xi)=e^{\vartheta-\xi} \\ \mathcal{Y}(\vartheta,\xi)=-e^{\vartheta-\xi} \end{cases}.$$

which is the exact solution given in (43).

The two-dimensional plots of the approximate and exact solutions of $\mathcal{X}(\vartheta,\xi)$ and $\mathcal{Y}(\vartheta,\xi)$ using the KHDJM for system (41) with different values of $\alpha=\beta$ is given in

Figure 1. Furthermore, the values of approximate solutions obtained by KHDJM, exact solutions, and absolute errors of $\mathcal{X}(\vartheta,\xi)$ and $\mathcal{Y}(\vartheta,\xi)$ for system (41) with different values of $\alpha=\beta$ are provided in Table 2 and 3, respectively.

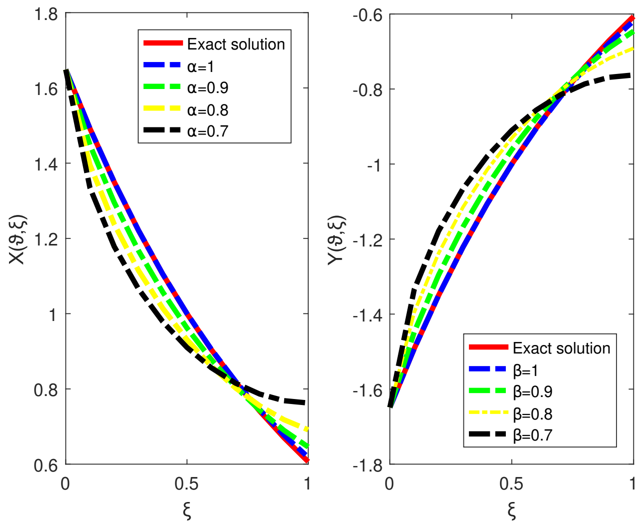

FIGURE 2. Two-dimensional plots representation of the approximate and exact solutions $\mathcal{X}(\vartheta,\xi)$ and $\mathcal{Y}(\vartheta,\xi)$ of system (41) at $\vartheta=1$.

ξ	$\alpha=0.7$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1$	Exact solution	$\lvert\mathcal{X}_{Exact}-\mathcal{X}_{KHDJM}\rvert$
0.01	1.5785	1.6050	1.6218	1.6323	1.6323	1.3716×10^{-12}
0.03	1.5022	1.5457	1.5774	1.6000	1.6000	3.3220×10^{-10}
0.05	1.4446	1.4967	1.5374	1.5683	1.5683	4.2580×10^{-9}
0.07	1.3961	1.4533	1.5001	1.5373	1.5373	2.2825×10^{-8}
0.09	1.3537	1.4137	1.4648	1.5068	1.5068	7.9928×10^{-8}

TABLE 2. Comparison of numerical values of the approximate and exact solutions $\mathcal{X}(\vartheta,\xi)$ of system (41) for distinct values of fractional parameter α.

ξ	$\beta=0.7$	$\beta=0.8$	$\beta=0.9$	$\beta=1$	Exact solution	$\lvert\mathcal{Y}_{Exact}-\mathcal{Y}_{KHDJM}\rvert$
0.01	-1.5785	-1.6050	-1.6218	-1.6323	-1.6323	1.3716×10^{-12}
0.03	-1.5022	-1.5457	-1.5774	-1.6000	-1.6000	3.3220×10^{-10}
0.05	-1.4446	-1.4967	-1.5374	-1.5683	-1.5683	4.2580×10^{-9}
0.07	-1.3961	-1.4533	-1.5001	-1.5373	-1.5373	2.2825×10^{-8}
0.09	-1.3537	-1.4137	-1.4648	-1.5068	-1.5068	7.9928×10^{-8}

TABLE 3. Comparison of numerical values of the approximate and exact solutions $\mathcal{Y}(\vartheta,\xi)$ of system (41) for distinct values of fractional parameters β.

6. Conclusions

The Khalouta-Daftardar-Jafari method was successfully applied to find the analytical solution of systems of nonlinear time-fractional partial differential equations. The reliability of the method and reduced computational workload give this method wider applicability. Also, the behavior of the solution can be formally determined by analytical approximate. Furthermore, we proved the convergence of the solutions. Finally, some numerical applications are included to demonstrate the validity and applicability of the proposed technique.

Systems of nonlinear time-fractional partial differential equations are a very hot and important topic, so it will be very interesting to use KHDJM to solve other types of systems and see if this mehtod is suitable and effective mathematical tool for resolve these systems.

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