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Infinity Weak Solutions for a Nonlocal Fractional Problem with Different Boundary Conditions

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Abstract. In this paper, we investigate two nonlinear equations of fractional type under different external conditions. The first part of this study aims to prove the existence of an infinite number of solutions for nonlocal elliptic problems with non-homogeneous Neumann boundary conditions. The proof is guaranteed by exploiting the correct oscillatory behavior of non-smooth terms. The second section of the paper examines a class of nonlocal elliptic problems in which non-smooth components exhibit a mixed effect of concave and convex nonlinearity at Dirichlet boundary conditions. The nonlinearities do not satisfy Ambrosetti-Rabinowitz and monotonicity conditions. Our framework is a Fractional Orlicz-Sobolev space. To establish the main result, we apply variational approaches paired with Ekeland's variational principle.

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1. Introduction

This work deals with the analysis of problems

$$\begin{cases} (-\Delta)^s_{a(.)}u + a(|u|)u = f_\lambda(x,u) & in\, \Omega, \\ B(u) = 0 & on\, \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $B(u)$ is a boundary conditions chosen later and Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $s \in (0,1)$, λ is a positive parameter, $(-\Delta)^s_{a(.)}$ is the nonlocal fractional $a(.)$-Laplacian operator introduced in [16] and defined as

$$(-\Delta)^s_{a(.)}u(x) = p.v. \int_{\mathbb{R}^N} a\big(|D^s u|\big) D^s u \frac{dy}{|x-y|^N}, \quad \text{for all } x \in \mathbb{R}^N,$$

where $D^s u = \dfrac{u(x)-u(y)}{|x-y|^s}$, is the s-Hölder quotient and $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing and right continuous function, with

$$a(0) = 0, \quad a(t) > 0 \quad \text{for } t > 0 \;\text{ and } \quad \lim_{t\to\infty} a(t) = \infty, \tag{1.2}$$

which partnered with the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} a(|t|)t & for\, t \neq 0 \\ 0 & for\; t = 0, \end{cases} \tag{1.3}$$

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is such that is an odd, increasing homeomorphism from $\mathbb{R}$ onto itself.
In this study, we employ appropriate variational methods in the fractional Orlicz-Sobolev space $W^s\mathbb{L}_A(\Omega)$ to establish the existence and multiplicity of a weak solution to the problem (1.1). The space $W^s\mathbb{L}_A(\Omega)$ was introduced by Bonder et al. in [16]. Currently, the fractional Orlicz-Sobolev space serves as an extension of the traditional fractional Sobolev space $W^{s,p}(\Omega)$ (see [1, 7, 23, 28, 43]). Consequently, several authors such as [3, 10, 11, 17, 26] have extended various properties of fractional Sobolev spaces to fractional Orlicz-Sobolev spaces. The applicability of fractional Orlicz-Sobolev spaces in various branches of mathematics has captured considerable interest and has been the subject of research in multiple directions. While it is impossible to cover every aspect of the subject in this study, we will present a few instances for those interested. For example, we refer to [19, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 46].

The aim of this paper is to investigate problem (1.1) under various boundary conditions. The first one is when $B(u) = \mathcal{N}_a(u)$, referred to as the non-homogeneous Neumann boundary condition, and (1.1) is rewritten as follows:

$$\begin{cases} (-\Delta)^s_{a(.)} u + a(|u|)u = f_\lambda(x,u) & in\,\Omega, \\ \mathcal{N}_a(u) = 0 & on\ \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.4}$$

where $\mathcal{N}_a$ is defined by

$$\mathcal{N}_a u(x) = \int_\Omega a\big(|D^s u|\big) D^s u \frac{dy}{|x-y|^N}, \qquad x \in \mathbb{R}^N\backslash\overline{\Omega}, \tag{1.5}$$

which can be considered as the natural generalization of the non-local derivative presented in [24]. Here, we choose the nonlinearity function $f_\lambda(x,u) = \lambda f(x,u)$, where $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x,0) = 0$ for almost every $x \in \overline{\Omega}$. The second case is when $B(u)$ represents Dirichlet-type boundary conditions, and (1.1) is rewritten as follows:

$$\begin{cases} (-\Delta)^s_{a(.)} u + a(|u|)u = f_\lambda(x,u) & in\,\Omega, \\ u = 0 & on\ \mathbb{R}^N\backslash\Omega. \end{cases} \tag{1.6}$$

Here, we consider the nonlinearity function f_λ with a slowly growing principal part, incorporating a critical Orlicz-Sobolev lower term related to the principal part. Specifically, $f_\lambda(x,u) = \lambda g(u) + f(x,u)$, where g is an odd, increasing homeomorphism from $\mathbb{R}$ to $\mathbb{R}$. In the past decades, problems associated with elliptic equations such as

$$-\Delta u = f_\lambda(x,u) \quad x \in \Omega,$$

have been extensively investigated with various types of nonlinearities. For instance, in [20], the authors examined the following problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^p & in\,\Omega, \\ u > 0 & in\,\Omega, \\ B(u) = 0 & on\ \partial\Omega, \end{cases} \tag{1.7}$$

where $B(u)$ represents a mixed Dirichlet-Neumann boundary condition. They established results regarding the existence and multiplicity of solutions for problem (1.7). Ambrosetti et al. in [6] demonstrated the existence and multiplicity of solutions to the semilinear elliptic boundary value problem with concave and convex nonlinearities, specifically,

$$\begin{cases} -\Delta u = \lambda u^q + u^p & in\, \Omega, \\ u > 0 & in\, \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases} \tag{1.8}$$

with $0 < q < 1 < p \leq (N+2)/(N-2)$. Afterward, many authors extended the results presented in (1.7) and (1.8) to the general class of functions known as Young functions, as seen in [40]. The authors investigated the following problem:

$$\begin{cases} -\text{div}(\phi(|\nabla u|)\nabla u) = g(u) + \lambda f(x,u) & in\, \Omega, \\ u = 0 & on\ \partial\Omega, \end{cases}$$

they used a variational approach to obtain a non-negative solution for the above problem. Also in [41], the authors show the existence and multiplicity of solutions for problems like

$$\begin{cases} -\text{div}(\phi(|\nabla u|)\nabla u) = \lambda f(x,u) & in\, \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases}$$

In [14] the authors established the multiplicity result for the following eigenvalue nonhomogeneous Neumann problem

$$\begin{cases} -\text{div}(\phi(|\nabla u|)\nabla u) + \phi(|u|)u = \lambda f(x,u) & in\, \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & on\ \partial\Omega. \end{cases}$$

Non-local elliptic problems involving fractional $a(\cdot)$-Laplacian operators with Dirichlet-type boundary conditions have become more common in recent years, as seen in [8, 31, 46, 47].
However, elliptic problems involving fractional $a(.)$-Laplacian operators and Neuman boundary conditions have lately been attacked by a few researchers, see [12].

Our first finding in this paper is to extended non-homogeneous Neumann problem (1.4) admits a sequence of pairwise different weak solutions on appropriate fractional space $\mathbb{W}$ for a specific interval of values of the parameter λ. The space $\mathbb{W}$ is defined by all mesurable functions $w : \mathbb{R}^N \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A(|D^s w|)d\mu + \int_\Omega A(w(x))dx < \infty, \quad \text{with} \quad \Omega^c = \mathbb{R}^N \setminus \Omega.$$

In the first part of this study, we utilize multiple critical point theorems established in [15], which can be viewed as extensions of Ricceri's variational principle [52].

Theorem 1.1. Let $J, I : X \to \mathbb{R}$ be two Gâteaux differentiable functionals in reflexive real Banach space X, such that I is sequentially weakly upper semicontinuous, J is strongly continuous, coercive and sequentially weakly lower semicontinuous. For each $r > \inf_X J$, let

$$\psi(r) := \inf_{w \in J^{-1}(-\infty,0)} \frac{\sup_{w \in J^{-1}(-\infty,r)} I(v) - I(u)}{r - J(w)},$$

and

$$\delta := \lim_{r \to (\inf_X J)^+} \inf \psi(r).$$

If $\delta < \infty$ then for each $\lambda \in (0, 1/\delta)$, one of the following alternative holds:
(a_1) A global minimum of J exists, as well as a local minimum of $h_\lambda := J - \lambda I$.
(a_2) A sequence $\{w_n\}$ of pairwise distinct critical points (local minima) of h_λ exists, that converges weakly to a global minimum of J with $\lim_{n\to+\infty} J(w_n) = \inf_X J$.

We refer to [14, 48] and reference therein for more Neuman problems.

Next, we examine a class of non-local elliptic problems (1.6) with concave and convex nonlinearities that do not satisfy the well-known Ambrosetti-Rabinowitz condition. This type of problem has not been thoroughly investigated in the fractional framework but has been studied in the classical framework, particularly in the Orlicz-Sobolev space. For instance, da Silva et al. in [22] explored (1.6) with nonlinearities of the form $\lambda a(x)|u|^{q-2}u + b(x)|u|^{l^*-2}u$, exhibiting critical behavior at infinity, where $l^* = Nl/N - l$ with $1 < l < N$, and $a, b : \Omega \to \mathbb{R}$ are two indefinite functions in sign. They obtained a ground-state solution using the well-known Nehari method. For other convex-concave nonlinearities, we refer the reader to [5, 13, 22, 44, 51] and references therein. In this study, we establish the existence of at least two nontrivial solutions to (1.6) for $\lambda > 0$ small enough. We employ functional techniques and variational approaches, coupled with the Ekeland variational principle, on $W^s\mathbb{L}_A(\Omega)$.

The following provides an overview of the organization of this work. We briefly review certain features of Orlicz and fractional Orlicz-Sobolev spaces in Section 2. Section 3 outlines the assumptions used in this study. In Section 4, we present the main results of the problems (1.4), (1.6), along with their proofs.

2. Some preliminary results

The reader is referred to [2, 4, 12, 16, 49] for more details on Orlicz and fractional Orlicz-Sobolev space.

We note by $\mathbf{N}$ the set of all N-functions and C_i ($i \in \mathbb{N}$) a positive constant. Let a be as in (1.2). We will use the following notation $A(t) = \int_0^t \varphi(r)\,\mathrm{d}r$, for every $t \in \mathbb{R}$, then, $A \in \mathbf{N}$ and its complementary $\overline{A}$ given by this relationship $\overline{A}(t) := \sup_{r\geq 0}\{tr - A(r)\}$, is also in $\mathbf{N}$. We point out that $A \in \Delta_2$ if for a certain constant $k > 0$,

$$A(2t) \leq k\,A(t), \quad \text{for every } t > 0. \tag{2.1}$$

We observe that A and $\overline{A}$ satisfies the following Young's inequality:

$$rt \leq A(r) + \overline{A}(t) \ \text{ for all } t,\, r \geq 0. \tag{2.2}$$

Recall that $A^* \in \mathbf{N}$ is defined by

$$(A^*)^{-1}(t) = \int_0^t \frac{A^{-1}(r)}{r^{\frac{N+s}{N}}} dr \text{ for } \ t \geq 0,$$

where we mention that

$$(H_0) \int_0^1 \frac{A^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t < \infty \ \text{ and } (H_\infty) \int_1^{+\infty} \frac{A^{-1}(t)}{t^{1+\frac{s}{N}}}\,\mathrm{d}t = +\infty, \text{for } s \in (0,1).$$

Let $(M, A) \in \mathbf{N}$. The notation $M \prec\prec A$ means that, for each $\varepsilon > 0$,

$$\frac{M(\varepsilon t)}{A(t)} \to 0 \quad \text{as } t \to \infty. \tag{2.3}$$

The Orlicz space $\mathbb{L}_A(\Omega)$ is defined as the mesurable functions $u : \Omega \to \mathbb{R}$ such that $\displaystyle\int_\Omega A\Big(d|u(x)|\Big)\,\mathrm{d}x < +\infty$ for some $d > 0$. The usual norm on $\mathbb{L}_A(\Omega)$ is

$$\|u\|_A = \inf\Big\{d > 0\,/\,\int_\Omega A\Big(\frac{|u(x)|}{d}\Big)\,\mathrm{d}x \le 1\Big\}.$$

Recall that, the Hölder inequality holds

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \le 2||u||_A||v||_{\overline{A}} \text{ for all } u \in \mathbb{L}_A(\Omega) \text{ and } v \in \mathbb{L}_{\overline{A}}(\Omega).$$

One major inequality in $\mathbb{L}_A(\Omega)$ is:

$$\int_\Omega A\Big(\frac{|u(x)|}{||u||_A}\Big)\,\mathrm{d}x \le 1 \text{ for all } u \in \mathbb{L}_A(\Omega) \setminus \{0\}. \tag{2.4}$$

After this, we list a few inequalities that will be used for our proofs. The proof is provided in [42].

Lemma 2.1. Let $A \in \mathbf{N}$, then these assertions are equivalent:
1)

$$1 < l := \inf_{t>0} \frac{ta(t)}{A(t)} \le \sup_{t>0} \frac{ta(t)}{A(t)} := m < +\infty. \tag{2.5}$$

2)

$$\min\{t^l, t^m\}A(\rho) \le A(\rho t) \le \max\{t^l, t^m\}A(\rho),\ \forall t, \rho \ge 0. \tag{2.6}$$

3) $A \in \Delta_2$.

Lemma 2.2. If $A \in \mathbf{N}$ satisfies (2.5) then we have

$$\min\{||u||_A^l, ||u||_A^m\} \le \int_\Omega A(|u|)dx \le \max\{||u||_A^l, ||u||_A^m\},\ \forall u \in \mathbb{L}_A(\Omega). \tag{2.7}$$

Lemma 2.3. Let $\overline{A}$ be the complement of A, $\bar{l} = \frac{l}{l-1}$ and $\overline{m} = \frac{m}{m-1}$, If $A \in \mathbf{N}$ and (2.5) hold, then $\overline{A}$ satisfies:
1)

$$\min\{t^{\bar{l}}, t^{\overline{m}}\}\overline{A}(\rho) \le \overline{A}(\rho t) \le \max\{t^{\bar{l}}, t^{\overline{m}}\}\overline{A}(\rho),\ \forall t, \rho \ge 0. \tag{2.8}$$

2)

$$\min\{||u||_{\overline{A}}^{\bar{l}}, ||u||_{\overline{A}}^{\overline{m}}\} \le \int_\Omega \overline{A}(|u|)dx \le \max\{||u||_{\overline{A}}^{\bar{l}}, ||u||_{\overline{A}}^{\overline{m}}\},\ \forall u \in \mathbb{L}_{\overline{A}}(\Omega). \tag{2.9}$$

Lemma 2.4. We have $A \prec\prec A^*$, i.e, $\displaystyle\lim_{t\to\infty} \frac{A(kt)}{A^*(t)} = 0$, $\forall k > 0$.

Remark 2.5. By Lemma 2.1, Lemma 2.3 and (2.5), we show that $(A, \overline{A}) \in \Delta_2$.

We now look at the definition of the fractional Orlicz-Sobolev spaces $W^s\mathbb{L}_A(\Omega)$, which defined as the mesurable functions $u \in \mathbb{L}_A(\Omega)$ such that

$$\int_{\Omega\times\Omega} A\big(d|D^s w|\big)|x-y|^{-N}dxdy < \infty \text{ for some } d > 0.$$

This space is equipped with the norm,

$$||u||_{s,A} = ||u||_A + [u]_{s,A}, \tag{2.10}$$

where $[.]_{s,A}$ is the Gagliardo semi-norm, defined by

$$[u]_{s,A} = \inf\left\{d > 0 : \int_{\Omega\times\Omega} A\Big(\frac{|D^s w|}{d}\Big)|x-y|^{-N}dxdy \leq 1\right\}.$$

We put $\Upsilon(w) := \int_{\Omega\times\Omega} A(|D^s w|)d\mu + \int_\Omega A(|w|)dx$, $\Psi_1(w) := \int_\Omega F(x,w)dx$ and $\Psi_2(w) := \int_\Omega G(w)dx$, for all $x \in \Omega$, $w \in W^s\mathbb{L}_A(\Omega)$.
$\Gamma(w) := \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A(|D^s w|)d\mu + \int_\Omega A(|w|)dx$, $\Phi(w) := \int_\Omega F(x,w)dx$. for all $x \in \Omega$, $w \in \mathbb{W}$, where $F(x,t) := \int_0^t f(x,r)dr$, $G(t) := \int_0^t g(r)dr$ and $d\mu = \frac{dxdy}{|x-y|^N}$.

Lemma 2.6. ([9] Lemma 3, [45] Lemma 3.4) The functions $\Upsilon, \Psi_{i=1,2} : W^s\mathbb{L}_A(\Omega) \to \mathbb{R}$ are well defined and its the $C^1(W^s\mathbb{L}_A(\Omega), \mathbb{R})$ and we have

$$\langle \Upsilon'(w), \overline{w}\rangle = \int_{\Omega\times\Omega} a\big(|D^s w|\big)D^s w D^s\overline{w}d\mu + \int_\Omega a\big(|w|\big)w\overline{w}dx, \tag{2.11}$$

$$\langle \Psi_1'(w), \overline{w}\rangle = \int_\Omega f(x,w)\overline{w}dx,$$

$$\langle \Psi_2'(w), \overline{w}\rangle = \int_\Omega g(w)\overline{w}dx,$$

for all $\overline{w} \in W^s\mathbb{L}_A(\Omega)$.

Lemma 2.7. ([53] Proposition 4.1) The functions $\Gamma, \Phi : \mathbb{W} \to \mathbb{R}$ are well defined and its the $C^1(\mathbb{W}, \mathbb{R})$ and we have

$$\langle \Gamma'(w), \overline{w}\rangle = \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} a\big(|D^s w|\big)D^s w D^s\overline{w}d\mu + \int_\Omega a\big(|w|\big)w\overline{w}dx, \tag{2.12}$$

$$\langle \Phi'(w), \overline{w}\rangle = \int_\Omega f(x,w)\overline{w}dx,$$

for all $\overline{w} \in \mathbb{W}$.

Proposition 2.8. [9] The following norms

$$||w||_{s,A} = ||w||_A + [w]_{s,A},$$
$$||w||_{max} = \max\{||w||_A, [w]_{s,A}\},$$
$$||w|| = \inf\{\lambda > 0 : \Upsilon\Big(\frac{w}{\lambda}\Big) \leq 1\}, \tag{2.13}$$

are equivalents on $W^s\mathbb{L}_A(\Omega)$. i.e,

$$||w|| \leq 2||w||_{max} \leq 2||w||_{s,A} \leq 4||w||. \tag{2.14}$$

To deal with problem (1.6) under consideration, we choose

$$W_0^s\mathbb{L}_A(\Omega) := \big\{u \in W^{s,A}(\mathbb{R}^N) : u = 0 \; a.e \; \mathbb{R}^N \setminus \Omega\big\}.$$

In these spaces the generalized Poincaré inequality reads as follows (see [10])

$$||u||_A \leq C_1[u]_{s,A}, \; \forall u \in W_0^s\mathbb{L}_A(\Omega). \tag{2.15}$$

We have $(W_0^s\mathbb{L}_A(\Omega), [u]_{s,A})$ is a Banach space whose norm is equivalent to $||u||_{s,A}$. Also is a separable (resp. reflexive) space if and only if $A \in \Delta_2$ (resp. $(A, \overline{A}) \in \Delta_2 \times \Delta_2$). Furthermore if $A \in \Delta_2$ and $A(\sqrt{t})$ is convex, then the space $W_0^s\mathbb{L}_A(\Omega)$ is

uniformly convex (see [16]).
The fractional $a(\cdot)$-Laplacian operator specified in (1) is defined between $W_0^s\mathbb{L}_A(\Omega)$ and its dual space $(W_0^s\mathbb{L}_A(\Omega))^*$. This is confirmed by ([16], Theorem 6.12), where the following expression is derived:

$$\langle \mathcal{G}'(w), v\rangle \quad = \int_{\Omega\times\Omega} a\big(|D^s w|\big) D^s w D^s v d\mu = \langle (-\Delta)^s_{a(.)} w, v\rangle, \tag{2.16}$$

for all $w, v \in W_0^s\mathbb{L}_A(\Omega)$, where $\mathcal{G}(w) := \displaystyle\int_{\Omega\times\Omega} A(|D^s w|)d\mu$.
Lastly, the following proposition will be useful in the subsequent developments.

Lemma 2.9. ([12] Lemma 3.6, Lemma 4.1) The following properties are true:
1)

$$\mathcal{G}\left(\frac{u}{[u]_{s,A}}\right) \leq 1, \text{ for all } u \in W_0^s\mathbb{L}_A(\Omega)\setminus\{0\}.$$

2)

$$\min\{[u]^l_{s,A}, [u]^m_{s,A}\} \leq \mathcal{G}(u) \leq \max\{[u]^l_{s,A}, [u]^m_{s,A}\}, \text{ for all } u \in W_0^s\mathbb{L}_A(\Omega).$$

3)

$$\min\{||u||^l, ||u||^m\} \leq \Upsilon(u) \leq \max\{||u||^l, ||u||^m\}, \text{ for all } u \in W_0^s\mathbb{L}_A(\Omega).$$

Lemma 2.10. [18] Suppose that $A(\sqrt{t})$ is convex, $u_k \rightharpoonup u$ in $W_0^s\mathbb{L}_A(\Omega)$ and

$$\limsup \langle \mathcal{G}'(u_k), u_k - u\rangle \leq 0.$$

Then $u_k \to u \in W_0^s\mathbb{L}_A(\Omega)$.

Now, recall that the natural space to look for (weak) solutions of (1.4) is given by

$$\mathbb{W} := \Big\{u : \mathbb{R}^N \to \mathbb{R} \text{ mesurable } : \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A(|D^s w|)d\mu + \int_\Omega A(w(x))dx < \infty\Big\}.$$

This is a reflexive Banach space with respect to the norm (see [24])

$$||w||_{s,A,*} = ||w||_A + [w]_{s,A,*} \tag{2.17}$$

where

$$[w]_{s,A,*} := \inf\Big\{d > 0 : \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A\Big(\frac{|D^s w|}{d}\Big)d\mu \leq 1\Big\}.$$

Remark 2.11. $\Omega\times\Omega \subset \mathbb{R}^{2N}\setminus(\Omega^c)^2$, then $||w||_{s,A} \leq ||w||_{s,A,*}$ for all $w \in \mathbb{W}$.

Lemma 2.12. Let $w \in \mathbb{W}$. Then

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A(|D^s w|)d\mu + \int_\Omega A(|w|)dx \geq ||w||^l_{s,A,*}, \quad \text{if} \quad ||w||_{s,A,*} < 1.$$

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A(|D^s w|)d\mu + \int_\Omega A(|w|)dx \geq ||w||^m_{s,A,*}, \quad \text{if} \quad ||w||_{s,A,*} > 1.$$

Proof. by some argument in lemma 2.3 in [50], we proof this Lemma. □

Proposition 2.13. Let $w \in \mathbb{W}$ and assume that $\Gamma(w) \leq r$, for some $0 < r < 1$. Then, one has $||w||_{s,A,*} < 1$.

Proof. Let $w \in \mathbb{W}$. By (2.13), if $\Gamma(w) \leq r$ holds, then $||w||_{s,A,*} \leq 1$. Now, claim that $||w||_{s,A,*} \neq 1$. Arguing by contradiction, assume that there exists $w \in \mathbb{W}$ with $||w||_{s,A,*} = 1$ and $\Gamma(w) \leq r$ holds. Let us take $\beta \in (0,1)$, for all $x \in \Omega$, we have

$$\begin{aligned} \Gamma(w) = \Gamma\Big(\frac{\beta}{\beta}w\Big) &= \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A\Big(\frac{\beta}{\beta}|D^s w|\Big)d\mu + \int_\Omega A\Big(\frac{\beta}{\beta}|w(x)|\Big)dx \\ &\geq \beta^m\Bigg[\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A\Big(\frac{|D^s w|}{\beta}\Big)d\mu + \int_\Omega A\Big(\frac{|w(x)|}{\beta}\Big)dx\Bigg]. \end{aligned} \tag{2.18}$$

Set $v = \frac{w(x)}{\beta}$. Then we have $||v||_{s,A,*} = \frac{1}{\beta} > 1$. By Lemma 2.12, we infer that

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A\big(|D^s v|\big)d\mu + \int_\Omega A\big(|v(x)|\big)dx \geq ||v||^l_{s,A,*} > 1. \tag{2.19}$$

Combining (2.18) and (2.19) we deduce that

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A\big(|D^s w|\big)d\mu + \int_\Omega A\big(|w(x)|\big)dx \geq \beta^m.$$

Letting $\beta \nearrow 1$ in the above inequality we obtain

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} A\big(|D^s w|\big)d\mu + \int_\Omega A\big(|w(x)|\big)dx \geq 1.$$

that contradicts condition $\Gamma(w) \leq r$. The proof is complete. $\square$

3. Hypotheses

Related to functions A, f and g, our hypotheses are the following:
(A_1):

$$1 < l_i := \inf_{t>0} \frac{t\varphi(t)}{A(t)} \leq \sup_{t>0} \frac{t\varphi(t)}{A(t)} := n_i < +\infty.$$

(A_2): The function $t \to A(\sqrt{t})$ where $t \in [0,+\infty)$ is convex.
(A_3): There exist $\tau(x) \in L^1(\Omega)$ such that, for all $\sigma \in [0,1]$,

$$\overline{A}(\sigma t) \leq C_2\overline{A}(t) + \tau(x), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\overline{A}(x,t) := mA(t) - t\varphi(t)$. For the function f:
(f_1) $|f(x,t)| \leq C_3(1 + h(|t|))$, for all $x \in \Omega$ and $t \in \mathbb{R}$, where $h : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphism function from $\mathbb{R}$ to $\mathbb{R}$, $H(t) := \int_0^t h(r)dr$ satisfies $H \prec\prec A^*$ and $h_0 := \inf_{t>0} \frac{th(t)}{H(t)} > m$.
(f_2) $\limsup_{t\to 0} \frac{f(x,t)}{\varphi(|t|)} < \frac{1}{\lambda_1}$ uniformly for almost all $x \in \Omega$ where λ_1 is defined in Lemma 2.3 [8].
(f_3) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{m-2}t} = +\infty$ uniformly for almost all $x \in \Omega$.
(f_4) There exist $C_2 \geq 1$ and $\rho(x) \in L^1(\Omega)$ such that, for all $\sigma \in [0,1]$,

$$\overline{F}(x,\sigma t) \leq C_2\overline{F}(x,t) + \rho(x), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\overline{F}(x,t) := tf(x,t) - mF(x,t)$.
For the function g:

(g_1) $G \prec A(\infty)$, $\lim_{t\to 0}\frac{A(t)}{G(t)} = 0$.
We set

$$B := \lim_{\zeta\to 0^+}\inf \frac{\int_\Omega \max\limits_{t\leq\zeta} F(x,t)dx}{\zeta^m} \qquad \text{and} \qquad D := \lim_{\zeta\to 0^+}\sup \frac{\Phi(\zeta)}{\zeta^l}.$$

The embedding below (cf. [10, 12]) will be used in this paper:

$$W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} \mathbb{L}_B(\Omega) \quad \text{and} \quad \mathbb{W} \overset{\text{cpt}}{\hookrightarrow} \mathbb{L}_B(\Omega) \quad \text{if} \quad B \prec\prec A^*.$$

In particular, by Lemma 2.4, (g_1) and (f_1) we have, $G \prec A \prec\prec A^*$ and $H \prec\prec A^*$. Then

$$W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} \mathbb{L}_A(\Omega), \quad W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} \mathbb{L}_G(\Omega) \quad \text{and} \quad W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} \mathbb{L}_H(\Omega). \tag{3.1}$$

Moreover, if $s'l > N$. Then

$$W_0^s\mathbb{L}_A(\Omega) \overset{\text{cpt}}{\hookrightarrow} L^\infty(\Omega),$$

i.e, there exists a constant $c > 0$ such that

$$|w|_\infty \leq c||w|| \quad w \in W_0^s\mathbb{L}_A(\Omega), \tag{3.2}$$

where $|w|_\infty := \sup\limits_{x\in\overline{\Omega}} |w(x)|$ and $0 < s' < s < 1$.

At this point we set the definition of our weak solution, we say that $w \in \mathbb{W}$ is a weak solution for problem (1.4) if

$$\int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} a\big(|D^s w|\big) D^s w D^s \overline{w} d\mu + \int_\Omega a\big(|w|\big) w\overline{w} dx - \lambda \int_\Omega f(x,w)\overline{w} dx = 0,$$

for all $\overline{w} \in \mathbb{W}$ and $w \in W_0^s\mathbb{L}_A(\Omega)$ is a weak solution for problem (1.6) if

$$\int_{\Omega\times\Omega} a\big(|D^s w|\big) D^s w D^s \overline{w} d\mu + \int_\Omega a\big(|w|\big) w\overline{w} dx - \lambda \int_\Omega g(w)\overline{w} dx - \int_\Omega f(x,w)\overline{w} dx = 0,$$

for all $\overline{w} \in W_0^s\mathbb{L}_A(\Omega)$.

4. Main results

Our main results are stated below

Theorem 4.1. Let $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, be a continuous function, $A \in \mathbf{N}$ satisfies (A_1)-(A_3) and let $\alpha > 0$ such that

$$\lim_{t\to 0^+} \frac{A(t)}{t^l} < \alpha. \tag{4.1}$$

Further, assume

$$\lim_{\zeta\to 0^+}\inf \frac{\int_\Omega \max\limits_{t\leq\zeta} F(x,t)dx}{\zeta^m} < \frac{(2c)^{-m}}{\alpha|\Omega|} \lim_{\zeta\to 0^+}\sup \frac{\Phi(\zeta)}{\zeta^l}. \tag{4.2}$$

Then, for every λ belonging to

$$\Big]\alpha|\Omega|D^{-1}, (2c)^{-m}B^{-1}\Big[,$$

problem (1.4) admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $\mathbb{W}$.

Theorem 4.2. Given A satisfies (A_1)-(A_3), f satisfies (f_1)-(f_4) and g satisfies (g_1). Then there exists $\lambda_* > 0$ such that, for each $\lambda \in (0, \lambda_*)$, problem (1.6) has two nontrivial weak solutions.

For any $\lambda > 0$, we defined the Euler functional $\mathcal{L}_\lambda : \mathbb{W} \to \mathbb{R}$ associated to problem (1.4) by

$$\mathcal{L}_\lambda(w) := \Gamma(w) - \lambda\Phi(w), \text{ for all } w \in \mathbb{W}.$$

We can observe that, according to Lemma 2.7 we have that $\mathcal{L}_\lambda \in C^1(\mathbb{W}, \mathbb{R})$ with the derivative given by

$$\langle \mathcal{L}'_\lambda(w), \overline{w}\rangle = \int_{\mathbb{R}^{2N}\setminus(\Omega^c)^2} a\big(|D^s w|\big) D^s w D^s \overline{w} d\mu + \int_\Omega a\big(|w|\big) w\overline{w} dx - \lambda \int_\Omega f(x, w)\overline{w} dx. \tag{4.3}$$

And the Euler functional $\mathcal{I}_\lambda : W_0^s \mathbb{L}_A(\Omega) \to \mathbb{R}$ associated to problem (1.6) is as follows

$$\mathcal{I}_\lambda(w) := \Upsilon(w) - \lambda\Psi_1(w) - \Psi_2(w), \text{ for all } w \in W_0^s\mathbb{L}_A(\Omega).$$

By Lemma 2.6 we have that $\mathcal{I}_\lambda \in C^1(W_0^s\mathbb{L}_A(\Omega), \mathbb{R})$ with the derivative given by

$$\begin{aligned} \langle \mathcal{I}'_\lambda(w), \overline{w}\rangle &= \int_{\Omega\times\Omega} a\big(|D^s w|\big) D^s w D^s \overline{w} d\mu + \int_\Omega a\big(|w|\big) w\overline{w} dx \\ &\quad -\lambda \int_\Omega g(w)\overline{w} dx - \int_\Omega f(x, w)\overline{w} dx. \end{aligned} \tag{4.4}$$

Hence, finding weak solutions for problem (1.4) (resp. (1.6)) is equivalent to find critical points for the functional $\mathcal{L}_\lambda$ (resp. $\mathcal{I}_\lambda$).

Proof of Theorem 4.1. We can seek for weak solutions of problem (1.4) by applying Theorem 1.1. For that let $\{c_n\}$ be a real sequence such that $\lim_{n\to\infty} c_n = 0$ and

$$B := \lim_{n\to\infty} \frac{\displaystyle\int_\Omega \max_{t\le c_n} F(x, t) dx}{c_n^m}.$$

Put $r_n = \left(\frac{c_n}{2c}\right)^m$ for all $n \in \mathbb{N}$. Then, by Lemmas 2.12 and Proposition 2.13, we can deduce that

$$\Big\{v \in \mathbb{W} : \Gamma(v) < r_n\Big\} \subseteq \Big\{v \in \mathbb{W} : ||v||_{s,A,*} < \frac{c_n}{2c}\Big\}.$$

Because of the Remark 2.11 and (3.2), we infer that

$$|v|_\infty \le c||v||_{s,A,*}. \tag{4.5}$$

According to (2.14),

$$|v(x)| \le |v|_\infty \le c||v||_{s,A} \le 2c||v||_{s,A,*} \le c_n, \qquad \forall x \in \overline{\Omega}.$$

Hence

$$\Big\{v \in \mathbb{W} : ||v||_{s,A,*} < \frac{c_n}{2c}\Big\} \subseteq \Big\{v \in \mathbb{W} : |v| < c_n\Big\}.$$

Considering that $\Gamma(u_0) = 0$ and $\Phi(u_0) = 0$, where $u_0(x) = 0$ for all $x \in \Omega$, for all $n \in \mathbb{N}$ one has

$$\begin{aligned}\psi(r_n) &= \inf_{\Gamma(v)<r_n} \frac{\sup_{\Gamma(v)<r_n} \Phi(v) - \Phi(u)}{r_n - \Gamma(u)} \leq \frac{\sup_{\Gamma(v)<r_n} \Phi(v)}{r_n} = \frac{\sup_{\Gamma(v)<r_n} \int_\Omega F(x,v)dx}{r_n} \\ &\leq \frac{\int_\Omega \max_{|t|<c_n} F(x,t)dx}{r_n} \\ &\leq (2c)^m \times \frac{\int_\Omega \max_{|t|<c_n} F(x,t)dx}{c_n^m}.\end{aligned}$$

Therefore, from the assumption (4.2) one has $B < +\infty$, we obtain

$$\delta \leq \lim_{n\to\infty} \inf \psi(r_n) \leq (2c)^m B < +\infty.$$

Now, take

$$\lambda \in \Big]\alpha|\Omega|D^{-1}, (2c)^{-m}B^{-1}\Big[,$$

At this point we will show that 0, that is the unique global minimum of Γ, is not a local minimum of $\mathcal{L}_\lambda$. For this goal, let $\{\zeta_n\}$ be a real sequence of positive numbers such that $\lim_{n\to\infty} \zeta_n = 0$ and

$$\lim_{n\to\infty} \frac{\Phi(\zeta_n)}{\zeta_n} = D. \tag{4.6}$$

From the Neuman boundary condition,

$$\mathcal{N}_a w_n(x) = \int_\Omega a\big(|D^s w_n|\big) D^s w_n \frac{dy}{|x-y|^N} = 0, \qquad x \in \mathbb{R}^N \backslash \overline{\Omega},$$

and the continuous of the function a, implies that $D^s w_n = \frac{w_n(x)-w_n(y)}{|x-y|^s} = 0$. Hence $w_n(x) = w_n(y)$ for all $x, y \in \Omega \times \Omega$. Now, For each $n \in \mathbb{N}$, put $w_n(z) := \zeta_n$, for all $z \in \Omega$. Clearly $w_n \in \mathbb{W}$, for each $n \in \mathbb{N}$ and $w_n \to 0$ as $n \to \infty$. Hence

$$\Gamma(w_n) = \int_{\mathbb{R}^{2N}\backslash(\Omega^c)^2} A(|D^s w_n|)\mu + \int_\Omega A(|w_n|)dz = \int_\Omega A(|w_n|)dz = A(\zeta_n)|\Omega|.$$

Moreover, from hypothesis (4.1), taking into account that $\lim_{n\to\infty} w_n = 0$ one

$$A(w_n) < \alpha w_n^l,$$

for every $n \geq n_0$. If $D < \infty$, let $\epsilon \in \Big]\frac{\alpha|\Omega|}{\lambda D}, 1\Big[$. By (4.6) there exists n_ϵ such that

$$\Phi(\zeta_n) > \epsilon D \zeta_n^l, \qquad \forall n > n_\epsilon.$$

Hence

$$\begin{aligned}\mathcal{L}_\lambda(w_n) &= \Gamma(w_n) - \lambda\Phi(w_n) \\ &\leq \alpha w_n^l|\Omega| - \lambda\epsilon D w_n^l = w_n^l(\alpha|\Omega| - \lambda\epsilon D) < 0,\end{aligned}$$

for every $n \geq \max\{n_0, n_\epsilon\}$. On the other hand, if $D = +\infty$ let us consider $\beta > \frac{\alpha|\Omega|}{\lambda}$. By (4.6) there exists n_β such that

$$\Phi(\zeta_n) > \beta\zeta_n^l, \qquad \forall n > n_\beta.$$

Moreover,

$$\begin{aligned}\mathcal{L}_\lambda(w_n) &= \Gamma(w_n) - \lambda\Phi(w_n)\\ &\leq \alpha w_n^l|\Omega| - \lambda\beta w_n^l = w_n^l(\alpha|\Omega| - \lambda\beta) < 0,\end{aligned}$$

for every $n \geq \{n_0, v_\beta\}$. As a consequence, $\mathcal{L}_\lambda < 0$ for any n large enough. Since $\mathcal{L}_\lambda(0) = \Gamma(0) - \lambda\Phi(0) = 0$, i.e 0 isn't a local minimum of $\mathcal{L}_\lambda$. Further, because Γ has zero as a unique global minimum. By Theorem 1.1 we have the existence of a sequence $\{v_n\}$ of critical points of the functional $\mathcal{L}_\lambda$, pairwise distinct, such that $\lim_{n\to\infty}\Gamma(v_n) = 0$. By the Lemma 2.12 we have $||v_n||_{s,A,*}^m \leq \Gamma(v_n)$ for every n large enough. Thus $\lim_{n\to\infty}||v_n|||_{s,A,*} = 0$ and this completes the proof.

Proof of Theorem 4.2. First, consider X as a real Banach space and $\mathcal{I} \in C^1(X,\mathbb{R})$. We say $\mathcal{I}$ satisfies the C_c condition if any sequence $\{w_n\} \subset X$ such that $\mathcal{I}(w_n) \to c$ and $||\mathcal{I}'(w_n)||_{X^*}(1 + ||w_n||) \to 0$ as $n \to \infty$ has a convergent subsequence. $\{w_n\}$ is called a Cerami sequence at the level $c \in \mathbb{R}$. Now, we can mentioned the following Lemma

Lemma 4.3. ([21]) Let $\mathcal{I} \in C^1(X,\mathbb{R})$ fulfilling the C_c conditions. If
(i) $\mathcal{I}(\theta) = 0$.
(ii) There exist two constants $\tau > 0$, $\eta > 0$, such that $\mathcal{I}(w) \geq \eta$ for any $w \in X$ with $||w|| = \tau$.
(iii) There exists a function $\tilde{w} \in X$ such that $||\tilde{w}|| > \tau$ and $\mathcal{I}(\tilde{w}) < 0$.
Then $\mathcal{I}$ has a critical value $c \geq \eta$, i.e., there exists $u \in X$ such that $\mathcal{I}(u) = c$ and $\mathcal{I}'(u) = \theta$.

Lemma 4.4. Given that (A_1), (f_1), (f_2) and (g_1) hold, then there exist positive constants λ_*, τ, η, such that, for each $\lambda \in (0, \lambda_*)$, $J_\lambda(w) \geq \eta$, for any $w \in W_0^s\mathbb{L}_A(\Omega)$, with $||w|| = \tau$.

Proof. By (3.1), there exists C_4 such that,

$$||w||_G \leq C_4||w||, \quad ||w||_H \leq C_4||w||, \quad \forall w \in W_0^s\mathbb{L}_A(\Omega). \tag{4.7}$$

Let $0 < \tau < \min\{1, 1/C_4\}$ for any $w \in S_\tau := \{w \in W_0^s\mathbb{L}_A(\Omega) : ||w|| = \tau\}$, using (4.7) we infer that $||w||_G < 1$ and $||w||_H < 1$. Moreover, by (2.4) we deduced $\displaystyle\int_\Omega G(w)dx < 1$. On other hand, by using condition (f_2), there exist $\epsilon_0 \in (0,1)$ and $\delta > 0$ such that

$$|F(x,t)| \leq (\frac{1-\epsilon_0}{\lambda_1})A(t), \quad \forall x \in \Omega, |t| < \delta. \tag{4.8}$$

Since $H(t)/t$ is increasing on $[\delta, \infty)$. By condition (f_1) which combined with (4.8), we get

$$|F(x,t)| \leq \Big(\frac{1-\epsilon_0}{\lambda_1}\Big)A(t) + C_5H(t), \quad \forall x \in \Omega,\ t \in \mathbb{R}. \tag{4.9}$$

According to, (4.9),

$$\begin{aligned}\mathcal{I}_\lambda(w) &= \int_{\Omega\times\Omega} A(|D^s w|)d\mu + \int_\Omega A(|w|)dx - \lambda\int_\Omega G(w)dx - \int_\Omega F(x,w)dx.\\ &\geq \int_{\Omega\times\Omega} A(|D^s w|)d\mu + \int_\Omega A(|w|)dx - \lambda\int_\Omega G(w)dx - \int_\Omega |F(x,w)|dx.\\ &\geq \int_{\Omega\times\Omega} A(|D^s w|)d\mu + \int_\Omega A(|w|)dx - \lambda - (\frac{1-\epsilon_0}{\lambda_1})\int_\Omega A(w)dx - C_5\int_\Omega H(w)dx.\end{aligned}$$

Using Lemma 2.3 in [8], Lemma 2.9,

$$
\begin{aligned}
\mathcal{I}_\lambda(w) \geq\ & \int_{\Omega\times\Omega} A\big(|D^s w|\big)d\mu + \int_\Omega A\big(|w|\big)dx - \lambda - (1-\epsilon_0)\int_{\Omega\times\Omega} A\big(|D^s w|\big)d\mu \\
& -C_5\int_\Omega H(w)dx \\
>\ & \epsilon_0 \int_{\Omega\times\Omega} A\big(|D^s w|\big)d\mu + \epsilon_0\int_\Omega A\big(|w|\big)dx - \lambda - C_5||w||_H^{h_0} \\
\geq\ & \epsilon_0\Psi(w) - \lambda - C_6||w||^{h_0} \\
\geq\ & \epsilon_0||w||^m - \lambda - C_6||w||^{h_0} \\
=\ & ||w||^m(\epsilon_0 - C_6||w||^{h_0-m}) - \lambda.
\end{aligned}
\tag{4.10}
$$

Denote $\psi(\tau) = \epsilon_0 - C_6\tau^{h_0-m}$, by $h_0 > m$, we get, $\lim_{\tau\to 0^+}\psi(\tau) \to \epsilon_0 > 0$. Therefore, choose $\tau > 0$ small enough such that $\psi(\tau) > \epsilon_0/2$. Set $\lambda_* := \epsilon_0\tau^m/4 > 0$, $\eta := \epsilon_0\tau^m/4$. For all $w \in S_\tau$ and $\lambda \in (0, \lambda_*)$, applying (4.10), we obtain

$$\mathcal{I}_\lambda(w) \geq \epsilon_0\tau^m/4 > 0 = \eta.$$

□

Lemma 4.5. Given that (A_1), (f_1), (f_3), and (g_1) hold. Then, for any $\lambda > 0$, $\tau > 0$, there exists a function $w_\lambda, \tilde{w}_\lambda \in W_0^s\mathbb{L}_A(\Omega)$ such that
(i) $||w_\lambda|| > \tau$ and $\lim_{t\to+\infty}\mathcal{I}_\lambda(w_\lambda) = -\infty$.
(ii) $||\tilde{w}_\lambda|| < \tau$ and $\lim_{t\to 0}\mathcal{I}_\lambda(\tilde{w}_\lambda) < 0$.

Proof. Let $K > 0$. It follows from (f_3) that there exists a constant $C_K > 0$ such that

$$F(x,t) \geq K|t|^m - C_K \quad \forall (x,t) \in \Omega\times\mathbb{R}. \tag{4.11}$$

Let us choose a compact set $S, \tilde{S} \subset \Omega$ where $|S|, |\tilde{S}| > 0$, it is possible to define $w_0, \tilde{w}_0 \in C_c^\infty(\Omega)\setminus\{0\}$ such that $w_0(x) = 1$ for $x \in S$, $0 \leq w_0(x) \leq 1$ for $x \in \Omega$ and $\tilde{w}_0 = 1$ for $x \in \tilde{S}$, $0 \leq \tilde{w}_0(x) \leq 1$ for $x \in \Omega$. Thus by Theorem 7 in [10] we have $w_0, \tilde{w}_0 \in W_0^s\mathbb{L}_A(\Omega)$. Let $t > 1$ be large enough to make $||tw_0|| > 1$, by (4.11), the Proposition 2.8 and the Lemma 2.9, we have

$$
\begin{aligned}
\mathcal{I}_\lambda(tw_0) &= \int_{\Omega\times\Omega} A(t|D^s w_0|)d\mu + \int_\Omega A\big(t|w_0|\big)dx - \lambda\int_\Omega G(tw_0)dx - \int_\Omega F(x,tw_0)dx \\
&\leq t^m[w_0]_{s,A}^m + t^m||w_0||_A^m - Kt^m\int_\Omega |w_0|^m dx + C_K|\Omega| \\
&\leq 2t^m||w_0||_{max}^m - Kt^m\int_\Omega |w_0|^m dx + C_K|\Omega| \\
&\leq t^m\Big(2||w_0||_{max}^m - K\int_S |w_0|^m dx\Big) + C_K|\Omega| \\
&= t^m\Big(2||w_0||_{max}^m - K|S|\Big) + C_K|\Omega|.
\end{aligned}
$$

From Proposition 2.8,

$$\mathcal{I}_\lambda(tw_0) \quad \leq t^m\Big(2||w_0||^m - K|S|\Big) + C_K|\Omega|.$$

Since $K > 0$ is arbitrary we can choose $K := \dfrac{3||w_0||^m}{|S|}$,

$$\mathcal{I}_\lambda(tw_0) \quad \leq -t^m ||w_0||^m + C_K|\Omega|.$$

Due to $||w_0|| > 0$, we infer that $\mathcal{I}_\lambda(tw_0) \to -\infty$ as $t \to +\infty$. Taking t large enough such that $t > \max\{1, \frac{\tau+1}{||w_0||}\}$, set $w_\lambda = tw_0$, which completes the proof. For (ii). Take $t \in (0, \delta)$ where δ given in (4.8) such tha $||t\tilde{w}_0|| < 1$ and $||t\tilde{w}_0||_G < 1$, by (4.8) we have

$$|F(x, t\tilde{w}_0)| \leq (\frac{1-\epsilon_0}{\lambda_1})A(t\tilde{w}_0), \quad \forall x \in \Omega.$$

From Lemma 2.3 in [8], we have

$$\begin{aligned}
\mathcal{I}_\lambda(t\tilde{w}_0) = &\int_{\Omega\times\Omega} A\big(t|D^s\tilde{w}_0|\big)d\mu + \int_\Omega A\big(t|\tilde{w}_0|\big)dx - \lambda\int_\Omega G(t\tilde{w}_0)dx - \int_\Omega F(x, t\tilde{w}_0)dx \\
\leq &\int_{\Omega\times\Omega} A\big(t|D^s\tilde{w}_0|\big)d\mu + \int_\Omega A\big(t|\tilde{w}_0|\big)dx - \lambda\int_\Omega G(t\tilde{w}_0)dx \\
&+ (\frac{1-\epsilon_0}{\lambda_1})\int_\Omega A(t\tilde{w}_0)dx \\
\leq &(2+\lambda_1-\epsilon_0)\int_{\Omega\times\Omega} A\big(t|D^s\tilde{w}_0|\big)d\mu - \lambda\int_\Omega G(t\tilde{w}_0)dx
\end{aligned}$$

According to Lemma 2.9, we infer that

$$\begin{aligned}
\mathcal{I}_\lambda(t\tilde{w}_0) \leq &\quad (2+\lambda_1-\epsilon_0)A(t)\int_{\Omega\times\Omega} \max\{|D^s\tilde{w}_0|^l, |D^s\tilde{w}_0|^m\}d\mu - \lambda\int_{\tilde{S}} G(t)dx \\
\leq &\quad (2+\lambda_1-\epsilon_0)A(t)||\tilde{w}_0||^l_{W^{s,l}(\Omega)} - \lambda G(t)|\tilde{S}| \\
= &\quad G(t)\Big[\tilde{C}\tfrac{A(t)}{G(t)} - \lambda|\tilde{S}|\Big],
\end{aligned} \tag{4.12}$$

where $\tilde{C} = (2+\lambda_1-\epsilon_0)||\tilde{w}_0||^l_{W^{s,l}(\Omega)}$. According to (g_1), we get the result for a small t and for $\tilde{w}_\lambda = t\tilde{w}_0 < \tau$. □

Lemma 4.6. Given that (A_1), (g_1), and (f_1)-(f_4) hold. Then, for every $\lambda > 0$, the functional $\mathcal{I}_\lambda$ satisfies C_c-condition for any $c > 0$.

Proof. Given $\lambda > 0$, $c > 0$. Let $\{u_n\} \subset W^s_0\mathbb{L}_A(\Omega)$ be a Cerami sequence at the level c of $\mathcal{I}_\lambda$, i.e.,

$$\mathcal{I}_\lambda(u_n) \to c \quad \text{and} \quad ||\mathcal{I}'_\lambda(u_n)||_{(W^s_0\mathbb{L}_A(\Omega))^*}(1+||u_n||) \to 0 \quad \text{as } n \to \infty. \tag{4.13}$$

First, we shall show that $\{u_n\}$ is bounded. Otherwise, there is a subsequence, still denoted by $\{u_n\}$, such that $\lim_{n\to\infty} ||u_n|| = \infty$ and $\|u_n\| > 1$ for all $n \in \mathbb{N}$. We denote $w_n(x) = \frac{u_n(x)}{||u_n||}$, $x \in \Omega$. Then $\{w_n\} \subset W^s_0\mathbb{L}_A(\Omega)$ and $||w_n(x)|| = 1$ for every $n \in \mathbb{N}$. On other hand, assume that there exists $\{w\} \subset W^s_0\mathbb{L}_A(\Omega)$ such that $w_n \rightharpoonup w$. From (3.1), it follows that

$$||w_n - w||_{L^1} \to 0, \quad ||w_n - w||_G \to 0, \quad ||w_n - w||_H \to 0 \quad \text{as } n \to \infty. \tag{4.14}$$

$$w_n(x) \to w(x) \quad \text{a.e}\, x \in \Omega,\ n \to \infty. \tag{4.15}$$

Claim: $w(x) = 0$ a.e $x \in \Omega$.
$|\Omega_0| := \{x \in \Omega : w(x) \neq 0\} > 0$. Given $x \in \Omega_0$, (4.15) implies that $|u_n(x)| =$

$|w_n(x)| \times ||u_n|| \to \infty$ as $n \to \infty$. Furthermore, by (f_3) we obtain that, for given $x \in \Omega_0$,

$$\frac{F(x,u_n(x))}{||u_n||^m} = \frac{F(x,u_n(x))}{|u_n|^m}|w_n(x)|^m \to \infty, \quad n \to \infty. \tag{4.16}$$

From (f_3) and the continuity of F on $\Omega \times \mathbb{R}$, there exists a constant $C > 0$ such that

$$F(x,t) \geq C \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

which implies that

$$\frac{F(x,u_n(x)) - C}{||u_n||^m} = \frac{F(x,u_n(x)) - C}{|u_n|^m}|w_n(x)|^m \geq 0, \quad \forall x \in \Omega. \tag{4.17}$$

From (4.13), it follows that

$$c+o(1) = \mathcal{I}_\lambda(u_n) = \int_{\Omega\times\Omega} A(|D^s u_n|)d\mu + \int_\Omega A(|u_n|)dx - \lambda \int_\Omega G(|u_n|)dx - \int_\Omega F(x,u_n)dx.$$

Dividing the above equality by $||u_n||^m$, by using Lemma 2.9 and the fact that $||u_n|| > 1$,

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n(x))}{||u_n||^m}dx = \frac{\liminf_{n\to\infty} \int_\Omega F(x,u_n(x))dx}{||u_n||^m} \\
&= \liminf_{n\to\infty} \left[\frac{\int_{\Omega\times\Omega} A(|D^s u_n|)d\mu + \int_\Omega A(|u_n|)dx}{||u_n||^m} - \frac{\lambda \int_\Omega G(|u_n|)dx}{||u_n||^m} - \frac{c+o(1)}{||u_n||^m} \right] \\
&\leq \liminf_{n\to\infty} \left[\frac{\int_{\Omega\times\Omega} A(|D^s u_n|)d\mu + \int_\Omega A(|u_n|)dx}{||u_n||^m} - \frac{c+o(1)}{||u_n||^m} \right] \\
&\leq 1.
\end{aligned} \tag{4.18}$$

By Fatou's lemma and (4.15),(4.16),(4.17) and (4.18)

$$\begin{aligned}
\infty &= \int_{\Omega_0} \lim_{n\to\infty} \frac{F(x,u_n(x)) - C}{||u_n||^m}dx \\
&\leq \liminf_{n\to\infty} \int_{\Omega_0} \frac{F(x,u_n(x)) - C}{||u_n||^m}dx \leq \liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n(x)) - C}{||u_n||^m}dx \\
&\leq \liminf_{n\to\infty} \frac{\int_\Omega F(x,u_n(x))dx}{||u_n||^m} - \limsup_{n\to\infty} \frac{\int_\Omega C_1 dx}{||u_n||^m} \\
&= \liminf_{n\to\infty} \frac{\int_\Omega F(x,u_n(x))dx}{||u_n||^m} \leq 1.
\end{aligned} \tag{4.19}$$

Consequently, we get a contradiction, which implies that $w(x) = 0$ a.e. $x \in \Omega$. Since $\mathcal{I}_\lambda(tu_n)$ is continuous on $[0,1]$ for each $n \in N$, there exists $t_n \in [0,1]$ such that

$$\mathcal{I}(t_n u_n) = \max_{t\in[0,1]} \mathcal{I}(tu_n).$$

Due to $||\mathcal{I}'_\lambda(u_n)||_{(W_0^s\mathbb{L}_A(\Omega))^*}(1 + ||u_n||) \to 0$, we deduce

$$\langle \mathcal{I}'_\lambda(u_n)(t_n u_n), t_n u_n \rangle \to 0, \quad n \to \infty. \tag{4.20}$$

Take $\{s_k\}_{k=1}^{\infty} \subset (1,\infty)$ with $s_k \to +\infty$ as $k \to \infty$. Then, for each $n,k \in \mathbb{N}$, one has $||s_k u_n|| = s_k > 1$. From (4.14) and the claim, combining conditions (g_1) and (f_1), we deduce

$$\begin{aligned} \int_\Omega F(x, s_k w_n(x))dx \quad &\leq C \int_\Omega [|s_k w_n| + H(s_k wn)]dx \\ &\leq C(||s_k w_n||_{L^1} + ||s_k w_n||_H) \to 0, \quad n \to \infty, \end{aligned} \tag{4.21}$$

and

$$\int_\Omega G(s_k w_n(x))dx \leq ||s_k w_n||_G \to 0, \quad n \to \infty. \tag{4.22}$$

Due to $\lim_{n\to\infty} ||u_n|| = \infty$, given $k \in \mathbb{N}$, there exists $n_k \geq k$. For all $n \geq n_k \geq k$, one has $||u_n|| > s_k$, i.e, $0 < \frac{s_k}{||u_n||} < 1$. From the fact that $||s_k w_n|| > 1$, Lemma 2.9 and (4.21), (4.22), for large $n \in \mathbb{N}$,

$$\begin{aligned} &\mathcal{I}_\lambda(t_n u_n) = \max_{t\in[0,1]} \mathcal{I}_\lambda(t_n u_n) \geq \mathcal{I}_\lambda(\frac{s_k}{||u_n||} u_n) = \mathcal{I}_\lambda(s_k w_n) \\ &= \int_{\Omega\times\Omega} A(s_k|D^s w_n|)d\mu + \int_\Omega A(s_k|w_n|)dx - \lambda \int_\Omega G(s_k|w_n|)dx - \int_\Omega F(x, s_k w_n)dx \\ &\geq \|s_k w_n\|^l - \lambda \int_\Omega G(s_k|w_n|)dx - \int_\Omega F(x, s_k w_n)dx \\ &\geq \frac{1}{2}\|s_k w_n\|^l = \frac{1}{2}s_k^l. \end{aligned} \tag{4.23}$$

Let $s_k = \|u_k\|^\eta > 1$, where $\eta \in (\frac{m}{l}, +\infty)$ is a constant. For all $n \geq n_k \geq k$, one has

$$\mathcal{I}_\lambda(t_n u_n) \geq \frac{1}{2}||u_k||^{\eta l}. \tag{4.24}$$

Applying (4.20), for large $n \in \mathbb{N}$

$$\begin{aligned} \mathcal{I}_\lambda(t_n u_n) &= \mathcal{I}_\lambda(t_n u_n) - \frac{1}{m}\langle \mathcal{I}'_\lambda(t_n u_n), \overline{u}_n \rangle + o(1) \\ &= \int_{\Omega\times\Omega} A(t_n|D^s u_n|)d\mu + \int_\Omega A(t_n|u_n|)dx - \lambda \int_\Omega G(t_n|u_n|)dx \\ &- \int_\Omega F(x, t_n u_n)dx - \frac{1}{m}\int_{\Omega\times\Omega} a\big(t_n|D^s u|\big)|t_n D^s u_n|^2 d\mu \\ &- \int_\Omega a\big(t_n|u_n|\big)(t_n u_n)^2 dx + \frac{\lambda}{m}\int_\Omega g(t_n u_n) t_n u_n dx + \frac{1}{m}\int_\Omega t_n u_n f(x, u_n)dx \\ &= \frac{1}{m}\int_{\Omega\times\Omega} \overline{A}(t_n|D^s u_n|)dx + \frac{1}{m}\int_\Omega \overline{F}(x, t_n u_n)dx \\ &+ \frac{\lambda}{m}\int_\Omega \big[\underbrace{g(t_n u_n)t_n u_n - mG(t_n u_n)}_{<0}\big]dx + o(1). \end{aligned}$$

Due to (f_3), (f_4), and (A_2),

$$\mathcal{I}_\lambda(t_n u_n) \leq \frac{1}{m}\int_{\Omega\times\Omega} \big[C_2\overline{A}(|D^s u_n|) + \tau(x)\big]d\mu + \frac{1}{m}\int_\Omega \big[C_2\overline{F}(x, u_n) + \rho(x)\big]dx + o(1)$$

$$
\begin{aligned}
&= \frac{C_2}{m}\Big[\int_{\Omega\times\Omega}\overline{A}(|D^s u_n|)d\mu + \int_\Omega \overline{F}(x,u_n)dx\Big] + C_3' + o(1)\\
&= C_2\mathcal{I}_\lambda(u_n) - \frac{C_2}{m}\langle \mathcal{I}_\lambda'(u_n),u_n\rangle + C_2\lambda\int_\Omega\Big[G(u_n) - \frac{1}{m}u_n g(u_n)\Big]dx + C_3' + o(1)\\
&\le C_2 c + C_2\lambda\big(1-\frac{1}{m}\big)\int_\Omega G(2u_n)dx + C_3' + o(1)\\
&\le C_4 + C_4\int_\Omega A(2u_n)dx \le C_4 + C_4||u_n||^m.
\end{aligned}
$$

Combined with (4.24), we have $\frac{1}{2}||u_k||^{\eta l} - C_4||u_n||^m \le C_4$. Letting $k\to\infty$, then $n\ge n_k\ge k\to\infty$. From $\eta l > m$, we get $\infty\le C_2$. This contradiction shows that $\{u_n\}$ is bounded, that is, $\sup_{n\in\mathbb{N}}||u_n|| := K_0 < \infty$.
Taking into account the reflexivity of $W_0^s\mathbb{L}_A(\Omega)$ and the Eberlein-Smulian theorem, we may assume that $u_n \rightharpoonup u \in W_0^s\mathbb{L}_A(\Omega)$. By using (3.1), we obtain

$$
||u_n-u||_{L^1}\to 0,\quad ||u_n-u||_G\to 0,\quad ||u_n-u||_H\to 0,\quad as\quad n\to\infty. \tag{4.25}
$$

Using (f_1) and Hölder's inequality, we have

$$
\begin{aligned}
&\Big|\lambda\int_\Omega g(u_n)(u_n-u)dx + \int_\Omega f(x,u_n)(u_n-u)dx\Big|\\
&\qquad\le \lambda\textstyle\int_\Omega |g(u_n)(u_n-u)|dx + \displaystyle\int_\Omega\big[C|u_n-u| + C|h(x,u_n)(u_n-u)|\big]dx\\
&\qquad\le 2\lambda\|g(u_n)\|_{\tilde{G}}\|u_n-u\|_G + C\|u_n-u\|_{L^1} + 2C||h(u_n)||_{\overline{H}}||u_n-u||_H.
\end{aligned}
\tag{4.26}
$$

Now, we will show that both $||g(u_n)||_{\overline{G}}$ and $||h(u_n)||_{\overline{H}}$ are bounded. Applying Lemma 2.1

$$
\begin{aligned}
\int_\Omega \overline{G}(g(u_n))dx \quad &\le \int_\Omega u_n g(u_n) \le \int_\Omega G(2u_n)dx\\
&\le C_4 + \int_\Omega A(2u_n)dx \le C_4 + C_4 + ||u_n||^m < \infty.
\end{aligned}
\tag{4.27}
$$

The definition of $||.||_{\overline{G}}$ yields that $||g(u_n)||_{\overline{G}} \le C_4 + C_4K_0^m$. On the other hand, due to $H \prec\prec A^*$, then for all $\alpha>0$ there exists K_α such that

$$
H(t)\le A^*(\alpha t) + K_\alpha,\qquad t\ge 0. \tag{4.28}
$$

By Lemma 2.4 [42], we have $m^* := \sup_{t>0}\frac{tA_*'(t)}{A_*(t)} \le \frac{Nm}{N-m} < \infty$. Since $W_0^s\mathbb{L}_A(\Omega)\hookrightarrow \mathbb{L}_{A_*}(\Omega)$,

$$
\begin{aligned}
\int_\Omega \overline{H}(h(u_n))dx \le \int_\Omega H(2u_n)dx &\le K_\alpha|\Omega| + \int_\Omega A_*(u_n)dx\\
&\le C_5 + C_5||u_n||_{A^*}^{m^*} \le C_6 + C_6K_0^{m^*} < \infty.
\end{aligned}
$$

Hence, $||h(u_n)||_{\overline{H}} \le C_6 + C_6K_0^{m^*} < \infty$. Combining (4.25) and (4.26), we have

$$
\int_\Omega g(u_n)(u_n-u)dx + \int_\Omega f(x,u_n)(u_n-u)dx \to 0,\quad as\ n\to\infty. \tag{4.29}
$$

From (4.13), it follows that $\int_{\Omega\times\Omega} a\big(|D^s u_n|\big)|D^s w_n||D^s w_n - D^s u|^2 d\mu \to 0$ as $n\to\infty$. Lemma 2.10 implies that $\lim_{n\to\infty}||u_n-u|| = 0$. Therefore, $\mathcal{I}_\lambda$ satisfies C_c-condition. $\square$

Conclusion of the proof of Theorem 4.2. Let $\lambda^* > 0$, $\eta > 0$ and $\tau > 0$ are constants defined in Lemma 4.4. For all $\lambda \in (0, \lambda^*)$, Lemma 4.5, Lemma 4.6 show that the functional $\mathcal{I}_\lambda$ satisfies all the assumptions of Lemma 4.3. Then $\mathcal{I}_\lambda$ has a critical value $c \geq \eta > 0$. Thus problem (1.4) has a nontrivial weak solution w with $\mathcal{I}_\lambda(w) = c$. We now prove that there is another weak solution $\tilde{w} \neq w$. Let $B_\tau := \{w \in W_0^s\mathbb{L}_A(\Omega) : ||w|| \leq \tau\}$, $U_\tau := \{W_0^s\mathbb{L}_A(\Omega) \in W_0^s\mathbb{L}_A(\Omega) : ||w|| < \tau\}$. Applying Lemma 4.5, we deduce that

$$-\infty < \tilde{c} := \inf_{B_\tau} \mathcal{I}_\lambda(w) < 0.$$

For every $\sigma \in (0, \inf_{S_\tau} \mathcal{I}_\lambda(w) - \inf_{U_\tau} \mathcal{I}_\lambda(w))$ by the Ekeland variational principle [25], there exists $w_\sigma \in B_\tau$ such that

$$\mathcal{I}_\lambda(w_\sigma) \leq \inf_{B_\tau} \mathcal{I}_\lambda(w) + \sigma$$

and

$$\mathcal{I}_\lambda(w_\sigma) < \mathcal{I}_\lambda(w) + \sigma||w_\sigma - w||, \quad \forall w \neq w_\sigma. \tag{4.30}$$

Therefore,

$$\mathcal{I}_\lambda(w_\sigma) \leq \inf_{B_\tau} \mathcal{I}_\lambda(w) + \sigma < \inf_{U_\tau} \mathcal{I}_\lambda(w) + \inf_{S_\tau} \mathcal{I}_\lambda(w) - \inf_{U_\tau} \mathcal{I}_\lambda(w) = \inf_{S_\tau} \mathcal{I}_\lambda(w),$$

which implies $w_\sigma \in U_\tau$. Now, $\forall v \in B_1$, take $h \in (0, \tau - ||w_\sigma||)$, then $w_\sigma + hv \in B_\tau$. By (4.30), we have

$$\mathcal{I}_\lambda(w_\sigma) - \mathcal{I}_\lambda(w_\sigma + hv) \leq \sigma h||v||.$$

Dividing the above inequality by h and letting $h \to 0^+$, one has

$$\langle \mathcal{I}'_\lambda(w_\sigma), v\rangle \geq -\sigma||v||.$$

Replacing v with $-v$ in the above inequality, we deduce $\langle \mathcal{I}'_\lambda(w_\sigma), v\rangle \leq \sigma||v||$. Therfore $\langle \mathcal{I}'_\lambda(w_\sigma), v\rangle \geq \sigma$. Summarizing, there exist $\{\tilde{w}_n\}_{n=1}^\infty \subset U_\tau$ such that $\mathcal{I}_\lambda(\tilde{w}_n) \to \tilde{c}$ and $||\mathcal{I}_\lambda(\tilde{w}_n)|| \leq \frac{1}{n} \to 0$ as $n \to \infty$. From the Eberlein-Smulian theorem, we may assume $\tilde{w}_n$ converges to $\tilde{w} \in B_\tau$. 4.29 and Lemma 2.10 implies that $\lim_{n\to\infty} ||\tilde{w}_n - \tilde{w}|| = 0$. Since $\mathcal{I}_\lambda \in C^1(W_0^s\mathbb{L}_A(\Omega), \mathbb{R})$ and $||\mathcal{I}'_\lambda(\tilde{w}_n)|| \to 0$, one has $\mathcal{I}'_\lambda(\tilde{w}) = \lim_{n\to\infty} \mathcal{I}'_\lambda(\tilde{w}_n) = \theta$ and $\mathcal{I}'_\lambda(\tilde{w}) = \tilde{c}$, so $\tilde{w} \neq \theta$ and $\tilde{w} \neq w$, which completes the proof.

References

[1] G. Abdeljabbar, Multiplicity of Nontrivial Solutions of a Class of Fractional p-Laplacian Problem, *Zeitschrift fur Analysis und ihre Anwendungen* **34** (2015), no. 3, 309–319.

[2] R.A. Adams, J.F. Fournier, *Sobolev spaces*, Second edition, Elsevier/Academic Press, Amsterdam, 2003.

[3] A. Alberico, A. Cianchi, L. Pick, L. Slavikova, Boundedness of functions in fractional Orlicz-Sobolev spaces, *Nonlinear Analysis* **230** (2023), 113231.

[4] K.B. Ali, M. Hsini, K. Kefi, N.T. Chung, On a nonlocal fractional p (.,.)-Laplacian problem with competing nonlinearities, *Complex Analysis and Operator Theory* **13** (2019), no. 3, 1377–1399.

[5] A. Ambrosetti, J.Garcia Azorero, I. Peral, Multiplicity results for some nonlinear elliptic equations, *J. Funct. Anal.* **137** (1996), 219–242.

[6] A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519–543.

[7] E. Arhrrabi, H. El-Houari, Fractional Sobolev space: Study of Kirchhoff-Schrödinger systems with singular nonlinearity, CUBO, A Mathematical Journal **26** (2024), no. 3, 407–430.

[8] E. Azroul, A. Benkirane, M. Srati, M. Shimi, Existence of solutions for a nonlocal Kirchhoff type problem in Fractional Orlicz-Sobolev spaces, arXiv preprint arXiv:1901.05216, (2019).

[9] E. Azroul, A. Benkirane, M. Srati, Nonlocal eigenvalue type problem in fractional Orlicz-Sobolev space, *Advances in Operator Theory* **5** (2020), no. 4, 1599–1617.

[10] E. Azroul, A. Benkirane, M. Srati, Existence of solutions for a non-local type problem in Fractional Orlicz Sobolev Spaces, *Adv. Oper. Theory* **5** (2020), 1350–1375.

[11] A. Bahrouni, S. Bahrouni, M. Xiang, On a class of nonvariational problems in fractional Orlicz-Sobolev spaces, *Nonlinear Analysis* **190** (2020), 111595.

[12] S. Bahrouni, A.M. Salort, Neumann and Robin type boundary conditions in Fractional Orlicz-Sobolev spaces, *ESAIM: Control, Optimisation and Calculus of Variations* **27** (2021), S15.

[13] R. Biswas, S. Tiwari, Nehari manifold approach for fractional p (.)-Laplacian system involving concave-convex nonlinearities, *Electronic J. of Differential Equations* **2020** (2020), no. 98, 1–29.

[14] G. Bonanno, G.M. Bisci, V. Radulescu, Existence of three solutions for a non-homogeneous Neumann problem through Orlicz-Sobolev spaces, *Nonlinear Analysis: Theory, Methods & Applications* **74** (2011), no. 14, 4785–4795.

[15] G. Bonanno, G.M. Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Boundary Value Problems* **2009** (2009), Art. 670675, 1–20.

[16] J.F. Bonder, A.M. Salort, Fractional order Orlicz-Sobolev Spaces, *Journal of Functional Analysis* **277** (2019), no. 2, 333–367.

[17] J.F. Bonder, A. Salort, H. Vivas, Interior and up to the boundary regularity for the fractional g-Laplacian: the convex case, *Nonlinear Analysis* **223** (2022), Art. 113060.

[18] A. Boumazourh, M. Srati, Leray-Schauder's solution for a nonlocal problem in a fractional Orlicz-Sobolev space, *Moroccan J. of Pure and Appl. Anal.* (MJPAA) **6** (2020), no. 1, 42-52.

[19] L.S. Chadli, H. El-Houari, H. Moussa, Multiplicity of solutions for nonlocal parametric elliptic systems in fractional Orlicz-Sobolev spaces, *J. of Elliptic and Parabolic Eq.* **9** (2023), 1131–1164.

[20] E. Colorado, I. Peral, Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions, *J. of Functional Analysis* **199** (2003), no. 2, 468–507.

[21] D.G. Costa, C.A. Magalhaes, Existence results for perturbations of the p-Laplacian, *Nonlinear Analysis: Theory, Methods & Applications* **24** (1995), no. 3, 409–418.

[22] E.D. da Silva, M.L.M. Carvalho, J.V. Goncalves, C. Goulart, Critical quasilinear elliptic problems using concave-convex nonlinearities, *Annali di Matematica Pura ed Applicata* **198** (2019), no. 3, 693–726.

[23] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bulletin des sciences mathématiques* **136** (2012), no. 5, 521–573.

[24] S. Dipierro, X. Ros-Oton, E. Valdinoci, Nonlocal problems with Neumann boundary conditions, *Revista Matematica Iberoamericana* **33** (2017), no. 2, 377–416.

[25] I. Ekeland, On the variational principle, *Journal of Mathematical Analysis and Applications* **47** (1974), no. 2, 324-353.

[26] H. El-Houari, L.S. Chadli, H. Moussa, Existence of a solution to a nonlocal Schrödinger system problem in fractional modular spaces, *Advances in Operator Theory* **7** (2022), no. 1, 1–30.

[27] H. El-Houari, M. Hicham, S. Kassimi, H. Sabiki, Fractional Musielak spaces: a class of nonlocal problem involving concave-convex nonlinearity, *J. of Elliptic and Parabolic Eq.* **10** (2024), 87–125.

[28] H. El-Houari, L.S. Chadli, M. Hicham, Nehari manifold and fibering map approach for fractional $p(.)$-Laplacian Schrödinger system, *SeMA Journal* **81** (2024), 729–751.

[29] H. El-Houari, L.S. Chadli, H. Moussa, Multiple solutions in fractional Orlicz-Sobolev Spaces for a class of nonlocal Kirchhoff systems, *Filomat* **38** (2024), no. 8, 2857–2875.

[30] H. El-Houari, H. Moussa, L.S. Chadli, A class of elliptic inclusion in fractional Orlicz-Sobolev spaces, *Complex Variables and Elliptic Equations* **69** (2022), no. 5, 755–772.

[31] H. El-Houari, L.S. Chadli, H. Moussa, A class of non local elliptic system in non reflexive fractional Orlicz-Sobolev spaces, *Asian-European Journal of Mathematics* 16 (2023), no. 7, 2350114 (2023).

[32] H. El-Houari, H. Moussa, L.S. Chadli, Ground State Solutions for a Nonlocal System in Fractional Orlicz-Sobolev Spaces, *International Journal of Differential Equations* **2022** (2022), 1–16.

[33] H. El-Houari, H. Sabiki, H. Moussa, On topological degree for pseudomonotone operators in fractional Orlicz-Sobolev spaces: study of positive solutions of non-local elliptic problems, *Advances in Operator Theory* **9** (2024), no. 2, Art. 16.

[34] H. El-Houari, L.S. Chadli, H. Moussa, A weak solution to a non-local problem in fractional Orlicz-Sobolev spaces, *Asia Pac. J. Math.* **10** (2023), no. 2.
[35] H. El-Houari, M. Hicham, H. Sabiki, Multiplicity and concentration properties of solutions for double-phase problem in fractional modular spaces, *J. Elliptic Parabol. Equ.* **10** (2024), 755–801.
[36] H. El-Houari, H. Moussa, H. Sabiki, Fractional Musielak spaces: a class of non-local elliptic system involving generalized nonlinearity, *Rendiconti del Circolo Matematico di Palermo Series* **2** (2024), 1–26.
[37] H. El-Houari, H. Moussa, H. Sabiki, Fractional Musielak spaces: study of nonlocal elliptic problem with Choquard-logarithmic nonlinearity, *Complex Variables and Elliptic Equations* (2024), 1-24.
[38] H. El-Houari, H. Moussa, On a class of generalized Choquard system in fractional Orlicz-Sobolev Spaces, *Journal of Mathematical Analysis and Applications* **540** (2024), no. 1, 128563.
[39] H. El-Houari, S. Hajar, H. Moussa, Multivalued Elliptic Inclusion in Fractional Orlicz-Sobolev Spaces, *Complex Analysis and Operator Theory* **18** (2024), no. 4, 94.
[40] N. Fukagai, M. Ito, K. Narukawa, Quasilinear elliptic equations with slowly growing principal part and critical Orlicz-Sobolev nonlinear term, *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* **139** (2009), no. 1, 73–106.
[41] N. Fukagai, K. Narukawa, On the existence of multiple positive solutions of quasilinear elliptic eigenvalue problems, *Annali di Matematica Pura ed Applicata* **186** (2007), 539–564.
[42] N. Fukagai, M. Ito, K. Narukawa, Positive solutions of quasilinear elliptic equations with critical Orlicz-Sobolev nonlinearity on $\mathbb{R}^N$, *Funkcial. Ekvac.* **49** (2006), 235–267.
[43] A. Ghanmi, K. Saoudi, The Nehari manifold for a singular elliptic equation involving the fractional Laplace operator, *Fractional Differential Calculus* **6** (2016), no. 2, 201–217.
[44] A. Ghanmi, Nontrivial solutions for Kirchhoff-type problems involving the p(x)-Laplace operator, *Rocky Mountain J. Math.* **48** (2018), no. 4, 1145-1158.
[45] M. Garcia-Huidobro, V.K. Le, R. Manasevich, K. Schmitt, On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting, *Nonlinear Differential Equations and Applications NoDEA* **6** (1999), no. 2, 207–225.
[46] E.H. Hamza, L.S. Chadli, H. Moussa, Existence of ground state solutions of elliptic system in Fractional Orlicz-Sobolev Spaces, *Results in Nonlinear Analysis* **5** (2022), no. 2, 112–130.
[47] H. El-houari, L.S. Chadli, H. Moussa, On a class of fractional $\Gamma(\cdot)$-Kirchhoff-Schrödinger system type, *CUBO, A Mathematical Journal* **26** (2024), no. 1, 53–73.
[48] S. Heidarkhani, G. Caristi, M. Ferrara, Perturbed Kirchhoff-type Neumann problems in Orlicz-Sobolev spaces, *Computers & Mathematics with Applications* **71** (2016), no. 10, 2008–2019.
[49] M.A. Krasnosel'skii, Y.B. Rutickii, Convex functions and Orlicz spaces, (Vol. 9), Groningen, Noordhoff, 1961.
[50] A. Kristaly, M. Mihăilescu, V. Rădulescu, Two non-trivial solutions for a non-homogeneous Neumann problem: an Orlicz-Sobolev space setting, *Proceedings of the Royal Society of Edinburgh Section A: Mathematics* **139** (2009), no. 2, 367–379.
[51] Q.J. Lou, Multiplicity of Positive Solutions for Kirchhoff Systems, *Bulletin of the Malaysian Mathematical Sciences Society* **43** (2020), no. 5, 3529–3556.
[52] B. Ricceri, A general variational principle and some of its applications, *Journal of Computational and Applied Mathematics* **113** (2000), no. 1-2, 401–410.
[53] A.M. Salort, Eigenvalues and minimizers for a non-standard growth non-local operator, *Journal of Differential Equations* **268** (2020), no. 9, 5413–5439.

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