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Statistical Convergence of Complex Uncertain Sequence of Order (α, β)

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Abstract. In this paper, the concept of asymptomatic density of order (α, β) (where (α, β) are real numbers such that $0 < \alpha \leq \beta \leq 1$) has been used to introduce the concepts of statistical convergence of order (α, β) for complex uncertain sequences: the notions of statistical convergence in mean of order (α, β), statistical convergence in measure of order (α, β), statistical convergence in distribution of order (α, β), almost surely statistical convergence of order (α, β), uniformly almost surely statistical convergence of order (α, β) for complex uncertain sequences. Also, relationships among those introduced concepts have been studied.

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1. Introduction

The idea of convergence of sequence is one of the widely discussed topics in mathematics. It took a long time to have the modern definition of convergence used in calculas. However, mathematicians were not satisfied with this usual concept of convergence. A more general idea of convergence called statistical convergence was proposed by Fast [5] and Schoenberg [13]. The notion of asymptomatic density or natural density of any subset of natural numbers has been introduced to define the statistical convergence of a sequence of real or complex numbers. For a more detailed study related to statistical convergence, one may refer to Connor [4], Friday [6], and Salat [14]. While applying statistical convergence in approximation theory, Gadjiev and Orhan [7] introduced the idea of the order of statistical convergence of a sequence. The idea of the order of statistical convergence has been generalized by Altinok and Et [2] using two parameters α, β, where α and β are two real numbers such that $0 < \alpha \leq \beta \leq 1$. In [2], it has been shown that the statistical convergence of order (α, β) is well defined for $\alpha \leq \beta$, but not well defined for $\alpha > \beta$. The statistical convergence of order (α, β) reduces to usual statistical convergence when $\alpha = \beta = 1$.

In real life, we often face indeterminacy, i.e., our lack of ability to determine the outcome of a particular phenomenon. When a large amount of sample is available, then we can fit a probability distribution to understand the nature of that phenomenon. However, due to technical or economic difficulties, an appropriate amount of data is not always available, and in such a case, we can not fit the probability distribution of such phenomenon. To overcome this difficulty, Liu [8], introduced the idea

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of uncertainty theory with the notion of belief degree function. However, belief degree should not be misinterpreted as subjective probability [8]. Liu [8] defined uncertain space on which uncertain measure has been defined. He [8] has defined uncertain variables also. An uncertainty distribution of an uncertain variable has been defined in [8] to study the behavior of an uncertain variable. Liu [8] also introduced convergence concepts of uncertain sequence (sequence of uncertain variables): convergence in mean, convergence in measure, convergence in distribution, and almost surely convergence. Later Peng [12] introduced the idea of a complex uncertain variable. You [20] proposed a new idea of convergence called uniformly almost surely convergence and discussed its relationships with the convergence concepts introduced by Liu [8]. Many exciting results about these convergence concepts can be found in [3].

Most recently, Tripathy and Nath [16] have introduced the idea of statistical convergence of complex uncertain sequences, and they have studied relationships among those convergence concepts. This work has generalized the idea of convergence of complex uncertain sequences.

However, the idea of the density of a subset of natural numbers of order (α, β) has not yet been used in convergence concepts of uncertain sequences. In this paper, the idea of statistical convergence of complex uncertain sequences of order (α, β) (where α and β are two real numbers such that $0 < \alpha \leq \beta \leq 1$) has been introduced, and also relationships among those introduced concepts have been studied.

2. Definitions and Preliminaries

In this section few basic definitions, theorems, and fundamental concepts used throughout this paper will be procured. At first, let us have the definitions of natural or asymptomatic density of a subset of natural numbers, statistical convergence of a real or complex sequence and asymptomatic density of order (α, β) of a subset of natural numbers.

Definition 2.1. [14] The *asymptomatic (or natural) density* of a set $A \subseteq \mathbb{N}$ is defined as $\delta(A) = \lim_{n\to\infty} \dfrac{|\{k \leq n \mid k \in A\}|}{n}$, whenever the limit exists. The vertical bars denote the cardinality of the under lying set.

Definition 2.2. [14] A sequence (x_n) of real numbers is called *statistically convergent* to a number ξ provided that for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : | x_k - \xi | \geq \varepsilon\}) = 0$.

Definition 2.3. [2] For $0 < \alpha \leq \beta \leq 1$, the *asymptomatic density* of order (α, β) or the (α, β) density of a set $A \subseteq \mathbb{N}$ is defined as $\delta^{(\alpha,\beta)}(A) = \lim_{n\to\infty} \dfrac{|\{k \leq n \mid k \in A\}|^{\beta}}{n^{\alpha}}$, whenever the limit exists.
Now, let us have some axioms, basic definitions of uncertainty theory used in this paper.

Definition 2.4. [8] Let L be a $\sigma - algebra$ on a non-empty set Γ. A real valued set function M is called an *uncertain measure* if it satisfies the following axioms:
Axiom 1. $M(\Gamma) = 1$ **(Normality Axiom)**
Axiom 2. $M(A) + M(A^c) = 1$ **for each** $A \in L$ **(Duality Axiom)**
Axiom 3. $M(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} M\{A_n\}$, **for every countable sequence** $\{A_n\}$ **in** L,

(Subadditivity Axiom)
The triplet (Γ, L, M) is called an uncertainty space, and each element A in L is called an event.

Definition 2.5. [8] An *uncertain variable* ξ is a measurable function from an uncertainty space (Γ, L, M) to the set of real numbers, i. e. for any Borel set B of real numbers, the set $\{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$ is an event.

Definition 2.6. [8] The *uncertainty distribution* Φ of an uncertain variable ξ is defined by

$$\Phi(x) = M\{\xi \leq x\} = M(\gamma \in \Gamma \mid \xi(\gamma) \leq x),$$

for all $x \in \mathbb{R}$.

Definition 2.7. [8] Let ξ be an uncertain variable. The expected value of ξ is defined as

$$E(\xi) = \int_0^{\infty} M\{\xi \geq x\}dx + \int_{-\infty}^{0} M\{\xi \leq x\}dx,$$

provided that the integral exists.

Now, we shall procure concepts and theorems on complex uncertain variables, first proposed by Peng [12]. The complex uncertain variable is mainly used to model a complex uncertain quantity.

Definition 2.8. [12] A *complex uncertain variable* ζ is a measurable function from an uncertainty space (Γ, L, M) to the set of complex numbers, i.e. for any Borel set B of complex numbers, the set $\{\gamma \in \Gamma \mid \zeta(\gamma) \in B\}$ is an event.

Theorem 2.1. [12] *A variable ζ from an uncertainty space (Γ, L, M) to the set of complex numbers is a complex uncertain variable if and only if $Re(\zeta)$ and $Im(\zeta)$ are uncertain variables where $Re(\zeta)$ and $Im(\zeta)$ represent the real and the imaginary parts of ζ respectively.*

Definition 2.9. [12] The *complex uncertainty distribution* Φ of a complex uncertain variable ζ is a function from $\mathbb{C}$ to $[0, 1]$ defined by

$$\Phi(c) = M\{Re(\zeta) \leq Re(c), Im(\zeta) \leq Im(c)\},$$

for any $c \in \mathbb{C}$.

Theorem 2.2. [12] *A function $\Phi : \mathbb{C} \to [0, 1]$ is an uncertainty distribution if and only if it is increasing with respect to the real part $Re(c)$ and imaginary part $Im(c)$ such that*
(i) $\lim\limits_{x \to -\infty} \Phi(x + iy) \neq 1$, $\lim\limits_{y \to -\infty} \Phi(a + iy) \neq 1$, *for any* $a, b \in \mathbb{R}$
(ii) $\lim\limits_{x \to +\infty, y \to \infty} \Phi(x + iy) \neq 0$,
where $i = \sqrt{-1}$ *is the imaginary unit.*
Now, we shall mention the definitions of several types of convergence concepts of complex uncertain variables, first proposed by Chen, Ning and Wang [3].

Definition 2.10. [3] The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent almost surely* to ζ if there exists an event Λ with $M\{\Lambda\} = 1$ such that
$\lim\limits_{n \to \infty} \mid \zeta_n(\gamma) - \zeta(\gamma) \mid = 0$ for every $\gamma \in \Lambda$. In that case we write $\zeta_n \to \zeta$ a.s.

Definition 2.11. [3] The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent in measure* to ζ if $\lim_{n\to\infty} M\{\| \zeta_n - \zeta \| \geq \varepsilon\} = 0$ for every $\varepsilon > 0$.

Definition 2.12. [3] The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent in mean* to ζ if $\lim_{n\to\infty} E[\| \zeta_n - \zeta \|] = 0$.

Definition 2.13. [3] Let $\Phi_1, \Phi_2, \cdots$ be the complex uncertainty distributions of complex uncertain variables $\zeta_1, \zeta_2, \cdots$. respectively. The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent in distribution* to ζ if $\lim_{n\to\infty} \Phi_n(c) = \Phi(c)$ for all $c \in \mathbb{C}$ at which Φ is continuous.

Definition 2.14. [3] The complex uncertain sequence $\{\zeta_n\}$ is said to be *convergent uniformly almost surely (u.a.s.)* to ζ if there exists a sequence of events $\{E'_k\}$, $M\{E'_k\} \to 0$ such that $\{\zeta_n\}$ converges uniformly to ζ in $\Gamma - E'_k$, for any fixed $k \in \mathbb{N}$.

Finally, we are mentioning the definition of statistical convergence of complex uncertain sequence, first proposed by Tripathy and Nath [16].

Definition 2.15. [16] The complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent almost surely (s.a.s.)* to ζ if for every $\varepsilon > 0$ there exists an event Λ with $M\{\Lambda\} = 1$ such that $\lim_{n\to\infty} \frac{| \{k \leq n : | \zeta_k(\gamma) - \zeta(\gamma) | \geq \varepsilon\} |}{n} = 0$, for every $\gamma \in \Lambda$. In that case we write $\zeta_n \to \zeta$ *a.s.*

Definition 2.16. [16] The complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent in measure* to ζ if $\lim_{n\to\infty} \frac{| \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} |}{n} = 0$, for every ε, $\delta > 0$.

Definition 2.17. [16] The complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent in mean* to ζ if $\lim_{n\to\infty} \frac{| \{k \leq n : E(| \zeta_k(\gamma) - \zeta(\gamma) | \geq \varepsilon)\} |}{n} = 0$ for every $\varepsilon > 0$.

Definition 2.18. [16] Let $\Phi_1, \Phi_2, \cdots$ be the complex uncertainty distributions of complex uncertain variables $\zeta_1, \zeta_2, \cdots$. respectively. The complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent in distribution* to ζ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{| \{k \leq n : | \Phi_k(c) - \Phi(c) | \geq \varepsilon\} |}{n} = 0,$$

for all $c \in \mathbb{C}$ at which Φ is continuous.

Definition 2.19. [16] The complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent uniformly almost surely (u.a.s.)* to ζ if for every $\varepsilon > 0$ there exists $\delta > 0$ and a sequence of events $\{E'_k\}, M\{E'_k\} \to 0$ such that,

$$\lim_{n\to\infty} \frac{| \{k \leq n : | M(E'_k) - 0 | \geq \varepsilon\} |}{n} = 0$$

and

$$\lim_{n\to\infty} \frac{| \{k \leq n : | \zeta_k(\gamma) - \zeta(\gamma) | \geq \delta\} |}{n} = 0,$$

for every $\gamma \in \Gamma - E_k$ such that $M(E_k) \geq \varepsilon$.

3. Main Results

In this section, we have introduced five convergence concepts of order (α,β) (where α,β are real numbers such that $0<\alpha\leq\beta\leq 1$) for a complex uncertain sequence. We have also studied relationships among those introduced concepts.

3.1. Definitions.

Definition 3.1. A complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent in mean of order* (α,β) to ζ if $\lim_{n\to\infty}\frac{|\{k\leq n: E[\|\zeta_k-\zeta\|]\geq\varepsilon\}|^\beta}{n^\alpha}=0$, for every $\varepsilon>0$.

Definition 3.2. A complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent in measure of order* (α,β) to ζ if
$\lim_{n\to\infty}\frac{|\{k\leq n: M(\|\zeta_k-\zeta\|\geq\varepsilon)\geq\delta\}|^\beta}{n^\alpha}=0$, for every $\varepsilon,\delta\geq 0$.

Definition 3.3. Let $\Phi_1,\Phi_2,\cdots$ be the complex uncertainty distributions of complex uncertain variables $\zeta_1,\zeta_2,\cdots$. respectively. The complex uncertain sequence $\{\zeta_n\}$ is said to be statistically convergent in distribution of order (α,β) to ζ if for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{|\{k\leq n:|\Phi_k(c)-\Phi(c)|\geq\varepsilon\}|^\beta}{n^\alpha}=0,$$

for all $c\in\mathbb{C}$ at which Φ is continuous.

Definition 3.4. The complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent almost surely of order* (α,β) to ζ if there exist some α,β satisfying $0<\alpha\leq\beta\leq 1$, such that for every $\varepsilon>0$ there exists an event Λ with $M\{\Lambda\}=1$ such that for every $\gamma\ \varepsilon\ \Lambda$,

$$\lim_{n\to\infty}\frac{|\{k\leq n:|\zeta_k(\gamma)-\zeta(\gamma)|\geq\varepsilon\}|^\beta}{n^\alpha}=0.$$

Definition 3.5. A complex uncertain sequence $\{\zeta_n\}$ is said to be *statistically convergent uniformly almost surely of order* (α,β) to ζ if there exists some α,β satisfying $0<\alpha\leq\beta\leq 1$, such that for every $\varepsilon>0$ there exists $\delta>0$ and a sequence $\{E_k\}$ of events such that,

$$\lim_{n\to\infty}\frac{|\{k\leq n: M(E_k)\geq\varepsilon\}|^\beta}{n^\alpha}=0$$

and

$$\lim_{n\to\infty}\frac{|\{k\leq n:|\zeta_k(\gamma)-\zeta(\gamma)|\geq\varepsilon\}|^\beta}{n^\alpha}=0,$$

for every $\gamma\in\Gamma-E_k$ such that $M(E_k)\geq\varepsilon$.

3.2. Relationships among statistical convergence concepts of order (α,β).

Theorem 3.1. *For a complex uncertain sequence $\{\zeta_n\}$, statistical convergence in mean of order (α,β) to ζ implies statistical convergence of $\{\zeta_n\}$ in measure of order (α,β) to ζ.*

Proof. Let $\{\zeta_n\}$ be a complex uncertain sequence, statistically convergent in mean of order (α, β) to ζ. For every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\mid \{k \leq n : E[\| \zeta_k - \zeta \|] \geq \varepsilon\} \mid^\beta}{n^\alpha} = 0.$$

From Markov's inequality, we have

$$M(\| \zeta_k - \zeta \|) \geq \varepsilon) \leq \frac{E(\| \zeta_k - \zeta \|)}{\varepsilon},$$

for every $\varepsilon > 0$ and for all $k \in \mathbb{N}$. Hence, for every $\varepsilon > 0, \delta > 0$, and $n \in \mathbb{N}$,

$$\{k \leq n : M(\| \zeta_k - \zeta \|) \geq \varepsilon) \geq \delta\} \subseteq \{k \leq n : E[\| \zeta_k - \zeta \|] \geq \delta_\varepsilon\}$$

$$\Rightarrow \mid \{k \leq n : M(\| \zeta_k - \zeta \|) \geq \varepsilon) \geq \delta\} \mid \ \leq \ \mid \{k \leq n : E[\| \zeta_k - \zeta \|] \geq \delta_\varepsilon\} \mid$$

$$\Rightarrow \frac{\mid \{k \leq n : M(\| \zeta_k - \zeta \|) \geq \varepsilon) \geq \delta\} \mid^\beta}{n^\alpha} \leq \frac{\mid \{k \leq n : E[\| \zeta_k - \zeta \|] \geq \delta_\varepsilon\} \mid^\beta}{n^\alpha}.$$

Statistical convergence in mean of order (α, β) implies that the sequence on RHS converges to zero. Thus, we have

$$\lim_{n\to\infty} \frac{\mid \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid^\beta}{n^\alpha} = 0$$

Hence, statistically convergence in mean of order (α, β) implies statistical convergence of $\{\zeta_n\}$ in measure of order (α, β). □

Remark 3.1. However, statistical convergence in measure of order (α, β) does not imply statistical convergence in mean of order (α, β). Let us consider the following example.

Example 3.1. Let us consider the uncertainty space (Γ, L, M) where $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, ...\}$ and L is the power of set of Γ. Let us consider the uncertain measure M defined as

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \dfrac{1}{(n+1)} & \text{if } \sup\limits_{\gamma_n \in \Lambda} \dfrac{1}{(n+1)} < 0.5 \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{1}{(n+1)} & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{1}{(n+1)} < 0.5 \\ 0.5, & \text{otherwise,} \end{cases}$$

where $\Lambda \in L$. Let us consider a sequence $\{\zeta_n\}$ of complex uncertain variable defined as

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, \gamma = \gamma_n \\ 0, \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. Let ζ be another complex uncertain variable such that $\zeta \equiv 0$.
Hence,

$$\| \zeta_n(\gamma) - \zeta(\gamma) \| = \begin{cases} (n+1), \gamma = \gamma_n \\ 0, \text{otherwise.} \end{cases}$$

For $n \geq 2$ and $\varepsilon > 0$

$$M(\| \zeta_n - \zeta \| \geq \varepsilon) = M(\{\gamma \in \Gamma : \mid \zeta_n(\gamma) - \zeta(\gamma) \mid \geq \varepsilon\}) = \begin{cases} M\{\gamma_n\} = \frac{1}{(n+1)}, & (n+1) \geq \varepsilon, \\ 0, & \varepsilon > (n+1). \end{cases}$$

Hence, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} M\{\| \zeta_n - \zeta \| \geq \varepsilon\} = 0.$$

For every $\varepsilon > 0$ and $\delta > 0, | \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} |$ is a finite number and for $0 < \beta \leq 1, | \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} |^\beta$ is also a finite number. Hence, for every $\varepsilon > 0, \delta > 0$ and for any α, β satisfying $0 < \alpha \leq \beta \leq 1$,

$$\lim_{n\to\infty} \frac{| \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} |^\beta}{n^\alpha} = 0.$$

So, the complex uncertain sequence $\{\zeta_n\}$ statistically converges in measure of order (α, β) to ζ. For $n \geq 2$ the uncertainty distribution Φ_n of $\| \zeta_n - \zeta \|$ is

$$\Phi_n(x) = M(\| \zeta_n - \zeta \| \leq x) = \begin{cases} 0 & , x < 0, \\ 1 - \frac{1}{(n+1)} & , 0 \leq x < (n+1), \\ 1 & , x \geq (n+1), \end{cases}$$

where $x \in \mathbb{R}$. Now,

$$E(\| \zeta_n - \zeta \|) = \int_0^\infty M(\| \zeta_n - \zeta \| \geq x)dx,$$

if the integral exists. Observe that, for $P > (n+1)$

$$\int_0^P M(\| \zeta_n - \zeta \| \geq x)dx = \int_0^{n+1} \frac{1}{(n+1)}dx + \int_{n+1}^P 0.dx = 1$$

$$\Rightarrow \lim_{P\to\infty} [\int_0^P M(\| \zeta_n - \zeta \| \geq x)dx] = 1$$

$$E(\| \zeta_n - \zeta \|) = \int_0^\infty M(\| \zeta_n - \zeta \| \geq x)dx = 1, (n \geq 2).$$

Let us choose $\varepsilon = 0.5, \alpha = \beta = 1$. Then,

$$\lim_{n\to\infty} \frac{| \{k \leq n : E(\| \zeta_k - \zeta \|) \geq \frac{1}{2}\} |}{n} = 1 \neq 0.$$

Thus, the complex uncertain sequence $\{\zeta_n\}$ is not statistically convergent in mean of order (α, β) to ζ.

Theorem 3.2. *Assume complex uncertain sequence $\{\zeta_n\}$ with real part $\{\xi_n\}$ and imaginary part $\{\eta_n\}$ respectively, for $n = 1, 2, \cdots$ Suppose that, the complex uncertain sequence $\{\zeta_n\}$ statistically converges in measure of order (α, β) to $\zeta = \xi + i\eta$. Then uncertain sequences $\{\xi_n\}$ and $\{\eta_n\}$ statistically convergence in measure of order (α, β) to ξ and η respectively. Converse of the statement is true if $\beta = 1$ and $0 < \alpha \leq 1$.*

Proof. Let us assume that, for some α, β satisfying $0 < \alpha \leq \beta \leq 1$ and for any $\delta > 0, \varepsilon > 0$,

$$\lim_{n\to\infty} \frac{| \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} |^\beta}{n^\alpha} = 0. \tag{1}$$

For any $\varepsilon > 0$

$$\{\gamma \in \Gamma :| \zeta_n(\gamma) - \zeta(\gamma) | < \varepsilon\} \subseteq \{\gamma \in \Gamma :| \xi_n(\gamma) - \xi(\gamma) | < \varepsilon\}$$

$$\Rightarrow M\{\gamma \in \Gamma :| \xi_n(\gamma) - \xi(\gamma) | \geq \varepsilon\} \leq M\{\gamma \in \Gamma :| \zeta_n(\gamma) - \zeta(\gamma) | \geq \varepsilon\}$$

$$\Rightarrow M\{\| \xi_n - \xi \| \geq \varepsilon\} \leq M\{\| \zeta_n - \zeta \| \geq \varepsilon\}.$$

Hence for any $\delta > 0$,

$$\{k \le n : M(\| \xi_k - \xi \| \ge \varepsilon) \ge \delta\} \subseteq \{k \le n : M(\| \zeta_k - \zeta \| \ge \varepsilon) \ge \delta\}.$$

For every $n \in \mathbb{N}$,

$$| \{k \le n : M(\| \xi_k - \xi \| \ge \varepsilon) \ge \delta\} | \le | \{k \le n : M(\| \zeta_k - \zeta \| \ge \varepsilon) \ge \delta\} | .$$

For any α, β satisfying $0 < \alpha \le \beta \le 1$,

$$\Rightarrow \frac{| \{k \le n : M(\| \xi_k - \xi \| \ge \varepsilon) \ge \delta\} |^\beta}{n^\alpha} \le \frac{| \{k \le n : M(\{\| \zeta_k - \zeta \| \ge \varepsilon)\} \ge \delta\} |^\beta}{n^\alpha} \tag{2}$$

Combining (3.1), (3.2), and letting $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{| \{k \le n : M(\| \xi_k - \xi \| \ge \varepsilon) \ge \delta\} |^\beta}{n^\alpha} = 0.$$

Similarly, it can be proved that

$$\lim_{n\to\infty} \frac{| \{k \le n : M(\| \eta_k - \eta \| \ge \varepsilon) \ge \delta\} |^\beta}{n^\alpha} = 0.$$

To prove the converse part, let us assume that, for some α satisfying $0 < \alpha \le 1$ and for any $\varepsilon > 0, \delta > 0$,

$$\lim_{n\to\infty} \frac{| \{k \le n : M(\| \xi_k - \xi \| \ge \varepsilon) \ge \delta\} |}{n^\alpha} = 0$$

and

$$\lim_{n\to\infty} \frac{| \{k \le n : M(\| \eta_k - \eta \| \ge \varepsilon) \ge \delta\} |}{n^\alpha} = 0. \tag{3}$$

Since, $\| \zeta_k - \zeta \| = \sqrt{\| \xi_k - \xi \|^2 + \| \eta_k - \eta \|^2}$, for every $\varepsilon > 0$ and $n \in \mathbb{N}$ we have,

$$\{\| \zeta_n - \zeta \| \ge \varepsilon\} \subseteq \{\| \xi_n - \xi \| \ge \frac{\varepsilon}{\sqrt{2}}\} \cup \{\| \eta_n - \eta \| \ge \frac{\varepsilon}{\sqrt{2}}\}.$$

Applying subadditivity theorem we have,

$$M\{\| \zeta_n - \zeta \| \ge \varepsilon\} \le M\{\| \xi_n - \xi \| \ge \frac{\varepsilon}{\sqrt{2}}\} + M\{\| \eta_n - \eta \| \ge \frac{\varepsilon}{\sqrt{2}}\}.$$

For any $\delta > 0$, if $M\{\| \xi_n - \xi \| \ge \frac{\varepsilon}{\sqrt{2}}\} < \frac{\delta}{2}$ and $M\{\| \eta_n - \eta \| \ge \frac{\varepsilon}{\sqrt{2}}\} < \frac{\delta}{2}$, then $M\{\| \zeta_n - \zeta \| \ge \varepsilon\} < \delta$. Hence, for any $n \in \mathbb{N}$,

$$M\{\| \zeta_n - \zeta \| \ge \varepsilon\} < \delta$$

$$\Rightarrow M\{\| \xi_n - \xi \| \ge \frac{\varepsilon}{\sqrt{2}}\} \ge \frac{\delta}{2} \textit{ or } M\{\| \eta_n - \eta \| \ge \frac{\varepsilon}{\sqrt{2}}\} \ge \frac{\delta}{2}.$$

For any $n \in \mathbb{N}$,

$$\{k \le n : M\{(\| \zeta_k - \zeta \| \ge \varepsilon) \ge \delta\}\} \subseteq A(n) \cup B(n),$$

where $A(n) = \{k \le n : M(\| \xi_k - \xi \| \ge \frac{\varepsilon}{\sqrt{2}}) \ge \frac{\delta}{2}\}$ and $B(n) = \{k \le n : M(\| \eta_k - \eta \| \ge \frac{\varepsilon}{\sqrt{2}}) \ge \frac{\delta}{2}\}$. Hence,

$$| \{k \le n : M\{(\| \zeta_k - \zeta \| \ge \varepsilon) \ge \delta\}\} | \le | A(n) | + | B(n) | \tag{4}$$

since, $| A(n) \cup B(n) | \le | A(n) | + | B(n) |$. From (3.47) we have,

$$\frac{| \{k \le n : M\{(\| \zeta_k - \zeta \| \ge \varepsilon) \ge \delta\}\} |}{n^\alpha} \le \frac{| A(n) |}{n^\alpha} + \frac{| B(n) |}{n^\alpha}, \tag{5}$$

where α is a real number satisfying $0 < \alpha \leq 1$.
Combining (3.4) and (3.5) and letting $n \to \infty$ we have,

$$\lim_{n\to\infty} \frac{|\{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\}|^\beta}{n^\alpha} = 0.$$

□

Theorem 3.3. *Assume a complex uncertain sequence $\{\zeta_n\}$ with real part $\{\xi_n\}$ and imaginary part $\{\eta_n\}$ respectively, for $n = 1, 2, \cdots$ If the complex uncertain sequence $\{\zeta_n\}$ statistically converges in measure of order (α, β) to $\zeta = \xi + i\eta$, then the complex uncertain sequence $\{\zeta_n\}$ statistically converges in distribution of order (α, β) to $\zeta = \xi + i\eta$.*

Proof. Let, $c = a + ib$ be a point of continuity of Φ, the complex uncertainty distribution of ζ. Let Φ_n be the complex uncertainty distribution of ζ_n. For any $\alpha > a, \beta > b$, we have
$\{\xi_n \leq a, \eta_n \leq b\} = \{\xi_n \leq a, \eta_n \leq b, \xi \leq \alpha\} \cup \{\xi_n \leq a, \eta_n \leq b, \xi > \alpha\}$
$= \{\xi_n \leq a, \eta_n \leq b, \xi \leq \alpha, \eta \leq \beta\} \cup \{\xi_n \leq a, \eta_n \leq b, \xi \leq \alpha, \eta > \beta\} \cup \{\xi_n \leq a, \eta_n \leq b, \xi > \alpha, \eta \leq \beta\} \cup \{\xi_n \leq a, \eta_n \leq b, \xi > \alpha, \eta > \beta\}$
$\Rightarrow \{\xi_n \leq a, \eta_n \leq b\} \subseteq \{\xi \leq \alpha, \eta \leq \beta\} \cup \{\| \xi_n - \xi \| \geq \alpha - a\} \cup \{\| \eta_n - \eta \| \geq \beta - b\}$.
Taking Uncertain measure M on both side and using subadditivity axiom,

$$\Phi_n(a + ib) \leq \Phi(\alpha + i\beta) + M(\{\| \xi_n - \xi \| \geq \alpha - a\}) + M(\{\| \eta_n - \eta \| \geq \beta - b\}). \quad (6)$$

For every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{|\{k \leq n : M(\| \xi_k - \xi \| \geq \alpha - a) \geq \varepsilon\}|}{n} = 0$$

and

$$\lim_{n\to\infty} \frac{|\{k \leq n : M(\| \eta_k - \eta \| \geq \beta - b) \geq \varepsilon\}|}{n} = 0.$$

On R.H.S of (3.6) there are two sequences, statistically converging to zero. The sum of two sequences is also a sequence, $\{u_n\}$, statistically converging to zero. By Decomposition theorem, $u_n = x_n + y_n,\ n \in \mathbb{N},$ where $\{x_n\}$ and $\{y_n\}$ are two real sequences statistically converging to zero and statistically null respectively. Hence for all $n \in \mathbb{N}$ (3.6) implies that

$$\Phi_n(a + ib) \leq \Phi(\alpha + i\beta) + x_n + y_n.$$

Since, $x_n \to 0$, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $0 \leq x_n < \varepsilon$ for all $n \geq k$. Hence, for all $n \geq k$,

$$\Phi_n(c) \leq \Phi(\alpha + i\beta) + \varepsilon + y_n \Rightarrow \Phi_n(c) - y_n \leq \Phi(\alpha + i\beta) + \varepsilon$$
$$\Rightarrow \sup\{\Phi_n(c) - y_n : n \geq k\} \leq \sup\{\Phi(\alpha + i\beta) + \varepsilon : n \geq k\}.$$

For two subsets A, B of $\mathbb{R}$ existence of $\sup A$ and existence of $\sup B$, imply that $\sup(A + B) = \sup(A) + \sup(B)$. So, we have,

$$\sup\{\Phi_n(c) : n \geq k\} + \sup\{-y_n : n \geq k\} \leq \Phi(\alpha + i\beta) + \varepsilon.$$

Since, $\{y_n\}$ is statistically null and $0 \leq y_n \leq 1, \sup\{-y_n : n \geq k\} = 0$. Thus,

$$\sup\{\Phi_n(c) : n \geq k\} \leq \Phi(\alpha + i\beta) + \varepsilon \Rightarrow M_n \leq \Phi(\alpha + i\beta) + \varepsilon, (n \geq k),$$

where $M_n = \sup\{\Phi_i(c) : i \geq n\}, n \in \mathbb{N}$. Letting $n \to \infty$, we get,

$$\lim_{n\to\infty} M_n \leq \Phi(\alpha + i\beta) + \varepsilon.$$

For every $\varepsilon > 0$. Hence,
$$\lim_{n\to\infty} \sup \Phi_n(c) \le \Phi(\alpha + i\beta).$$
Letting $\alpha + i\beta \to a + ib$, from continuity of Φ we get,
$$\lim_{n\to\infty} \sup \Phi_n(c) \le \Phi(c). \tag{7}$$
For any $x < a$ and $y < b$,
$\{\xi \le x, \eta \le y\} = \{\xi_n \le a, \xi \le x, \eta \le y\} \cup \{\xi_n > a, \xi \le x, \eta \le y\}$
$= \{\xi_n \le a, \eta_n \le b, \xi \le x, \eta \le y\} \cup \{\xi_n \le a, \eta_n > b, \xi \le x, \eta \le y\} \cup \{\xi_n > a, \eta_n \le b, \xi \le x, \eta \le y\} \cup \{\xi_n > a, \eta_n > b, \xi \le x, \eta \le y\}$
$\Rightarrow \{\xi \le x, \eta \le y\} \subseteq \{\xi_n \le a, \eta_n \le b\} \cup \{\| \xi_n - \xi \| \ge a - x\} \cup \{\| \eta_n - \eta \| \ge b - y\}$.
Taking Uncertain measure M on both side and using subadditivity axiom,
$$\Phi(x + iy) \le \Phi_n(a + ib) + M(\{\| \xi_n - \xi \| \ge a - x\}) + M(\{\| \eta_n - \eta \| \ge b - y\})$$
$$\Rightarrow \Phi(x + iy) - p_n \le \Phi_n(a + ib) + q_n,$$
where $\{p_n\}$ and $\{q_n\}$ are two real sequences statistically converging to zero and statistically null respectively. Since, $p_n \to 0$, for every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $0 \le p_n < \varepsilon$ or $-\varepsilon < -p_n \le 0$, for all $n \ge k$. Hence, for all $n \ge k$,
$$\Phi(x + iy) - \varepsilon \le \Phi_n(c) + q_n$$
$$\Rightarrow \inf\{\Phi(x + iy) - \varepsilon : n \ge k\} \le \inf\{\Phi_n(c) + q_n : n \ge k\}$$
$$\Rightarrow \Phi(x + iy) - \varepsilon \le \inf\{\Phi_n(c) + q_n : n \ge k\}.$$
For two subsets A, B of $\mathbb{R}$ existence of $\inf(A)$ and existence of $\inf(B)$, imply that $\inf(A + B) = \inf(A) + \inf(B)$. Since, $\{q_n\}$ is statistically null and $0 \le q_n \le 1, \inf\{q_n : n \ge k\} = 0$. Thus, for every $\varepsilon > 0$,
$$\Phi(x + iy) - \varepsilon \le \inf\{\Phi_n(c) : n \ge k\}.$$
For all $n \ge k$, we have
$$\Phi(x + iy) - \varepsilon \le m_n,$$
where $m_n = \inf\{\Phi_i(c) : i \ge n\}, n \in \mathbb{N}$. Letting $n \to \infty$, we get
$$\Phi(x + iy) - \varepsilon \le \lim_{n\to\infty} m_n.$$
For every $\varepsilon > 0$. Hence,
$$\Phi(x + iy) \le \lim_{n\to\infty} m_n \Rightarrow \Phi(x + iy) \le \lim_{n\to\infty} \inf\{\Phi_n(c)\}.$$
Letting $x + iy \to c = a + ib$, from continuity of Φ we get,
$$\Phi(c) \le \lim_{n\to\infty} \inf\{\Phi_n(c)\}. \tag{8}$$
Combining (3.7) and (3.8) we have, $\lim\limits_{n\to\infty} \Phi_n(c) = \Phi(c)$, where $c \in \mathbb{C}$ is a point of continuity of ϕ. So, for every $\varepsilon > 0, | \{k \le n :| \Phi_k(c) - \Phi(c) |\ge \varepsilon\} |$ is a finite number. So, for $0 < \beta \le 1, | \{k \le n :| \Phi_k(c) - \Phi(c) |\ge \varepsilon\} |^\beta$ is also a finite number. For every $\varepsilon > 0$, and for any α, β satisfying $0 < \alpha \le \beta \le 1$, it can be concluded that,
$$\lim_{n\to\infty} \frac{| \{k \le n :| \Phi_k(c) - \Phi(c) |\ge \varepsilon\} |^\beta}{n^\alpha} = 0.$$
Hence, the complex uncertain sequence $\{\zeta_n\}$ statistically converges in distribution of order (α, β) to $\zeta = \xi + i\eta$. $\square$

Remark 3.2. However, statistical convergence in distribution of order (α, β) does not imply statistical convergence in measure of order (α, β).

Example 3.2. Let us consider an uncertainty space (Γ, L, M) where $\Gamma = \{\gamma_1, \gamma_2\}$ with $M(\gamma_1) = M(\gamma_2) = \frac{1}{2}$. A complex uncertain variable is defined as

$$\zeta(\gamma) = \begin{cases} i, \ \ \gamma = \gamma_1, \\ -i, \gamma = \gamma_2. \end{cases}$$

A sequence $\{\zeta_n\}$ of complex uncertain variable is defined as, $\zeta_n = -\zeta$ for all $n \in \mathbb{N}$ Hence,

$$\zeta_n(\gamma) = \begin{cases} -i, \gamma = \gamma_1, \\ i, \ \ \gamma = \gamma_2. \end{cases}$$

If $\zeta_n = \xi_n + i\eta_n$, then we have,

$$\xi_n(\gamma) = \begin{cases} 0, \gamma = \gamma_1, \\ 0, \gamma = \gamma_2 \end{cases}$$

and

$$\eta_n(\gamma) = \begin{cases} -1, \gamma = \gamma_1, \\ 1, \ \ \gamma = \gamma_2. \end{cases}$$

Now, the uncertainty distribution Φ_n of ζ_n is defined as

$$\Phi_n(c) = \Phi_n(a + ib) = M\{(\xi_n \leq a) \cap (\eta_n \leq b)\}$$

$$\Rightarrow \Phi_n(c) = \begin{cases} 0, a < 0, b \in \mathbb{R}, \\ 0, a \geq 0, b < -1, \\ \frac{1}{2}, a \geq 0, -1 \leq b < 1, \\ 1, a \geq 0, b \geq 1, \end{cases}$$

for all $c = a + ib \in \mathbb{C}$. Similarly, it can be shown that the uncertainty distribution Φ of ζ is same as Φ_n on $\mathbb{C}$ i.e. $\Phi_n(c) = \Phi(c)$ for all $c = a + ib \in \mathbb{C}$. Thus, for every $\varepsilon > 0$, and for any α, β satisfying $0 < \alpha \leq \beta \leq 1$,

$$\lim_{n\to\infty} \frac{| \{k \leq n :| \Phi_k(c) - \Phi(c) |\geq \varepsilon\} |^\beta}{n^\alpha} = 0.$$

Hence, the complex uncertain sequence $\{\zeta_n\}$ statistically converges in distribution of order (α, β) to ζ.

Now, let us check the convergence in measure of order (α, β). It is found that

$$\zeta_n(\gamma) - \zeta(\gamma) = \begin{cases} -2i, \gamma = \gamma_1, \\ 2i \ \ , \gamma = \gamma_2, \end{cases}$$

for all $n \in \mathbb{N}$. Thus, $\| \zeta_n(\gamma) - \zeta(\gamma) \| = 2$, for all $n \in \mathbb{N}$ and for $\gamma = \gamma_1, \gamma_2$. Now, for any $\varepsilon > 0$,

$$M[\| \zeta_n - \zeta \| \geq \varepsilon] = M\{\gamma \in \Gamma :| \zeta_n(\gamma) - \zeta(\gamma) |\geq \varepsilon\} = \begin{cases} 1, \varepsilon \leq 2, \\ 0, \varepsilon > 2. \end{cases}$$

Let $\alpha = \beta = 1, \varepsilon = 0.5$ and $\delta = 1$. Then,

$$\frac{| \{k \leq n : M\{\| \zeta_k - \zeta \| \geq \varepsilon\} \geq \delta\} |^\beta}{n^\alpha} = \frac{| \{k \leq n : M\{\| \zeta_k - \zeta \| \geq 0.5\} \geq 1\} |}{n} = \frac{n}{n} = 1.$$

Hence, $\lim_{n\to\infty} \frac{|\{k \leq n : M\{\|\zeta_k - \zeta\| \geq 0.5\} \geq 1\}|}{n} = 1 \neq 0$. So, the complex uncertain sequence $\{\zeta_n\}$ does not statistically converge in measure of order (α, β) to ζ.

3.3. Statistical convergence almost surely of order (α,β) and statistical convergence in measure of order (α,β). Statistical convergence almost surely of order (α,β) does not imply statistical converge in measure of order (α,β).

Example 3.3. Let us consider the uncertainty space (Γ, L, M) where $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, ...\}$ and L is the power of set of Γ. Let us consider the uncertain measure M defined as

$$M\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \dfrac{n}{(2n+1)} & \text{if } \sup\limits_{\gamma_n \in \Lambda} \dfrac{n}{(2n+1)} < 0.5, \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{n}{(2n+1)} & \text{if } \sup\limits_{\gamma_n \in \Lambda^c} \dfrac{n}{(2n+1)} < 0.5, \\ 0.5, \text{ otherwise,} \end{cases}$$

where $\Lambda \in L$. Let us consider a sequence $\{\zeta_n\}$ of complex uncertain variable defined as

$$\zeta_n(\gamma) = \begin{cases} ni, \gamma = \gamma_n, \\ 0, \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. Let ζ be another complex uncertain variable such that $\zeta \equiv 0$. Hence,

$$\| \zeta_n(\gamma) - \zeta(\gamma) \| = \begin{cases} n, \gamma = \gamma_n, \\ 0, \text{otherwise.} \end{cases}$$

Here, $\Lambda = \Gamma$ with $M\{\Lambda\} = 1$. For any $\gamma_j \in \Lambda (j \in \mathbb{N}), \varepsilon > 0$ and $0 < \beta \leq 1$,

$$|\{k \in n : (|\zeta_k(\gamma_j) - \zeta(\gamma_j)| \geq \varepsilon)\}|^\beta = \begin{cases} 0, n < j, \\ 1, n \geq j, j \geq \varepsilon, \\ 0, n \geq j, j < \varepsilon. \end{cases}$$

Hence, for any α, β satisfying $0 < \alpha < \beta \leq 1$

$$\lim_{n\to\infty} \frac{|\{k \in n : (|\zeta_k(\gamma_j) - \zeta(\gamma_j)| \geq \varepsilon)\}|^\beta}{n^\alpha} = 0.$$

So, $\{\zeta_n\}$ statistically converges almost surely of order (α, β) to ζ.
Now, for every $n \in \mathbb{N}$,

$$M\{\| \zeta_n - \zeta \| \geq \varepsilon\} = \begin{cases} \frac{n}{2n+1}, n \geq \varepsilon, \\ 0, n < \varepsilon. \end{cases}$$

Let us choose $\alpha = \beta = 1, \varepsilon = 1$ and $\delta = \frac{1}{3}$. Then,for every $n \in \mathbb{N}$,

$$\frac{|\{k \leq n : M(\| \zeta_k - \zeta \| \geq 1) \geq \frac{1}{3}\}|}{n} = \frac{|\{k \leq n : \frac{k}{2k+1} \geq \frac{1}{3}\}|}{n} = \frac{n}{n} = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{|\{k \leq n : M(\| \zeta_k - \zeta \| \geq 1) \geq \frac{1}{3}\}|}{n} = 1.$$

Hence, $\{\zeta_n\}$ does not statistically converge in measure of order (α, β) to ζ.
If $\zeta_n = \xi_n + i\eta_n$, then we have, $\xi_n(\gamma) = 0$ for any $\gamma = \gamma_j$ and

$$\eta_n(\gamma) = \begin{cases} n, \gamma = \gamma_n, \\ 0, \text{otherwise.} \end{cases}$$

Now, the uncertainty distribution Φ_n of ζ_n is defined as

$$\Phi_n(c) = \Phi_n(a+ib) = M\{(\xi_n \le a) \cap (\eta_n \le b)\}$$

$$\Rightarrow \Phi_n(c) = \begin{cases} 0, a < 0, b \in \mathbb{R}, \\ 0, a \ge 0, b < 0, \\ 1 - \frac{n}{(2n+1)}, a \ge 0, 0 \le b < n, \\ 1, a \ge 0, b \ge n, \end{cases}$$

for all $c = a + ib \in \mathbb{C}$. Now, the uncertainty distribution Φ of ζ is defined as

$$\Phi(c) = \Phi(a+ib) = M\{(\xi \le a) \cap (\eta \le b)\}$$

$$\Rightarrow \Phi(c) = \begin{cases} 0, a < 0 \text{ or } b < 0, \\ 1, a \ge 0, b \ge 0, \end{cases}$$

for all $c = a + ib \in \mathbb{C}$. Let $c = \frac{1}{2} + i\frac{1}{2}$. Then, for every $k \in \mathbb{N}$,

$$\mid \phi_k(\frac{1}{2} + i\frac{1}{2}) - \phi(\frac{1}{2} + i\frac{1}{2}) \mid = \mid 1 - \frac{k}{2k+1} - 1 \mid = \frac{k}{2k+1}.$$

Hence,

$$\frac{\mid \{k \le n : \mid \phi_k(\frac{1}{2} + i\frac{1}{2}) - \phi(\frac{1}{2} + i\frac{1}{2}) \mid \ge \frac{1}{3}\}}{n} = \frac{\mid \{k \le n : \frac{k}{2k+1} \ge \frac{1}{3}\} \mid}{n} = 1$$

$$\Rightarrow \lim_{n \to \infty} \frac{\mid \{k \le n : \frac{k}{2k+1} \ge \frac{1}{3}\} \mid}{n} = 1.$$

So, $\{\zeta_n\}$ does not statistically converge in distribution of order (α, β) to ζ i.e. statistical convergence almost surely of order (α, β) does not imply statistical converge in distribution of order (α, β).

3.4. Statistical convergence almost surely of order (α, β) and statistical convergence in mean of order (α, β). Statistical convergence almost surely of order (α, β) does not imply statistical convergence in mean of order (α, β).

Example 3.4. Let us consider the uncertainty space (Γ, L, M), where $\Gamma = \{\gamma_1, \gamma_2, \gamma_3, ...\}$ and L is the power of set of Γ. Let us consider the uncertain measure M defined as

$$M\{\Lambda\} = \sum_{\gamma_n \in \Lambda} \frac{1}{2^n},$$

where $\Lambda \in L$. Let us consider a sequence $\{\zeta_n\}$ of complex uncertain variable defined as

$$\zeta_n(\gamma) = \begin{cases} i2^n, \gamma = \gamma_n, \\ 0, \text{otherwise}, \end{cases}$$

for all $n \in \mathbb{N}$. Let ζ be another complex uncertain variable such that $\zeta \equiv 0$. Hence,

$$\mid\mid \zeta_n(\gamma) - \zeta(\gamma) \mid\mid = \begin{cases} 2^n, \gamma = \gamma_n, \\ 0, \text{otherwise}. \end{cases}$$

Here, $\Lambda = \Gamma$ with $M\{\Lambda\} = 1$. For any $\gamma_j \in \Lambda(j \in \mathbb{N}), \varepsilon > 0$ and $0 < \beta \leq 1$,

$$| \{k \leq n : (| \zeta_k(\gamma_j) - \zeta(\gamma_j) | \geq \varepsilon)\} |^{\beta} = \begin{cases} 0, n < j, \\ 1, n \geq j, 2^j \geq \varepsilon, \\ 0, n \geq j, 2^j < \varepsilon. \end{cases}$$

Hence, for any α, β satisfying $0 < \alpha < \beta \leq 1$

$$\lim_{n\to\infty} \frac{| \{k \leq n : (| \zeta_k(\gamma_j) - \zeta(\gamma_j) | \geq \varepsilon)\} |^{\beta}}{n^{\alpha}} = 0.$$

So, $\{\zeta_n\}$ statistically converges almost surely of order (α, β) to ζ.
Let $\xi_n =\| \zeta_n - \zeta \|$, $n \in \mathbb{N}$. So,

$$\xi_n(\gamma) =\| \zeta_n(\gamma) - \zeta(\gamma) \|= \begin{cases} 2^n, \gamma = \gamma_n, \\ 0, \text{otherwise}. \end{cases}$$

The uncertainty distribution Φ_n of $\xi_n =\| \zeta_n - \zeta \|$ is defined as

$$\Phi_n(x) = M\{\xi_n \leq x\} = \begin{cases} 0, x < 0, \\ 1 - \frac{1}{2^n}, 0 \leq x < 2^n, \\ 1, x \geq 2^n, \end{cases}$$

for every $x \in \mathbb{R}$. Now, for any $n \in \mathbb{N}$,

$$E[\xi_n] = E[\| \zeta_n - \zeta \|] = \int_0^{\infty} M(\| \zeta_n - \zeta \| \geq x)dx = \int_0^{2^n} \frac{1}{2^n} dx = 1.$$

Let us choose $\varepsilon = 1, \alpha = \beta = 1$. Then,

$$\lim_{n\to\infty} \frac{| \{k \in n : E(\| \zeta_k - \zeta \|) \geq 1\} |}{n} = 1 \neq 0.$$

So, $\{\zeta_n\}$ does not statistically converge in mean of order (α, β) to ζ i.e. almost surely statistical convergence of order (α, β) does not imply statistical convergence in mean of order (α, β).

3.5. Statistical convergence almost surely of order (α, β) and statistical convergence in distribution of order (α, β). Statistical convergence almost surely of order (α, β) does not imply statistical convergence in distribution of order (α, β) [see Example 3.3]. Let us consider the following example to show that statistical convergence in distribution of order (α, β) does not imply statistical convergence almost surely of order (α, β).

Example 3.5. Let us consider an uncertainty space (Γ, L, M) where $\Gamma = \{\gamma_1, \gamma_2\}$ with $M\{\gamma_1\} = M\{\gamma_2\} = \frac{1}{2}$. A complex uncertain variable is defined as

$$\zeta(\gamma) = \begin{cases} i \ \ , \gamma = \gamma_1, \\ -i, \gamma = \gamma_2. \end{cases}$$

A sequence $\{\zeta_n\}$ of complex uncertain variable is defined as, $\zeta_n = -\zeta$ for all $n \in \mathbb{N}$. Hence,

$$\zeta_n(\gamma) = \begin{cases} i \ \ , \gamma = \gamma_1, \\ -i, \gamma = \gamma_2. \end{cases}$$

It has been already shown in Example 2 that complex uncertain sequence $\{\zeta_n\}$ statistically converges in distribution of order (α, β) to ζ Now, let us check for almost surely statistical convergence of order (α, β). It is found that

$$\zeta_n(\gamma) - \zeta(\gamma) = \begin{cases} -2i, \gamma = \gamma_1, \\ 2i, \gamma = \gamma_2, \end{cases}$$

for all $n \in \mathbb{N}$. Thus, $\mid \zeta_n(\gamma) - \zeta(\gamma) \mid = 2$ for all $n \in \mathbb{N}$ and for $\gamma = \gamma_1, \gamma_2$. Now, for any $\varepsilon > 0$,

$$\mid \{k \leq n : (\mid \zeta_k(\gamma) - \zeta(\gamma) \mid \geq \varepsilon)\} \mid = \begin{cases} n, \varepsilon \leq 2, \\ 0, \varepsilon > 2. \end{cases}$$

Hence,

$$\lim_{n \to \infty} \frac{\mid \{k \leq n : (\mid \zeta_k(\gamma) - \zeta(\gamma) \mid \geq 1)\} \mid}{n} = 1 \neq 0.$$

So, the complex uncertain sequence $\{\zeta_n\}$ does not statistically almost surely (order (α, β)) to ζ.

3.6. Sufficient conditions for almost surely statistical convergence of order (α, β) and uniformly almost surely statistical convergence of order (α, β).

Proposition 3.4. *Let $\{\zeta_n\}$ be a sequence of complex uncertain variables and ζ be a complex uncertain variable. Suppose that for every $\varepsilon > 0, \delta > 0$ and for some α, β satisfying $0 < \alpha < \beta \leq 1$ we have*

$$\lim_{n \to \infty} \frac{\mid \{k \leq n : M(\bigcap_{j=k}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| \geq \varepsilon) \geq \delta\} \mid^{\beta}}{n^{\alpha}} = 0.$$

Then, $\{\zeta_n\}$ converges statistically almost surely (order (α, β)) to ζ.

Proof. Let ε be any positive real number. Then for every $\delta > 0$ we have

$$\lim_{n \to \infty} \frac{\mid \{k \leq n : M(\bigcap_{j=k}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| \geq \varepsilon) \geq \delta\} \mid^{\beta}}{n^{\alpha}} = 0,$$

for some α, β satisfying $0 < \alpha < \beta \leq 1$. Statistical convergence of order (α, β) implies usual statistical convergence. Thus,

$$\lim_{n \to \infty} \frac{\mid \{k \leq n : M(\bigcap_{j=k}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| \geq \varepsilon) \geq \delta\} \mid}{n} = 0$$

$$\Rightarrow \lim_{n \to \infty} \frac{\mid \{k \leq n : M(\bigcap_{j=k}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| \geq \varepsilon) < \delta\} \mid}{n} = 1.$$

If $p \in \mathbb{N}$ is such that $p \notin \mid \{k \in \mathbb{N} : M(\bigcap_{j=k}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| \geq \varepsilon) < \delta\} \mid = B(say)$, then $\{i \in \mathbb{N} : i \geq p\} \cap B = \phi$. So, B is a finite set with density 1 which is not possible. So, every natural number $k \in B$. For every $k \in \mathbb{N}$ and $\delta > 0$,

$$0 \leq M(\bigcap_{j=k}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| \geq \varepsilon) < \delta.$$

For fixed $k_0 \in \mathbb{N}$,

$$M(\bigcap_{j=k_0}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| \geq \varepsilon) = 0$$

$$\Rightarrow M(\bigcap_{j=k_0}^{\infty} \bigcup_{i=j}^{\infty} \| \zeta_i - \zeta \| < \varepsilon) = 1.$$

Let $\Lambda = \bigcup_{j=k_0}^{\infty} \bigcap_{i=j}^{\infty} \| \zeta_i - \zeta \| < \varepsilon$. Thus, we have an event Λ with $M\{\Lambda\} = 1$. For any $\gamma \in \Lambda$ there exists a natural number $k_1 \geq k_0$ such that $\gamma \in \bigcap_{i=k_1}^{\infty} (\| \zeta_i - \zeta \| < \varepsilon)$ i.e. $| \zeta_i(\gamma) - \zeta(\gamma) | < \varepsilon$ for all $i \geq k_1$. Thus, for any fixed $\gamma \in \Lambda$,

$$\{i \in \mathbb{N} : i \geq k_1\} \subseteq \{k \in \mathbb{N} : | \zeta_i(\gamma) - \zeta(\gamma) | < \varepsilon\}$$

$$\Rightarrow \{k \in \mathbb{N} : | \zeta_i(\gamma) - \zeta(\gamma) | \geq \varepsilon\} \subseteq \{i \in \mathbb{N} : i \leq k_1\}.$$

So, $\{k \in \mathbb{N} : \ | \zeta_i(\gamma) - \zeta(\gamma) | \geq \varepsilon\}$ is a finite set for every $\gamma \in \Lambda$ and every $\varepsilon > 0$. Thus, for every $\varepsilon > 0$ there is an event $\Lambda = \bigcup_{j=k_0}^{\infty} \bigcap_{i=j}^{\infty} (\| \zeta_i - \zeta \| < \varepsilon)$ with $M\{\Lambda\} = 1$, such that for every $\gamma \in \Lambda$

$$\lim_{n \to \infty} \frac{| \{k \leq n : | \zeta_k(\gamma) - \zeta(\gamma) | \geq \varepsilon\} |^\beta}{n^\alpha} = 0,$$

for some α, β satisfying $0 < \alpha < \beta \leq 1$. Hence, $\{\zeta_n\}$ statistically almost surely converges (order (α, β)) to ζ. □

Proposition 3.5. *Let $\{\zeta_n\}$ be a sequence of complex uncertain variables and ζ be a complex uncertain variable. Suppose that for every $\varepsilon > 0, \delta > 0$ and for some α, β satisfying $0 < \alpha < \beta \leq 1$ we have*

$$\lim_{n \to \infty} \frac{| \{k \leq n : M(\bigcup_{n=k}^{\infty} \{n : \| \zeta_n - \zeta \| \geq \varepsilon\}) \geq \delta\} |^\beta}{n^\alpha} = 0.$$

Then, $\{\zeta_n\}$ converges statistically almost surely (order (α, β)) to ζ.

Proof. Let δ be any positive real number. Then for every $\varepsilon > 0$ we have

$$\lim_{n \to \infty} \frac{| \{k \leq n : M(\bigcup_{n=k}^{\infty} \{n : \| \zeta_n - \zeta \| \geq \varepsilon\}) \geq \delta\} |^\beta}{n^\alpha} = 0,$$

for some α,β satisfying $0<\alpha<\beta\le 1$. Statistical convergence of order (α,β) implies usual statistical convergence. Thus,

$$\lim_{n\to\infty}\frac{\mid \{k\le n : M(\bigcup_{n=k}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\})\ge\delta\}\mid^{\beta}}{n}=1.$$

Since, $\mid \{k\le n : M(\bigcup_{n=k}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\})\ge\delta\}\mid$ is an infinite set and $\bigcup_{n=k}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\}$ is a monotonically decreasing sequence of sets, $\{k\in\mathbb{N} : M(\bigcup_{n=k}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\})\ge\delta\}$ contains every natural number excepting first few finite number of natural numbers. For each $i\in\mathbb{N}$, let $E_i=\bigcup_{n=i}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\}$. Thus we have a sequence $\{E_k\}$ of events. Now for any $p\in\mathbb{N}$

$$p\in\{k\in\mathbb{N} : M(\bigcup_{n=k}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\})<\delta\}\Rightarrow p\in\{k\in\mathbb{N} : M(E_k)<\delta\}$$

$$\Rightarrow\{k\in\mathbb{N} : M(\bigcup_{n=k}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\})<\delta\}\subseteq\{k\in\mathbb{N} : M(E_k)<\delta\}$$

$$\Rightarrow\{k\in\mathbb{N} : M(E_k)\ge\delta\}\subseteq\{k\in\mathbb{N} : M(\bigcup_{n=k}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\})\ge\delta\}.$$

The set on R.H.S is a finite set for every $\varepsilon>0$ and $\delta>0$. Hence, $\{k\in\mathbb{N} : M(E_k)\ge\delta\}$ being a finite set, we have

$$\lim_{n\to\infty}\frac{\mid\{k\le\mathbb{N} : M(E_k)\ge\delta\}\mid^{\beta}}{n^{\alpha}}=0,$$

for some α,β satisfying $0<\alpha<\beta\le 1$. Now, for fixed $i\in\mathbb{N}$

$$\Gamma-E_i=\bigcap_{n=i}^{\infty}\{n :\mid\mid \zeta_n-\zeta\mid\mid\ge\varepsilon\},$$

$\gamma\in\Gamma-E_i$ implies that $\mid\zeta_n(\gamma)-\zeta(\gamma)\mid<\varepsilon$ for all $n\ge i$. Thus, for fixed $i\in\mathbb{N}$ and any $\gamma\in\Gamma-E_i,\{n\in\mathbb{N} : n\ge i\}\subseteq\{k\in\mathbb{N} :\mid\zeta_k(\gamma)-\zeta(\gamma)\mid<\varepsilon\}$. Hence, for fixed $i\in\mathbb{N}$, arbitrary $\varepsilon>0$, and any $\gamma\in\Gamma-E_i,\{k\in\mathbb{N} :\mid\zeta_k(\gamma)-\zeta(\gamma)\mid\ge\varepsilon\}$ is always a finite set. So, for every $\gamma\in\Gamma-E_i$

$$\lim_{n\to\infty}\frac{\mid\{k\in\mathbb{N} : n :\mid\zeta_n(\gamma)-\zeta(\gamma)\mid\ge\varepsilon\}\mid^{\beta}}{n^{\alpha}}=0,$$

for some α,β satisfying $0<\alpha<\beta\le 1$. Thus, $\{\zeta_n\}$ converges statistically almost surely (order (α,β)) to ζ. $\square$

Theorem 3.6. *Let $\{\zeta_n\}$ be a sequence of complex uncertain variables and ζ be a complex uncertain variable. Suppose that for every $\varepsilon > 0, \delta > 0$ and for some α, β satisfying $0 < \alpha < \beta \leq 1$ we have*

$$\lim_{n\to\infty} \frac{\mid \{k \leq n : M(\bigcup_{n=k}^{\infty}\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \geq \delta\} \mid^{\beta}}{n^{\alpha}} = 0.$$

Then, $\{\zeta_n\}$ converges statistically almost surely (order (α, β)) to ζ.

Proof. For any $k \in \mathbb{N}, \varepsilon > 0$,

$$\bigcap_{j=k}^{\infty}\bigcup_{n=j}^{\infty}(\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \subseteq \bigcup_{n=k}^{\infty}(\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}).$$

Taking uncertain measure M we get,

$$\Rightarrow M(\bigcap_{j=k}^{\infty}\bigcup_{n=j}^{\infty}(\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \subseteq M(\bigcup_{n=k}^{\infty}(\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}).$$

Hence for any $\delta > 0$,

$$\{k \leq n : M(\bigcap_{j=k}^{\infty}\bigcup_{n=j}^{\infty}\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \geq \delta\} \subseteq \{k \leq n : M(\bigcup_{n=k}^{\infty}(\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \geq \delta\}$$

$$\Rightarrow \frac{\mid \{k \leq n : M(\bigcap_{j=k}^{\infty}\bigcup_{n=j}^{\infty}\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \geq \delta\} \mid^{\beta}}{n^{\alpha}}$$
$$\leq \frac{\mid \{k \leq n : M(\bigcup_{n=k}^{\infty}\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \geq \delta\} \mid^{\beta}}{n^{\alpha}}$$
$$\Rightarrow \lim_{n\to\infty} \frac{\mid \{k \leq n : M(\bigcap_{j=k}^{\infty}\bigcup_{n=j}^{\infty}\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \geq \delta\} \mid^{\beta}}{n^{\alpha}} = 0,$$

for some α, β satisfying $0 < \alpha < \beta \leq 1$. Thus from Proposition 1 it can be concluded that $\{\zeta_n\}$ converges statistically almost surely (order (α, β)) to ζ. □

4. Sufficient condition for statistical convergence in measure of order (α, β)

Theorem 4.1. *Let $\{\zeta_n\}$ be a sequence of complex uncertain variables and ζ be a complex uncertain variable. Suppose that for every $\varepsilon > 0, \delta > 0$ and for some α, β satisfying $0 < \alpha < \beta \leq 1$ we have*

$$\lim_{n\to\infty} \frac{\mid \{k \leq n : M(\bigcup_{n=k}^{\infty}\{n :\parallel \zeta_n - \zeta \parallel \geq \varepsilon\}) \geq \delta\} \mid^{\beta}}{n^{\alpha}} = 0.$$

Then, $\{\zeta_n\}$ converges statistically in measure (order (α, β)) to ζ.

Proof. For any $k \in \mathbb{N}, \varepsilon > 0$,

$$(\| \zeta_k - \zeta \| \geq \varepsilon) \subseteq \bigcup_{n=k}^{\infty} (\{n : \| \zeta_n - \zeta \| \geq \varepsilon\})$$

$$\Rightarrow M(\| \zeta_k - \zeta \| \geq \varepsilon) \subseteq M(\bigcup_{n=k}^{\infty} \{n : \| \zeta_n - \zeta \| \geq \varepsilon\})$$

$$\Rightarrow \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \subseteq \{k \leq n : M(\bigcup_{n=k}^{\infty} \{n : \| \zeta_n - \zeta \| \geq \varepsilon\}) \geq \delta\},$$

for any $\delta > 0$. Thus, for some α, β satisfying $0 < \alpha < \beta \leq 1$,

$$\Rightarrow \lim_{n \to \infty} \frac{| \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} |^{\beta}}{n^{\alpha}} = 0.$$

Hence, $\{\zeta_n\}$ converges statistically in measure (order (α, β)) to ζ. □

5. Some Results

Let $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$. Then,
(a) A complex uncertain sequence $\{\zeta_n\}$ statistically converges in mean of order (α_1, β_2) to ζ implies that $\{\zeta_n\}$ statistically converges in mean of order (α_2, β_1) to ζ.
Proof. The inequality

$$\frac{| \{k \leq n : E[\| \zeta_k - \zeta \|] \geq \varepsilon\} |^{\beta_1}}{n^{\alpha_2}} \leq \frac{| \{k \leq n : E[\| \zeta_k - \zeta \|] \geq \varepsilon\} |^{\beta_2}}{n^{\alpha_1}}$$

can be used to prove the result. □
(b) A complex uncertain sequence $\{\zeta_n\}$ statistically converges in measure of order (α_1, β_2) to ζ implies that $\{\zeta_n\}$ statistically converges in measure of order (α_2, β_1) to ζ.
Proof. The inequality

$$\frac{| \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon)\} |^{\beta_1}}{n^{\alpha_2}} \leq \frac{| \{k \leq n : M(\| \zeta_k - \zeta \| \geq \varepsilon)\} |^{\beta_2}}{n^{\alpha_1}}$$

can be used to prove the result. □
(c) A complex uncertain sequence $\{\zeta_n\}$ statistically converges in distribution of order (α_1, β_2) to ζ implies that $\{\zeta_n\}$ statistically converges in distribution of order (α_2, β_1) to ζ.
Proof. The inequality

$$\frac{| \{k \leq n : | \Phi_k(c) - \Phi(c) | \geq \varepsilon)\} |^{\beta_1}}{n^{\alpha_2}} \leq \frac{| \{k \leq n : | \Phi_k(c) - \Phi(c) | \geq \varepsilon)\} |^{\beta_2}}{n^{\alpha_1}}$$

can be used to prove the result. □
(d) A complex uncertain sequence $\{\zeta_n\}$ statistically converges almost surely of order (α_1, β_2) to ζ implies that $\{\zeta_n\}$ statistically converges almost surely of order (α_2, β_1) to ζ.
Proof. The inequality

$$\frac{| \{k \leq n : | \zeta_k(\gamma) - \zeta(\gamma) | \geq \varepsilon\} |^{\beta_1}}{n^{\alpha_2}} \leq \frac{| \{k \leq n : | \zeta_k(\gamma) - \zeta(\gamma) | \geq \varepsilon\} |^{\beta_2}}{n^{\alpha_1}}$$

can be used to prove the result. □

(e) A complex uncertain sequence $\{\zeta_n\}$ statistically converges uniformly almost surely of order (α_1, β_2) to ζ implies that $\{\zeta_n\}$ statistically converges uniformly almost surely of order (α_2, β_1) to ζ.

Proof. The inequalities

$$\frac{|\{k \le n : M(E_k) \ge \varepsilon\}|^{\beta_1}}{n^{\alpha_2}} \le \frac{|\{k \le n : M(E_k) \ge \varepsilon\}|^{\beta_2}}{n^{\alpha_1}}$$

and

$$\frac{|\{k \le n : |\zeta_k(\gamma) - \zeta(\gamma)| \ge \varepsilon\}|^{\beta_1}}{n^{\alpha_2}} \le \frac{|\{k \le n : |\zeta_k(\gamma) - \zeta(\gamma)| \ge \varepsilon\}|^{\beta_2}}{n^{\alpha_1}}$$

can be used to prove the result. □

6. Conclusion

This paper introduces ideas of statistical convergence of order (α, β) for a complex uncertain sequence, and relationship among those presented concepts have been studied. Sufficient conditions of almost surely statistical convergence of order (α, β), uniformly almost surely statistical convergence of order (α, β), and statistical convergence in measure of order (α, β) for complex uncertain sequence have been developed.

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