Annals of University of Craiova, Math. Comp. Sci. Ser.
Volume 32, 2005, Pages 179–187
ISSN: 1223-6934

Creep formulation of a bilateral contact problem with friction

Thierry-Vincent Hoarau-Mantel, Ángel Rodríguez-Arós, and Juan Manuel Viaño

Abstract. We consider a mathematical model which describes the bilateral contact between a deformable body and an obstacle, the so-called foundation. The body is assumed to have a viscoelastic behavior with long-term memory that we describe with a creep-type constitutive law. The contact takes into account the effects of friction, which are modelled with the Tresca's law. Also, the effects of inertia are neglected, thus a quasistatic model is considered. We present two alternative yet equivalent weak formulations of the problem and establish existence and uniqueness results for both formulations. The proofs are based on arguments on time-dependent variational inequalities and fixed point. We also study the behavior of the solution with respect to the creep operator and establish a convergence result.

2000 Mathematics Subject Classification. Primary 49J40; Secondary 74M15.
Key words and phrases. viscoelastic material, long-term memory, creep-type constitutive law, Tresca's friction law, variational formulations, weak solution.

1. Introduction

In a number of publications (see for instance [10, 15]), the viscoelastic models with long-term memory are presented in the so-called *creep formulation*, that is the strain tensor is determined by the history of the stress tensor. In this case we have

$$\varepsilon_{ij}(t) = \mathcal{C}_{ijkl}\sigma_{kl}(\boldsymbol{u}(t)) + \int_0^t \mathcal{D}_{ijkl}(t-s)\sigma_{kl}(\boldsymbol{u}(s))ds, \tag{1}$$

where $\mathcal{C} = (\mathcal{C}_{ijkl})$ represents the fourth order tensor of elastic compliance and $\mathcal{D} = (\mathcal{D}_{ijkl})$ is the creep compliance tensor. Since in this case the stress tensor cannot be eliminated, there is a need to use new variational formulations, different from that used in [11], in order to study the corresponding contact problem. In [13], two different but equivalent formulations of the Signorini unilateral frictionless contact problem for viscoelastic materials with long-term memory in its *creep* version were studied. Existence and uniqueness of the weak solutions for both formulations were obtained and also the continuous dependence of that solutions with respect to the creep operator was established.

All the references above are concerned with frictionless problems. But there is an increasing number of studies of frictional contact problems. A survey of contact problems taking into account friction can be found in [6] for viscoelastic and viscoplastic constitutive laws. Specifically, Tresca's friction law may be found in [5, 9], and more recently has been used in [1, 2, 14].

This work is intended to be a continuation of [13], since we use (1) as constituve law and follow the general structure of [13]. Nevertheless the notation and certains results introduced therein are recalled here as they are needed. We consider two different yet

Received: October 14, 2004.

equivalent variational formulations of the contact problem, which lead to evolutionary systems for the displacement and stress field. But in this case the contact problem is bilateral and the friction on the contact boundary is modelled with the Tresca's friction law. We prove the unique solvability of the systems and therefore we deduce the existence of the unique weak solution to the contact problem. The proofs are based on results on time-dependent variational inequalities and fixed point. We also discuss the continuous dependence of the solution with respect to the creep tensor and derive a convergence result.
The paper is organized as follows. In Section 2 we state the mechanical problem and present the notation and preliminary material. In Section 3 we list the assumptions imposed on the problem data and derive the variational formulations to the model. We present our main existence and uniqueness results in Section 4 and the convergence result in Section 5.

2. Problem statement and preliminaries

We consider a viscoelastic body which occupies a domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$ in applications) with outer Lipschitz surface Γ that is divided into three disjoint measurable parts Γ_i, $i = 1, 2, 3$, such that meas$(\Gamma_1) > 0$. Let $[0, T]$ be the time interval of interest, where $T > 0$, and let $\boldsymbol{\nu}$ denote the unit outer normal on Γ. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there. A volume force of density $\boldsymbol{f}_0$ acts in $\Omega \times (0, T)$ and surface tractions of density $\boldsymbol{f}_2$ act on $\Gamma_2 \times (0, T)$. We assume that the body forces and tractions vary slowly in time, so the inertial terms may be neglected in the equation of motion, leading to a quasistatic problem. The body is in bilateral contact with a rigid obstacle, the so-called foundation, thus only tangential sliding is allowed on $\Gamma_3 \times (0, T)$. Friction is modelled with the Tresca's law. With these assumptions, denoting by S_d the space of second order symmetric tensors on $\mathbb{R}^d$, the classical formulation of the contact problem of the viscoelastic body is the following.

Problem 2.1. *Find a displacement field* $\boldsymbol{u} : \Omega \times [0, T] \longrightarrow \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times [0, T] \longrightarrow S_d$ *such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \mathcal{C}\boldsymbol{\sigma}(t) + \int_0^t \mathcal{D}(t-s)\boldsymbol{\sigma}(s)\,ds \qquad \text{in } \Omega, \tag{2}$$

$$\operatorname{Div}\boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = \boldsymbol{0} \qquad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{4}$$

$$\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{f}_2(t) \qquad \text{on } \Gamma_2, \tag{5}$$

$$\left\{\begin{array}{l} u_\nu(t) = 0,\ |\boldsymbol{\sigma}_\tau(t)| \le g, \\ |\boldsymbol{\sigma}_\tau(t)| < g \ \Rightarrow\ \dot{\boldsymbol{u}}_\tau(t) = \boldsymbol{0}, \\ |\boldsymbol{\sigma}_\tau(t)| = g \ \Rightarrow\ \exists\,\lambda \ge 0 \ s.t.\ \boldsymbol{\sigma}_\tau(t) = -\lambda\dot{\boldsymbol{u}}_\tau(t) \end{array}\right. \qquad \text{on } \Gamma_3, \tag{6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \qquad \text{in } \Omega. \tag{7}$$

for all $t \in [0, T]$.

In (2)–(7) and below, in order to simplify the notation, we do not indicate explicitly the dependence of various functions on the variable $\boldsymbol{x} \in \Omega \cup \Gamma$. Equation (2) represents the creep viscoelastic constitutive law, equation (3) is the equilibrium equation, while conditions (4) and (5) are the displacement and traction boundary conditions, respectively. The first expression in (6) represents the bilateral contact condition, in which u_ν denotes the normal displacement, while the remaining expressions represent the Tresca's friction law, in which $\boldsymbol{\sigma}_\tau$ denotes the tangential stress on the contact surface and $\dot{\boldsymbol{u}}_\tau$ is the derivative with respect to the time variable of the tangential displacement.

To study the mechanical problem (2)–(7) we introduce the notation we shall use and some preliminary material. For further details we refer the reader to [6, 7]. We denote by "$\cdot$" and $|\cdot|$ the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$, respectively. Here and everywhere in this paper the indices i, j, k, l run between 1 and d and the summation convention over repeated indices is adopted.

We also use the spaces

$$H = \{\, \boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega) \,\}, \qquad Q = \{\, \boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \,\},$$
$$H_1 = \{\, \boldsymbol{u} = (u_i) \in H \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in Q \,\}, \quad Q_1 = \{\, \boldsymbol{\sigma} \in Q \mid \mathrm{Div}\,\boldsymbol{\sigma} \in H \,\},$$

where $\boldsymbol{\varepsilon} : H_1 \longrightarrow Q$ and $\mathrm{Div} : Q_1 \longrightarrow H$ are the *deformation* and *divergence* operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \mathrm{Div}\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces H, Q, H_1 and Q_1 are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \, dx, \qquad (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij} \tau_{ij} \, dx,$$
$$(\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_H.$$

The associated norms on these spaces are denoted by $\|\cdot\|_H$, $\|\cdot\|_Q$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{Q_1}$, respectively.

For every element $\boldsymbol{v} \in H_1$ we still write $\boldsymbol{v}$ for the trace $\gamma\boldsymbol{v}$ of $\boldsymbol{v}$ on Γ and we denote by v_ν and $\boldsymbol{v}_\tau$ the *normal* and *tangential* components of $\boldsymbol{v}$ on the boundary Γ given by

$$v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}, \quad \boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}. \tag{8}$$

We also denote by $\boldsymbol{\sigma\nu}$ the trace of the element $\boldsymbol{\sigma} \in Q_1$ on Γ and by σ_ν, $\boldsymbol{\sigma}_\tau$ its *normal* and *tangential* traces, respectively. Note that when $\boldsymbol{\sigma}$ is a regular (say C^1) function then

$$\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{9}$$

and the following Green's formula holds:

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + (\mathrm{Div}\,\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} \, da \qquad \forall\, \boldsymbol{v} \in H_1. \tag{10}$$

Keeping in mind the boundary conditions (4) and (6), we introduce the closed subspace of H_1 defined by

$$V = \{\ \boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0} \ \text{ a.e. on } \Gamma_1,\ v_\nu = 0 \ \text{ a.e. on } \Gamma_3\ \}. \tag{11}$$

Since $\mathrm{meas}\,(\Gamma_1) > 0$, Korn's inequality holds: there exists a constant $C_K > 0$ which depends only on Ω and Γ_1 such that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_Q \geq C_K \|\boldsymbol{v}\|_{H_1} \quad \forall\, \boldsymbol{v} \in V. \tag{12}$$

A proof of Korn's inequality (12) may be found in [8, p. 79]. Over the space V, we use the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q, \qquad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{13}$$

It follows from (12) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space.
We also need the space of fourth order tensor fields

$$\mathbf{Q}_\infty = \{ \ \mathcal{E} = (\mathcal{E}_{ijkl}) \mid \mathcal{E}_{ijkl} = \mathcal{E}_{jikl} = \mathcal{E}_{klij} \in L^\infty(\Omega)\},$$

which is a real Banach space with the norm

$$\|\mathcal{E}\|_{\mathbf{Q}_\infty} = \max_{0\le i,j,k,l\le d} \|\mathcal{E}_{ijkl}\|_{L^\infty(\Omega)}.$$

Finally, if $(X, \|\cdot\|_X)$ is a real Banach space, we denote by $C([0,T];X)$ the space of continuous functions from $[0,T]$ to X, which is also a Banach space with the usual norm $\|\cdot\|_{C([0,T];X)}$ and we denote by $W^{1,2}(0,T;X)$ the Sobolev space of continuous functions with time derivative L^2 -summable. If $(X, \|\cdot\|_X)$ is a real Banach space or, furthermore, $(X, (\cdot,\cdot)_X)$ is a real Hilbert space, the space $W^{1,2}(0,T;X)$ inherits that structure.

3. Variational formulations

In this section we list the assumptions imposed on the data and derive two variational formulations of the mechanical problem (2)–(7).
We assume that the elasticity and relaxation tensors satisfy

$$\mathcal{C} \in \mathbf{Q}_\infty, \tag{14}$$

$$\exists\ \alpha > 0 \text{ such that } \ \mathcal{C}\boldsymbol{\xi}\cdot\boldsymbol{\xi} \ge \alpha|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in S_d, \quad \text{a.e. in} \ \ \Omega, \tag{15}$$

$$\mathcal{D} \in W^{1,2}(0,T;\mathbf{Q}_\infty). \tag{16}$$

We also assume that the force and traction densities satisfy

$$\boldsymbol{f}_0 \in W^{1,2}(0,T;H), \qquad \boldsymbol{f}_2 \in W^{1,2}(0,T;L^2(\Gamma_2)^d). \tag{17}$$

Also, we assume that the friction bound verifies

$$g \in L^\infty(\Gamma_3), \ g \ge 0 \quad \text{a.e. on } \Gamma_3. \tag{18}$$

Finally, the initial condition is such that

$$\boldsymbol{u}_0 \in V. \tag{19}$$

Next, we denote by $\boldsymbol{f}(t)$ the element of V given by

$$(\boldsymbol{f}(t), \boldsymbol{v})_V = (\boldsymbol{f}_0(t), \boldsymbol{v})_H + (\boldsymbol{f}_2(t), \boldsymbol{v})_{L^2(\Gamma_2)^d} \quad \forall \boldsymbol{v} \in V, \ t \in [0,T], \tag{20}$$

and we note that conditions (17) imply

$$\boldsymbol{f} \in W^{1,2}(0,T;V). \tag{21}$$

We define the seminorm $j : V \to \mathbb{R}_+$ given by

$$j(\boldsymbol{v}) = \int_{\Gamma_3} g|\boldsymbol{v}_\tau| da \quad \forall\ \boldsymbol{v} \in V, \tag{22}$$

which is continuous with the norm of V. Finally, for all $t \in [0,T]$, we denote the set of *admissible stress fields* given by

$$\Sigma(t) = \{ \ \boldsymbol{\tau} \in Q \mid (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{v}) \ge (\boldsymbol{f}(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in V \ \}. \tag{23}$$

Using (8)–(11), (20) and (22) it is straightforward to show that if $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are two regular functions satisfying (3)–(7) then $\boldsymbol{u}(t) \in V$, $\boldsymbol{\sigma}(t) \in Q_1$, and

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \\ &\qquad \ge (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in V, \ \text{a.e. in } (0,\mathrm{T}). \end{aligned} \tag{24}$$

Taking now $\boldsymbol{v} = 2\boldsymbol{u}(t)$ and $\boldsymbol{v} = \boldsymbol{0}$ in (24), both in V, we find

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(\dot{\boldsymbol{u}}(t)) = (\boldsymbol{f}(t), \dot{\boldsymbol{u}}(t))_V \quad \text{a.e. in } (0, \mathrm{T}). \tag{25}$$

If we sum (24) and (25) we see, by (23), that $\boldsymbol{\sigma}(t) \in \Sigma(t)$ a.e. in $(0, T)$ and we obtain from (24) and (23) the following evolutionary variational inequality:

$$\boldsymbol{\sigma}(t) \in \Sigma(t), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma(t), \tag{26}$$

a.e. in $(0, T)$. The initial condition (7) is equivalent to $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$, where $\boldsymbol{\sigma}_0$ may be obtained as

$$\boldsymbol{\sigma}_0 = \mathcal{C}^{-1}\boldsymbol{\varepsilon}(\boldsymbol{u}_0). \tag{27}$$

Moreover, we assume that

$$\boldsymbol{\sigma}_0 \in \Sigma_0. \tag{28}$$

The inequalities (24), (26), combined with (2) and (7) or (27), lead us to consider the following two variational problems.

Problem 3.1. *Find a displacement field* $\boldsymbol{u} : [0, T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to Q_1$ *such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \mathcal{C}\boldsymbol{\sigma}(t) + \int_0^t \mathcal{D}(t - s)\boldsymbol{\sigma}(s)\, ds \quad \forall\, t \in [0, T], \tag{29}$$

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q + j(\boldsymbol{v}) - j(\dot{\boldsymbol{u}}(t)) \tag{30}$$
$$\geq (\boldsymbol{f}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall\, \boldsymbol{v} \in V \text{ a.e. in } (0, \mathrm{T}),$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{31}$$

Problem 3.2. *Find a displacement field* $\boldsymbol{u} : [0, T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0, T] \to Q_1$ *such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \mathcal{C}\boldsymbol{\sigma}(t) + \int_0^t \mathcal{D}(t - s)\boldsymbol{\sigma}(s)\, ds \quad \forall\, t \in [0, T], \tag{32}$$

$$\boldsymbol{\sigma}(t) \in \Sigma(t), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma(t), \tag{33}$$
$$\text{a.e. in } (0, \mathrm{T}),$$

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0. \tag{34}$$

We remark that Problems 3.1 and 3.2 are formally equivalent to the mechanical problem (2)–(7). Indeed, if $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ represents a regular solution of the variational problem 3.1 or 3.2, using the arguments of [5], it follows that $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ satisfies (2)–(7). For this reason, we may consider Problems 3.1 and 3.2 as *variational formulations* of the mechanical problem (2)–(7).

4. Existence and uniqueness results

The main results of this section concern the unique solvability and the equivalence of the variational problems 3.1 and 3.2. They are stated as follows.

Theorem 4.1. *Assume* (14)–(19) *and* (28). *Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *to Problem* 3.1. *Moreover, the solution satisfies*

$$\boldsymbol{u} \in W^{1,2}(0, T; V), \qquad \boldsymbol{\sigma} \in W^{1,2}(0, T; Q_1). \tag{35}$$

Theorem 4.2. *Assume* (14)–(19) *and* (28). *Let* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *be a couple of functions which satisfies* (35). *Then* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *is a solution of the variational problem* 3.1 *if and only if* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *is a solution of the variational problem* 3.2.

Theorem 4.3. *Assume* (14)–(19) *and* (28). *Then there exists a unique solution* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *to Problem* 3.2. *Moreover, the solution satisfies* (35).

Theorems 4.1 and 4.3 state the unique solvability of Problems 3.1 and 3.2, respectively, while Theorem 4.2 expresses the equivalence of these variational problems. From these theorems we conclude that the mechanical problem (2)–(7) has a unique *weak solution* which solves both Problems 3.1 and 3.2.

Notice also that the strong coupling between the integral equation (29) and the time-dependent variational inequality (30) make Problem 3.1 a rather difficult mathematical model. On the other hand, the coupling between the equation (32) and the inequality (33) is weak since, using (32), we can eliminate the displacement field to obtain a variational formulation of the problem, in term of the stress. For this reason, we start with the proof of Theorem 4.3. Moreover, since Theorem 4.1 is a consequence of Theorems 4.2 and 4.3, we only need to provide the proofs of Theorems 4.3 and 4.2. In the following we voluntary omit the details of the various proofs. We refer the reader to [12].

To prove Theorem 4.3 we suppose in what follows that the assumptions (14)–(19) and (28) hold. We first observe that the variational inequality (33) is defined over the time-dependent convex set $\Sigma(t)$. Let us use a change of variable to convert the variational inequality into one associated with a fixed convex set. To this end, let us define $\boldsymbol{\sigma}_f : [0,T] \to Q$ by

$$\boldsymbol{\sigma}_f(t) = \boldsymbol{\varepsilon}(\boldsymbol{f}(t)) \qquad \forall\, t \in [0,T]. \tag{36}$$

Then, using the regularity (21) of $\boldsymbol{f}$, we obtain $\boldsymbol{\sigma}_f \in W^{1,2}(0,T;Q)$. Moreover, since

$$(\boldsymbol{\sigma}_f(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q = (\boldsymbol{f}(t), \boldsymbol{v})_V \qquad \forall \boldsymbol{v} \in V,\ t \in [0,T], \tag{37}$$

it follows from (20) that

$$\mathrm{Div}\, \boldsymbol{\sigma}_f(t) + \boldsymbol{f}_0(t) = 0 \qquad \forall\, t \in [0,T],$$

and using (17) we find $\mathrm{Div}\, \boldsymbol{\sigma}_f \in L^2(0,T;H)$. We conclude that

$$\boldsymbol{\sigma}_f \in W^{1,2}(0,T;Q_1). \tag{38}$$

We can now express $\Sigma(t) = \Sigma_0 + \{\boldsymbol{\sigma}_f(t)\}$ for all $t \in [0,T]$, where

$$\Sigma_0 = \{\, \boldsymbol{\tau} \in Q \mid (\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_Q + j(\boldsymbol{v}) \geq 0 \quad \forall\, \boldsymbol{v} \in V \,\} \tag{39}$$

is the auxiliar convex set of *reference stress fields* which is time independent. Note that for any given $t \in [0,T]$, $\Sigma(t)$ is a translation of Σ_0. The value of the translation is that of $\boldsymbol{\sigma}_f(t)$. We now introduce

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_f; \tag{40}$$

$$\bar{\boldsymbol{\sigma}}_0 = \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_f(0); \tag{41}$$

and consider the following variational problem.

Problem 4.1. *Find* $\boldsymbol{u} : [0,T] \to V$ *and* $\bar{\boldsymbol{\sigma}} : [0,T] \to Q_1$ *such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) = \mathcal{C}\bar{\boldsymbol{\sigma}}(t) + \mathcal{C}\boldsymbol{\sigma}_f(t) + \int_0^t \mathcal{D}(t-s)\bar{\boldsymbol{\sigma}}(s)\, ds \tag{42}$$

$$+ \int_0^t \mathcal{D}(t-s)\boldsymbol{\sigma}_f(s)\, ds \quad \forall\, t \in [0,T],$$

$$\bar{\boldsymbol{\sigma}}(t) \in \Sigma_0,\ (\boldsymbol{\tau} - \bar{\boldsymbol{\sigma}}(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)))_Q \geq 0 \quad \forall\, \boldsymbol{\tau} \in \Sigma_0,\ \text{a.e. in } (0,T) \tag{43}$$

$$\bar{\boldsymbol{\sigma}}(0) = \bar{\boldsymbol{\sigma}}_0. \tag{44}$$

Using (38) and (40) we get the following result.

Lemma 4.1. *The couple* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *is a solution to Problem 3.2 with regularity* $\boldsymbol{u} \in W^{1,2}(0,T;V)$, $\boldsymbol{\sigma} \in W^{1,2}(0,T;Q_1)$ *iff the couple* $\{\boldsymbol{u}, \bar{\boldsymbol{\sigma}}\}$ *is a solution to Problem 4.1 with regularity* $\boldsymbol{u} \in W^{1,2}(0,T;V)$, $\bar{\boldsymbol{\sigma}} \in W^{1,2}(0,T;Q_1)$.

We turn now to Problem 4.1 and show its well-posedness by using a fixed point argument. To this end we introduce the set

$$\mathcal{W} = \{\boldsymbol{\eta} \in W^{1,2}(0,T;Q) | \boldsymbol{\eta}(0) = \mathbf{0}\}. \tag{45}$$

Let $\boldsymbol{\eta} \in \mathcal{W}$ and consider the following auxiliary problem.

Problem 4.2. *Find* $\boldsymbol{u}_\eta : [0,T] \to V$ *and* $\boldsymbol{\sigma}_\eta : [0,T] \to Q_1$ *such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(t)) = \mathcal{C}\boldsymbol{\sigma}_\eta(t) + \mathcal{C}\boldsymbol{\sigma}_f(t) + \boldsymbol{\eta}(t) \quad \forall t \in [0,T] \tag{46}$$

$$\boldsymbol{\sigma}_\eta(t) \in \Sigma_0,\ (\boldsymbol{\tau} - \boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\eta(t)))_Q \geq 0 \ \ \forall \boldsymbol{\tau} \in \Sigma_0, \text{ a.e. in } (0,\mathrm{T}), \tag{47}$$

$$\boldsymbol{\sigma}_\eta(0) = \bar{\boldsymbol{\sigma}}_0. \tag{48}$$

The unique solvability of Problem 4.2 is based on an abstract result which can be found in [3, p. 189] or [4, p. 72]. Then we have

Lemma 4.2. *Problem 4.2 has a unique solution* $\{\boldsymbol{u}_\eta, \boldsymbol{\sigma}_\eta\}$ *such that* $\boldsymbol{u}_\eta \in W^{1,2}(0,T;V)$ *and* $\boldsymbol{\sigma}_\eta \in W^{1,2}(0,T;Q_1)$.

In the next step we consider the operator $\Lambda : \mathcal{W} \to \mathcal{W}$ defined by

$$\Lambda\boldsymbol{\eta}(t) = \int_0^t \mathcal{D}(t-s)(\boldsymbol{\sigma}_\eta(s) + \boldsymbol{\sigma}_f(s))\, ds \ \forall \boldsymbol{\eta} \in \mathcal{W},\ t \in [0,T]. \tag{49}$$

Keeping in mind assumption (16) it is easy to check that if $\boldsymbol{\eta} \in \mathcal{W}$ then $\Lambda\boldsymbol{\eta} \in \mathcal{W}$, that is the operator Λ is well defined. We have the following result.

Lemma 4.3. *The operator* Λ *has a unique fixed point* $\boldsymbol{\eta}^* \in \mathcal{W}$.

The proof of Lemma 4.3 is based on various estimates and the Banach fixed point Theorem.

We now have all the ingredients to prove Theorem 4.3.

Proof of Theorem 4.3

Existence. Let $\boldsymbol{\eta}^* \in \mathcal{W}$ be the fixed point of Λ and let $\{\boldsymbol{u}_{\eta^*}, \boldsymbol{\sigma}_{\eta^*}\}$ be the solution of Problem 4.2 for $\boldsymbol{\eta} = \boldsymbol{\eta}^*$. We can prove that the couple $(\boldsymbol{u}, \bar{\boldsymbol{\sigma}})$, where $\boldsymbol{u} = \boldsymbol{u}_{\eta^*}$ and $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_{\eta^*}$ is a solution of Problem 4.1. To conclude the existence part of Theorem 4.3 we use Lemma 4.1.

Uniqueness. The uniqueness part of Theorem 4.3 follows now from the uniqueness of the fixed point of the operator Λ defined by (49). Alternatively, it can be obtained directly from the Gronwall's inequality and several algebra. □

Proof of Theorem 4.2.

In order to let this paper in a reasonable length we do not give the details of this proof, which is based on various algebra and convex analysis. We refer again to [12].

5. A convergence result

Now we investigate the behavior of the solution to Problems 3.1 and 3.2 when the creep compliance operator converges to zero. To this end, we assume in what follows that (14)–(17) hold and let $(\mathcal{D}_\theta)_{\theta>0}$ be a family of operators which satisfy

$$\mathcal{D}_\theta \in W^{1,2}(0,T;\mathbf{Q}_\infty) \quad \forall\theta > 0, \tag{50}$$

$$\lim_{\theta\to 0} \|\mathcal{D}_\theta - \mathcal{D}\|_{W^{1,2}(0,T;\mathbf{Q}_\infty)} = 0. \tag{51}$$

We consider the following variational problem.

Problem 5.1. *Find a displacement field* $\boldsymbol{u}_\theta : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma}_\theta : [0,T] \to Q_1$ *such that*

$$\boldsymbol{\varepsilon}(\boldsymbol{u}_\theta(t)) = \mathcal{C}\boldsymbol{\sigma}_\theta(t) + \int_0^t \mathcal{D}_\theta(t-s)\boldsymbol{\sigma}_\theta(s)\, ds \quad \forall t \in [0,T], \tag{52}$$

$$\boldsymbol{\sigma}_\theta(t) \in \Sigma(t), \quad (\boldsymbol{\tau} - \boldsymbol{\sigma}_\theta(t), \boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}_\theta(t)))_Q \geq 0 \quad \forall \boldsymbol{\tau} \in \Sigma(t), \quad \textit{a.e. in } (0,T). \tag{53}$$

$$\boldsymbol{\sigma}_\theta(0) = \boldsymbol{\sigma}_0. \tag{54}$$

Clearly, Problem 5.1 represents the variational formulation of the frictional contact problem (2)–(7) in the case when the viscoelastic constitutive law (2) is replaced by the viscoelastic constitutive law (52). Moreover, by Theorem 4.3 it follows that Problem 5.1 has a unique solution, denoted $\{\boldsymbol{u}_\theta, \boldsymbol{\sigma}_\theta\}$, with regularity (35).

We have the following convergence result.

Theorem 5.1. *Assume* (14)–(17)*,* (50) *and* (51)*. Let* $\{\boldsymbol{u}, \boldsymbol{\sigma}\}$ *and* $\{\boldsymbol{u}_\theta, \boldsymbol{\sigma}_\theta\}$ *be the solutions of Problems* 3.2 *and* 5.1*, respectively. Then*

$$\lim_{\theta\to 0} \{\|\boldsymbol{u}_\theta - \boldsymbol{u}\|_{C(0,T;V)} + \|\boldsymbol{\sigma}_\theta - \boldsymbol{\sigma}\|_{C(0,T;Q_1)}\} = 0. \tag{55}$$

Theorem 5.1 represents a continuous dependence result of the weak solution of the frictional contact problem (2)–(7) with respect to the creep compliance operator. In addition to the mathematical interest in this result, it is of importance in applications, as it indicates that small inaccuracies in the creep compliance operator lead to small inaccuracies in the solution. In particular, if $\mathcal{D} = \mathbf{0}$ and $\mathcal{D}_\theta \to \mathbf{0}$ in $W^{1,2}(0,T;\mathbf{Q}_\infty)$ as $\theta \to 0$, it follows that the solution of the Problem 5.1 converges to the solution of a frictional elastic contact problem. In other words, the elastic frictional contact problem problem may be viewed as a limiting case of viscoelastic frictionless contact problem.

References

[1] A. Amassad, M. Shillor and M. Sofonea, A quasistatic contact problem for an elastic perfectly plastic body with Tresca's friction, *Nonlinear Analysis TMA*, **35** (1999), 95–109.

[2] A. Amassad and M. Sofonea, Analysis of a quasistatic viscoplastic problem involving Tresca friction law, *Discrete and Continuous Dynamical Systems*, **4** (1998), 55–72.

[3] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Editura Academiei, Bucharest-Noordhoff, Leyden, 1976.

[4] H. Brezis, *Opérateurs maximaux monotones et semigroups de contractions dans les espaces de Hilbert*, North-Holland, 1973.

[5] G. Duvaut and J.L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1976.

[6] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, Americal Mathematical Society-Intl. Press, 2002.

[7] I.R. Ionescu and M. Sofonea, *Functional and Numerical Methods in Viscoplasticity*, Oxford University Press, Oxford, 1993.

[8] J. Nečas and I. Hlavaček, *Mathematical Theory of Elastic and Elastoplastic Bodies: An Introduction*, Elsevier, Amsterdam, 1981.

[9] P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhäuser, Boston, 1985.

[10] A. C. Pipkin, *Lectures in Viscoelasticity Theory*, Applied Mathematical Sciences **7**, George Allen & Unwin Ltd. London, Sprionger-Verlag New York, 1972.

[11] A. D. Rodríguez–Arós, M. Sofonea and J. M. Viaño, A Signorini Frictionless Contact Problem for Viscoelastic Materials with Long-term Memory, *Numerical Mathematics and Advanced Applications - ENUMATH 2001*, Eds. F. Brezzi, A. Buffo, S. Corsaro and A. Murli, Springer, Milano, 2003, 327–336.

[12] M. Sofonea, T.-V. Hoarau-Mantel, J.M. Viaño and Á. Rodríguez-Arós, Creep Formulation of A Quasistatic Frictional Contact Problem, *Advances in Nonlinear Variational Inequalities*, **7** (2004), 1–25.

[13] M. Sofonea, A. D. Rodríguez–Arós and J. M. Viaño, Creep Formulation of the Signorini Frictionless Contact Problem. *Advances in Nonlinear Variational Inequalities*, **6** (2003), 23–24.

[14] M. Sofonea, A. D. Rodríguez–Arós and J. M. Viaño, A Class of Integro-Differential Variational Inequalities with Applications to Viscoelastic Contact, preprint.

[15] C. Truesdell (Ed.), *Mechanics of Solids, Vol III : Theory of Viscoelasticity, Plasticity, Elastic Waves and Elastic Stability*, Springer-Verlag, Berlin, 1973.

(T.-V. Hoarau-Mantel, Á. Rodríguez-Arós, and J.M. Viaño) Universidade de Santiago de Compostela, Departamento de Matemática Aplicada, 15782 Santiago de Compostela, España, Tel. 00-34-981-563100 Ext:13188 & Fax: 00-34-981-597054

E-mail address: `hoarau@usc.es, angelaros@usc.es and maviano@usc.es`