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An identification problem

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Abstract. This paper has the purpose to study an inverse problem related to a specific boundary value problem of rainfall type infiltration into an isotropic, homogeneous, unsaturated porous medium, in which saturation can be partially or totally reached after sometime.

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1. Statement of the problem

1.1. Mathematical model. Specific applications in soil sciences, hydrology or agriculture may require the study of inverse problems related to water infiltration in soils. This paper has the purpose to study an inverse problem related to a boundary value problem of rainfall type infiltration into an isotropic, homogeneous, unsaturated porous medium, in which saturation can be partially or totally reached after sometime.

The goal is to determine the rain rate that produced a certain moisture of the soil $\theta^0(x)$, measured in the flow domain, at the time T

$$\theta^0(x) = \theta^{observed}(x,T).$$

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N = 1,2,3)$ with the boundary $\partial\Omega \overset{notation}{=} \Gamma$ piecewise smooth, let $(0,T)$ be a finite time interval and let $x \in \Omega$ represent the vector $x = (x_1, x_2, x_3)$.

We consider Ω to be the cylinder $\Omega = \{x; (x_1,x_2) \in D,\ 0 < x_3 < L\}$ where D is an open bounded subset of $\mathbb{R}^{N-1}$ with smooth boundary and we assume that Γ is composed of the disjoint boundaries Γ_u, Γ_{lat} and Γ_b, all sufficiently smooth, where $\Gamma_u = \{x \in \Gamma; x_3 = 0\}$, $\Gamma_b = \{x \in \Gamma; x_3 = L\}$, $\Gamma = \Gamma_u \cup \Gamma_{lat} \cup \overline{\Gamma}_b$.

We also denote $\Gamma_\alpha = \Gamma_{lat} \cup \overline{\Gamma}_b$, where $\Gamma_u \cap \Gamma_\alpha = \emptyset$.

We shall deal with the diffusive form of the mathematical model of a rainfall water infiltration into a soil with the boundary Γ_α semipermeable, consisting in the Richards' equation with initial and boundary data

$$\frac{\partial\theta}{\partial t} - \Delta\beta^*(\theta) + \frac{\partial K(\theta)}{\partial x_3} = f \text{ in } Q = \Omega \times (0,T), \tag{1}$$

$$\theta(x,0) = \theta_0(x) \text{ in } \Omega, \tag{2}$$

$$(K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu = u \text{ on } \Sigma_u = \Gamma_u \times (0,T), \tag{3}$$

$$(K(\theta)i_3 - \nabla\beta^*(\theta)) \cdot \nu = \alpha\beta^*(\theta) + f_0 \text{ on } \Sigma_\alpha = \Gamma_\alpha \times (0,T), \tag{4}$$

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where ν is the outward normal to Γ, i_3 is the unit vector along Ox_3 downwards directed, $\alpha : \Gamma_\alpha \to [\alpha_m, \alpha_M] \subset \mathbb{R}_+$ is a positive and continuous functions on Γ_α, $-u$ is the rain rate on Σ_u, positive due to the downward orientation of the Ox_3 axis, f_0 is given on Σ_α and β^*, K and β are defined as follows (see [8])

$$\beta^*(\theta) = \begin{cases} \rho\theta, & \theta \le 0 \\ \int_0^\theta \beta(\xi)d\xi, & 0 < \theta < \theta_s \\ [K_s^*, +\infty), & \theta = \theta_s, \end{cases}, \; K_s^* = \lim_{\theta \nearrow \theta_s} \beta^*(\theta) \tag{5}$$

$$K(\theta) := \begin{cases} 0, & \theta \le 0 \\ K(\theta), & 0 < \theta \le \theta_s \end{cases}, \; \beta(\theta) := \begin{cases} 0, & \theta \le 0 \\ K(\theta), & 0 < \theta \le \theta_s. \end{cases} \tag{6}$$

Here:

$K : [0, \theta_s] \to [0, K_s]$ is the nonlinear *hydraulic conductivity* which is monotonically increasing and Lipschitz (in particular we can assume $K \in C^2([0, \theta_s])$;

$\beta : [0, \theta_s) \to [\rho, +\infty)$ is the nonlinear *water diffusivity* which is differentiable, monotonically increasing and convex (in particular $\beta \in C^2([0, \theta_s))$.

For the mathematical study reasons these functions have been extended to the negative axis, by continuity (see again [8]).

The functions β^* and K satisfy:

(i) $(\beta^*(\theta) - \beta^*(\overline{\theta}))(\theta - \overline{\theta}) \ge \rho(\theta - \overline{\theta})^2$, $\forall \theta, \overline{\theta} \in (-\infty, \theta_s]$

(ii) $\lim_{\theta \to -\infty} \beta^*(\theta) = -\infty$.

(ii_K) $\left|K(\theta) - K(\overline{\theta})\right| \le M \left|\theta - \overline{\theta}\right|$, $\forall \theta, \overline{\theta} \le \theta_s$.

1.2. Control problem. We shall work under the realistic assumption, that is the boundedness of the rain rate. So, let $R \in L^\infty(\Sigma_u)$ be the upper bound of the rain rate. The problem is to determine the water supply rate (rain rate) from the moisture observation at the final time T.

For this case we envisage the fact that the flux $-u$ may be produced at its turn by a mechanism natural (in the case of a rain) or artificial (in the case of an irrigation e.g.) which obeys a certain law. In this case we choose a simple law, by replacing in fact the flux u by its velocity q assumed to be bounded by two given functions a and b.

$$\frac{du}{dt} = q, \; u(0) = u_0, \; -R(x, 0) \le u_0 \le 0 \text{ a.e. on } \Gamma_u.$$

The initial data u_0 may be known, so fixed, or may be unknown, hence arbitrary, case that will be considered here. So, the identification problem to be solved is

$$(P) \qquad \min_{(u_0, q) \in U_T} \int_\Omega (\theta(x, T) - \theta^0(x))^2 dx,$$

where the admissible set reads as

$$U_T = \Big\{(u_0, q) \in L^\infty(\Gamma_u) \times L^\infty(\Sigma_u); \; \frac{du}{dt} = q, \; u(0) = u_0, u_0 \in L^2(\Gamma_u), \\ -R(x, 0) \le u_0(x) \le 0 \text{ a.e. on } \Gamma_u \Big\}, \tag{7}$$

where

$$q \in L^\infty(\Sigma_u), \; a(x) \le q(x, t) \le b(x) \text{ a.e. on } \Sigma_u,$$

$a, b \in L^2(\Gamma_u)$, $a(x) < b(x)$ a.e. on Γ_u.

For the present study we assume in addition that the rain rates are regular, i.e., $u \in L^2(0,T;H^1(\Gamma_u))$.

2. Functional framework

Let $V = H^1(\Omega)$, with the norm defined by

$$\|\psi\|_V = \left(\int_\Omega |\nabla \psi|^2 \, dx + \int_{\Gamma_\alpha} \alpha(x) \, |\psi|^2 \, d\sigma \right)^{1/2}. \tag{8}$$

$V' = (H^1(\Omega))'$ is its dual endowed with the scalar product

$$\langle \theta, \overline{\theta} \rangle_{V'} = (\theta, \psi), \ \forall \theta, \overline{\theta} \in V', \tag{9}$$

where $\psi \in V$ satisfies the boundary value problem

$$-\Delta \psi = \overline{\theta}, \ \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma_\alpha, \ \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \Gamma_u, \tag{10}$$

We set

$$D(A) = \{\theta \in L^2(\Omega); \ \exists \eta \in V \text{ and } \eta(x) \in \beta^*(\theta(x)) \text{ a.e. } x \in \Omega\}$$

and we define the multivalued operator $A : D(A) \subset V' \to V'$, by

$$(A\theta, \psi) = \int_\Omega \left(\nabla \eta \cdot \nabla \psi - K(\theta) \frac{\partial \psi}{\partial x_3} \right) dx + \int_{\Gamma_\alpha} \alpha \eta \psi d\sigma, \ \forall \psi \in V.$$

Moreover, we define $B \in L(L^2(\Gamma_u); V')$ and $f_\Gamma \in L^2(0,T;V')$ by

$$Bu(\psi) = -\int_{\Gamma_u} u \psi d\sigma, \ \forall \psi \in V$$

$$f_\Gamma(t)(\psi) = -\int_{\Gamma_\alpha} f_0 \psi d\sigma, \ \forall \psi \in V.$$

and with these notations we introduce the Cauchy problem

$$\frac{d\theta}{dt} + A\theta \ni f + Bu + f_\Gamma \text{ a.e. } t \in (0,T) \tag{11}$$

$$\theta(0) = \theta_0(x) \text{ in } \Omega, \tag{12}$$

whose strong solution, if exists, satisfies (1)-(4) in the sense of distributions.

In order to prove the solution existence and uniqueness, the multivalued function β^* was approximated by the continuous function

$$\beta^*_\varepsilon(\theta) = \begin{cases} \beta^*(\theta), & \theta < \theta_s \\ K^*_s + (\theta - \theta_s)/\varepsilon, & \theta \geq \theta_s, \end{cases} \tag{13}$$

for each $\varepsilon > 0$, so that, besides the properties (i) (for $\theta \in \mathbb{R}$), (ii), $\beta^*_\varepsilon(\theta)$ satisfies also

(iii) $\lim_{\theta \to \infty} \beta^*_\varepsilon(\theta) = +\infty$.

Also, in the approximating problem we extended K to the right of the saturation value by the constant value K_s.

However, the proof of the existence of the free boundary requires some stronger assumptions which apply for a smoother approximation β^*_ε of class C^3 a.e. on $\mathbb{R}$. So, we introduce the following function

$$\beta_\varepsilon^*(\theta) = \begin{cases} \rho_m\theta,\ \theta < -\delta \\ \beta_{ext}^*(\theta),\ \ -\delta \le \theta < 0 \\ \beta^*(\theta),\ 0 \le \theta < \theta_s - \varepsilon \\ \beta_{int}^*(\theta),\ \theta_s - \varepsilon \le \theta < \theta_s \\ \beta^*(\theta_s - \varepsilon) + \dfrac{K_s^* - \beta^*(\theta_s - \varepsilon)}{\varepsilon}[\theta - (\theta_s - \varepsilon)],\ \theta > \theta_s \end{cases} \tag{14}$$

where $\rho_m = (\beta_{ext}^*)'(-\delta) > 0$ and $\delta > 0$. Here, β_{ext}^* and β_{int}^* are polynomial functions of class C^3 on the corresponding definition intervals.

Correspondingly, the approximating problem reads as

$$\frac{d\theta_\varepsilon}{dt} + A_\varepsilon\theta_\varepsilon = f + Bu + f_\Gamma \text{ a.e. } t \in (0,T) \tag{15}$$

$$\theta_\varepsilon(0) = \theta_0(x) \text{ in } \Omega, \tag{16}$$

where $A_\varepsilon : D(A_\varepsilon) \subset V' \to V'$ is the single-valued operator defined by

$$(A_\varepsilon\theta, \psi) = \int_\Omega \left(\nabla\beta_\varepsilon^*(\theta)\cdot\nabla\psi - K(\theta)\frac{\partial\psi}{\partial x_3}\right)dx + \int_{\Gamma_\alpha} \alpha\beta_\varepsilon^*(\theta)\psi d\sigma,\ \forall\psi \in V$$

with the domain

$$D(A_\varepsilon) = \{\theta \in L^2(\Omega); \beta_\varepsilon^*(\theta) \in V\}.$$

Obviously the strong solution to (15)-(16) is the solution in the sense of distributions to the boundary value problem

$$\frac{\partial\theta_\varepsilon}{\partial t} - \Delta\beta_\varepsilon^*(\theta_\varepsilon) + \frac{\partial K(\theta_\varepsilon)}{\partial x_3} = f \text{ in } Q = \Omega \times (0,T) \tag{17}$$

$$\theta_\varepsilon(x,0) = \theta_0(x) \text{ in } \Omega \tag{18}$$

$$(K(\theta_\varepsilon)i_3 - \nabla\beta_\varepsilon^*(\theta_\varepsilon))\cdot\nu = u \text{ on } \Sigma_u = \Gamma_u \times (0,T) \tag{19}$$

$$(K(\theta_\varepsilon)i_3 - \nabla\beta^*(\theta_\varepsilon))\cdot\nu = \alpha\beta_\varepsilon^*(\theta_\varepsilon) + f_0 \text{ on } \Sigma_\alpha = \Gamma_\alpha \times (0,T). \tag{20}$$

3. Existence in the state system

Theorem 3.1. (existence of the solution to the approximating problem) *Let*

$$f \in L^2(0,T;V'),\ u \in L^2(\Sigma_u),\ f_0 \in L^2(\Sigma_\alpha), \tag{21}$$

$$\theta_0 \in L^2(\Omega);\ \theta_0 \le \theta_s \textit{ a.e. on } \Omega. \tag{22}$$

Then, problem (15)-(16) *has, for each* $\varepsilon > 0$, *a unique strong solution*

$$\theta_\varepsilon \in L^2(0,T;V) \cap W^{1,2}(0,T;V'),\ \beta_\varepsilon^*(\theta) \in L^2(0,T;V) \tag{23}$$

that satisfies the estimates

$$\|\theta_\varepsilon(t)\|_{V'}^2 + \int_0^t \|\theta_\varepsilon(\tau)\|^2 d\tau \le \gamma_1(\alpha_m)\left(\|\theta_0\|_{V'}^2 + \int_0^T \|f(\tau)\|_{V'}^2 d\tau + \right. \\ \left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau\right) \tag{24}$$

and

$$\|\theta_\varepsilon(t)\|^2 \le \int_\Omega j_\varepsilon(\theta_\varepsilon(t))dx + \int_0^t \left\|\frac{d\theta_\varepsilon}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \int_0^t \|\beta_\varepsilon^*(\theta_\varepsilon(\tau))\|_V^2 d\tau \le$$
$$\le \gamma_2(\alpha_m)\left(\int_\Omega j_\varepsilon(\theta_0)dx + \int_0^T \|f(\tau)\|_{V'}^2 d\tau + \right. \tag{25}$$
$$\left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau\right).$$

In the above estimates $\alpha_m = \min_{x\in\Gamma_\alpha} \alpha(x)$, $\gamma_1(\alpha_m) = O(1/\alpha_m)$, $\gamma_2(\alpha_m) = O(1/\alpha_m)$ as $\alpha_m \to 0$ and

$$j_\varepsilon(r) = \int_0^r \beta_\varepsilon^*(\xi)d\xi.$$

Theorem 3.2. (existence of the solution to the original problem) *Let* f, u, f_0 *and* θ_0 *satisfy* (21)-(22). *Then there exists a unique solution* θ *to the exact problem* (11)-(12) *with the following properties*

$$\theta \in L^2(0,T;V) \cap W^{1,2}(0,T;V'),\ \beta^*(\theta) \in L^2(0,T;V), \tag{26}$$

$$\theta \le \theta_s \ \text{ a.e. in } Q, \tag{27}$$

$$\|\theta(t)\|_{V'}^2 + \int_0^t \|\theta(\tau)\|^2 d\tau \le \gamma_1(\alpha_m)\left(\|\theta_0\|_{V'}^2 + \int_0^T \|f(\tau)\|_{V'}^2 d\tau + \right. \tag{28}$$
$$\left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau\right)$$

and

$$\|\theta(t)\|^2 \le \int_\Omega j(\theta(t))dx + \int_0^t \left\|\frac{d\theta}{d\tau}(\tau)\right\|_{V'}^2 d\tau + \int_0^t \|\eta(\tau)\|_V^2 d\tau \le$$
$$\le \gamma_2(\alpha_m)\left(\int_\Omega j(\theta_0)dx + \int_0^T \|f(\tau)\|_{V'}^2 d\tau + \right. \tag{29}$$
$$\left. + \int_0^T \|u(\tau)\|_{L^2(\Gamma_u)}^2 d\tau + \int_0^T \|f_0(\tau)\|_{L^2(\Gamma_\alpha)}^2 d\tau\right),$$

where $\eta \in \beta^*(\theta)$ *a.e. on* Q *and* $j : \mathbb{R} \to (-\infty, \infty]$ *is defined by*

$$j(r) = \begin{cases} \int_0^r \beta^*(\xi)d\xi, & r \le \theta_s \\ +\infty, & r > \theta_s, \end{cases} \qquad \beta^*(\theta_s) = \lim_{\substack{\xi\to\theta_s \\ \xi<\theta_s}} \beta^*(\theta_s) = K_s^*.$$

The proofs of the previous theorems are based on the results given in [2], [5], [7] and [8].

The next two theorems give some further regularity of the approximating solution.

Theorem 3.3. *Assume*

$$f \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(Q), \tag{30}$$

$$u \in W^{1,2}(0,T;L^2(\Gamma_u)) \cap L^\infty(\Sigma_u) \cap L^2(0,T;H^1(\Gamma_u)), \tag{31}$$

$$f_0 \in W^{1,2}(0,T;L^2(\Gamma_\alpha)) \cap L^\infty(\Sigma_\alpha) \cap L^2(0,T;H^1(\Gamma_\alpha)), \tag{32}$$

$$\theta_0 \in H^1(\Omega),\ \theta_0 \le \theta_s \text{ a.e. on } \Omega. \tag{33}$$

Then, the solution θ_ε *to problem* (15)-(16) *satisfies for each* $\varepsilon > 0$

$$\theta_\varepsilon \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \tag{34}$$

$$\beta_\varepsilon^*(\theta_\varepsilon) \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)). \tag{35}$$

Theorem 3.4. *Assume*

$$f \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(Q), \tag{36}$$

$$u \in W^{1,2}(0,T;L^2(\Gamma_u)) \cap L^\infty(\Sigma_u) \cap L^2(0,T;H^1(\Gamma_u)),, \tag{37}$$

$$f_0 \in W^{1,2}(0,T;L^2(\Gamma_\alpha)) \cap L^\infty(\Sigma_\alpha) \cap L^2(0,T;H^1(\Gamma_\alpha)),, \tag{38}$$

$$\theta_0 \in H^2(\Omega),\ \theta_0 \le \theta_s \text{ a.e. on } \Omega. \tag{39}$$

Then, problem (15)-(16) *has, for each* $\varepsilon > 0$, *a unique solution*

$$\theta_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega)), \tag{40}$$

$$\beta_\varepsilon^*(\theta_\varepsilon) \in W^{1,\infty}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;V) \cap L^\infty(0,T;H^2(\Omega). \tag{41}$$

The proofs of these theorems are presented in [9].

4. Identification problem in the case of a final time observation

We shall resume the identification problem

$$(P) \qquad \min_{(u_0,q)\in U_T} \int_\Omega (\theta(x,T) - \theta^0(x))^2 dx \tag{42}$$

where U_T is the admissible set for the rain rate

$$U_T = \Big\{ (u_0,q) \in L^\infty(\Gamma_u) \times L^\infty(\Sigma_u);\ \frac{du}{dt} = q,\ u(0) = u_0,$$
$$u_0 \in L^2(\Gamma_u),\ \ -R(x,0) \le u_0(x) \le 0 \text{ a.e. on } \Gamma_u \Big\}, \tag{43}$$
$$R \in L^\infty(\Sigma_u),\ q\ \in L^\infty(\Sigma_u),\ a(x) \le q(x,t) \le b(x) \text{ a.e. on } \Sigma_u$$

and $\theta^0(x)$ is the (unique) available observation at the final time T.

For $(u_0,q) \in U_T$, u is given by

$$u(x,t) = u_0(x) + \int_0^t q(x,s)ds \tag{44}$$

and it belongs to an interval determined function of a, b and R.

Moreover, we assume that we deal with rain rates $u \in L^2(0,T;H^1(\Gamma_u))$.

We may cite some general results related to identification in parabolic boundary value problems i.e. [1], [3], [6], [10], [11].

Theorem 4.1. (existence of the solution to problem (P)). *Problem* (P) *has at least one solution.*

Proof. Let $d = \min\limits_{u\in U_T} \int_\Omega (\theta(x,T)dx - \theta^0(x))^2 dx$ and let $\{(u_{0n}, q_n)\} \subset U_T$ be a minimizing sequence, i.e.,

$$d \le \int_\Omega (\theta_n(x,T) - \theta^0(x))^2 dx \le d + \frac{1}{n},\ n \ge 1, \tag{45}$$

where θ_n is the solution to the problem (11)-(12) corresponding to $u = u_n$, given by

$$u_n(x,t) = u_{0n}(x) + \int_0^t q_n(x,s)ds,$$

with $q_n \in L^2(\Sigma_u)$ and $a(x) \le q_n \le b(x)$ a.e. on Σ_u. Then, on subsequences, $u_{0n} \to \widetilde{u}$ weakly in $L^2(\Gamma_u)$ and $q_n \to \widetilde{q}$ weakly in $L^2(\Sigma_u)$. Using the previous relationship for u_n, these imply that

$$u_n \to \widetilde{u} \text{ weakly in } W^{1,2}(0,T;L^\infty(\Gamma_u)) \text{ and } \widetilde{u} \in U_T.$$

By Theorem 3.2, for each n, θ_n satisfies estimate (29), so it follows that there exists a subsequence of $\{\theta_n\}$, such that

$$\begin{aligned} \theta_n &\to \widetilde{\theta} \quad \text{weakly in } W^{1,2}(0,\,T\,;\,V') \cap L^2(0,\,T;\,V) \\ \theta_n &\to \widetilde{\theta} \quad \text{strongly in } L^2(0,\,T;\,L^2(\Omega)) \end{aligned}$$

and $\widetilde{\theta}$ is the solution to variational inequality (11)-(12) with $u = \widetilde{u}$.

Moreover, one can prove that

$$\theta_n(T) \to \widetilde{\theta}(T) \text{ weakly in } L^2(\Omega). \tag{46}$$

By Arzela theorem it follows that, for each $t \in [0,T]$, the set $\{(\theta_n(t),y)\}$ is compact in $C([0,T];L^2(\Omega))$, implying that on a subsequence that $\theta_n(t) \to \widetilde{\theta}(t)$ weakly in $L^2(\Omega)$, $\forall t \in [0,T]$. In particular we get (46).

This last together with (45) implies, by weakly lower semicontinuity, that

$$d \le \int_\Omega (\widetilde{\theta}(x,T) - \theta^0(x))^2 dx \le$$

$$\le \lim_{n\to\infty} \inf \int_\Omega (\theta_n(x,T) - \theta^0(x))^2 dx \le d$$

showing that $\widetilde{u}$ is a solution to problem (P).

Now, we introduce a family of approximating problems (P_ε)

$$(P_\varepsilon) \qquad \min_{(u_0,q)\in U_T} \int_\Omega (\theta(x,T) - \theta^0(x))^2 dx \tag{47}$$

subject to (15)-(16) with $\beta_\varepsilon^*(\theta)$ the smoother approximation of class $C^3(\mathbb{R})$. We remind also that we consider only rain rates $u \in L^2(0,T;H^1(\Gamma_u))$.

Obviously, still by Theorem 4.1, this problem has at least a solution.

Theorem 4.2. *Assume that θ_0, f and f_0 satisfy conditions of Theorem* 3.4 *and $R \in L^\infty(\Sigma_u)$. Let $((u_{0\varepsilon}, q_\varepsilon), \theta_\varepsilon)$ be a solution to approximating problem (P_ε). Then, there are subsequences of $\{(u_{0\varepsilon}, q_\varepsilon\}$ and $\{\theta_\varepsilon\}$, such that*

$$u_{0\varepsilon} \to u_0^* \textit{ weak star in } L^\infty(\Gamma_u),$$

$$q_\varepsilon \to q^* \textit{ weak star in } L^\infty(\Sigma_u),$$

$$u_\varepsilon \to u^* \textit{ weak star in } W^{1,\infty}(0,T;L^\infty(\Sigma_u)),$$

$$\theta_\varepsilon \to \theta^* \textit{ strongly in } L^2(Q) \textit{ and weakly in } L^\infty(0,T;\,V) \cap W^{1,2}(Q),$$

$$\theta_\varepsilon(T) \to \theta^*(T) \textit{ strongly in } L^2(\Omega),$$

where $(u_0^*, q^*) \in U_T$ and θ^* is the solution to (11)-(12) *with $u = u^*$.*

Moreover, (u_0^*, q^*) is a solution to (P) and $\lim_{\varepsilon\to 0}(P_\varepsilon) = (P)$.

Proof. Let $(\widetilde{u_0}, \widetilde{q}) \in U_T$ be a solution to problem (P), let $\widetilde{u}$ be given by (44) and let θ^ε be the solution to the approximating problem (15)-(16) where $u = \widetilde{u}$. By the optimality of $(u_{0\varepsilon}, q_\varepsilon)$ in problem (P_ε) we have

$$\int_\Omega (\theta_\varepsilon(x,T) - \theta^0(x))^2 dx \leq \int_\Omega (\theta^\varepsilon(x,T) - \theta^0(x))^2 dx.$$

Since θ^ε is the solution of the approximating problem corresponding to $\widetilde{u}$, it follows from Theorem 3.1 that θ^ε is bounded in $L^2(0,T;V)$ and θ'_ε is bounded in $L^2(0,T;V')$. But V is compact in $L^2(\Omega)$ and by Theorem 3.3 (see [9]) we have

$$\|\theta_\varepsilon(t)\|_V^2 \leq \text{constant}, \ \forall t \in [0,T], \tag{48}$$

so it follows that $\{\theta^\varepsilon(t)\}$ is compact in $L^2(\Omega)$, for any $t \in [0,T]$. Hence it follows that $\theta^\varepsilon(t) \to \widetilde{\theta}(t)$ strongly in $L^2(\Omega)$, $\forall t \in [0,T]$ and consequently

$$\theta^\varepsilon(T) \to \widetilde{\theta}(T) \text{ strongly in } L^2(\Omega), \tag{49}$$

where $\widetilde{\theta}$ is the solution to variational inequality (11)-(12) with $u = \widetilde{u}$.

From these relationships we get

$$\begin{aligned} \lim_{\varepsilon\to 0} \sup \int_\Omega (\theta_\varepsilon(x,T) - \theta^0(x))^2 dx \leq \lim_{\varepsilon\to 0} \sup \int_\Omega (\theta^\varepsilon(x,T) - \theta^0(x))^2 dx \leq \\ \leq \int_\Omega (\widetilde{\theta}(x,T) - \theta^0(x))^2 dx = \min(P). \end{aligned} \tag{50}$$

On the other hand, since $\{(u_{0\varepsilon}, q_\varepsilon)\} \subset U_T$, there exists a subsequence of $\{(u_{0\varepsilon}, q_\varepsilon)\}$, still denoted in the same way, such that

$$u_{0\varepsilon} \to u_0^* \text{ weak star in } L^\infty(\Gamma_u),$$

$$q_\varepsilon \to q^* \text{ weak star in } L^\infty(\Sigma_u),$$

$$u_\varepsilon \to u^* \text{ weak star in } W^{1,\infty}(0,T;L^\infty(\Gamma_u)),$$

where u_ε is given by (44) with $(u_{0\varepsilon}, q_\varepsilon)$.

Then, it follows that on a subsequence of $\{\theta_\varepsilon\}$, still denoted in the same way, $\theta_\varepsilon \to \theta^*$ strongly in $L^2(Q)$, where θ^* is the solution to variational inequality (11)-(12) with $u = u^*$.

Moreover, in a same way as before, using (48) applied to $\theta_\varepsilon(t)$, we have $\theta_\varepsilon(T) \to \theta^*(T)$ strongly in $L^2(\Omega)$.

Therefore we obtain that

$$\min(P) \leq \int_\Omega (\theta^*(x,T) - \theta^0(x))^2 dx \leq \lim_{\varepsilon\to 0} \inf \int_\Omega (\theta_\varepsilon(x,T) - \theta^0(x))^2 dx. \tag{51}$$

By (50) and (51) we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega (\theta_\varepsilon(x,T) - \theta^0(x))^2 dx = \int_\Omega (\theta^*(x,T) - \theta^0(x))^2 dx = \min(P_T).$$

This completes the proof.

5. The necessary conditions of optimality for the approximating problem

Proposition 5.1. *Assume that* θ_0, f *and* f_0 *satisfy conditions of Theorem* 3.4 *and* β_ε^* *is given by* (14). *Let* $((u_{0\varepsilon}^*, q^*), \theta_\varepsilon^*)$ *be optimal for approximating problem* (P_ε). *Then*

$$\begin{cases} u_{0\varepsilon}^*(x) = -R(x,0) & \text{on } [p_\varepsilon(x_1,x_2,0,t) > 0], \\ u_{0\varepsilon}^*(x) \in [-R(x,0),0] & \text{on } [p_\varepsilon(x_1,x_2,0,t) = 0], \\ u_{0\varepsilon}^*(x) = 0 & \text{on } [p_\varepsilon(x_1,x_2,0,t) < 0] \end{cases} \tag{52}$$

and

$$\begin{cases} q_\varepsilon^*(x,t) = a(x) & \text{on } \left[\int_t^T p_\varepsilon(x,s)ds > 0\right], \\ q_\varepsilon^*(x,t) \in [a(x),b(x)] & \text{on } \left[\int_t^T p_\varepsilon(x,s)ds = 0\right], \\ q_\varepsilon^*(x,t) = b(x) & \text{on } \left[\int_t^T p_\varepsilon(x,s)ds < 0\right]. \end{cases} \tag{53}$$

Proof. Let $(u_{0\varepsilon}^*, q_\varepsilon^*) \in U_T$. It follows then that

$$u_\varepsilon^*(x,t) = u_{0\varepsilon}^*(x) + \int_0^t q_\varepsilon^*(x,s)ds \tag{54}$$

that implies obviously that $u_\varepsilon^* \in W^{1,2}(0,T;L^\infty(\Sigma_u))$.

We introduce the variation $(u_{0\varepsilon}^\lambda, q_\varepsilon^\lambda)$

$$\begin{aligned} u_{0\varepsilon}^\lambda(t,x) &= u_{0\varepsilon}^*(t,x) + \lambda(v_{0\varepsilon}(t,x) - u_{0\varepsilon}^*(t,x)), \\ q_\varepsilon^\lambda(t,x) &= q_\varepsilon^*(t,x) + \lambda(q_\varepsilon(t,x) - q_\varepsilon^*(t,x)) \end{aligned}$$

for $(v_{0\varepsilon}, q_\varepsilon) \in U_T$, $\lambda > 0$. Consequently we get

$$u_\varepsilon^\lambda(t,x) = u_\varepsilon^*(t,x) + \lambda(v_\varepsilon(t,x) - u_\varepsilon^*(t,x)),$$

where

$$v_\varepsilon(x,t) = v_{0\varepsilon}(x) + \int_0^t q_\varepsilon(x,s)ds.$$

We define Y_ε by

$$Y_\varepsilon = \lim_{\lambda\to 0} \frac{\theta_\varepsilon^{u_\varepsilon^*+\lambda w_\varepsilon} - \theta_\varepsilon^{u_\varepsilon^*}}{\lambda}, \text{ where } w_\varepsilon = v_\varepsilon - u_\varepsilon^*$$

and we write the system in variations

$$\frac{\partial Y_\varepsilon}{\partial t} - \Delta(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon) + \frac{\partial}{\partial x_3}\,(K'(\theta_\varepsilon^*)Y_\varepsilon) = 0 \text{ in } Q, \tag{55}$$

$$Y_\varepsilon(x,0) = 0 \text{ in } \Omega, \tag{56}$$

$$(K_\varepsilon'(\theta_\varepsilon^*)Y_\varepsilon i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon)) \cdot \nu = w_\varepsilon \text{ on } \Sigma_u, \tag{57}$$

$$(K(\theta_\varepsilon^*)Y_\varepsilon i_3 - \nabla(\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon)) \cdot \nu = \alpha\beta_\varepsilon(\theta_\varepsilon^*)Y_\varepsilon \text{ on } \Sigma_\alpha. \tag{58}$$

Under the hypotheses of Theorem 3.4 the problem (55)-(58) has a unique solution

$$Y_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V) \tag{59}$$

$$\frac{dY_\varepsilon}{dt} \in L^2(0,T;V'). \tag{60}$$

Then, we set as the dual system

$$\frac{\partial p_\varepsilon}{\partial t} + \beta_\varepsilon(\theta_\varepsilon^*)\Delta p_\varepsilon + K'(\theta_\varepsilon^*)\frac{\partial p_\varepsilon}{\partial x_3} = 0 \text{ in } Q, \tag{61}$$

$$p_\varepsilon(x,T) = -(\theta_\varepsilon^*(x,T) - \theta^0(x)) \text{ in } \Omega, \tag{62}$$

$$\nabla p_\varepsilon \cdot \nu = 0 \text{ on } \Sigma_u, \tag{63}$$

$$\alpha p_\varepsilon + \nabla p_\varepsilon \cdot \nu = 0 \text{ on } \Sigma_\alpha. \tag{64}$$

Under the same assumptions as before, the dual system has a unique solution

$$p_\varepsilon \in C([0,T];L^2(\Omega)) \cap L^2(0,T;V) \tag{65}$$

$$\frac{dp_\varepsilon}{dt} \in L^2(0,T;V'). \tag{66}$$

The proofs of these two results are done in a similar way as in [9].

Now we multiply equation (61) by p_ε and integrate over Q. Taking into account the conditions for Y_ε and (61)-(64) we obtain

$$\int_\Omega p_\varepsilon(x,T)Y_\varepsilon(x,T)dx + \int_0^T \int_{\Gamma_u} p_\varepsilon w_\varepsilon d\sigma dt = 0. \tag{67}$$

By the assumption that $(u_\varepsilon^*, \theta_\varepsilon^*)$ is optimal we have

$$\int_\Omega (\theta_\varepsilon^\lambda(x,T) - \theta^0)^2 dx \geq \int_\Omega (\theta_\varepsilon^*(x,T) - \theta^0)^2 dx$$

and from here we deduce that

$$\int_\Omega Y_\varepsilon(x,T)(\theta_\varepsilon^*(x,T) - \theta^0)dx \geq 0. \tag{68}$$

Hence from (62), (67) and (68) we deduce the condition

$$\int_0^T \int_{\Gamma_u} p_\varepsilon w_\varepsilon d\sigma dt = -\int_\Omega p_\varepsilon(x,T)Y_\varepsilon(x,T)dx \geq 0,$$

or, in fact

$$\int_0^T \int_{\Gamma_u} (-p_\varepsilon)(u_\varepsilon^* - v_\varepsilon)d\sigma dt \geq 0. \tag{69}$$

Therefore we obtain

$$\int_0^T \int_{\Gamma_u} (-p_\varepsilon)[u_{0\varepsilon}^* - v_{0\varepsilon} + \int_0^t (q_\varepsilon^*(x,s) - q_\varepsilon(x,s))ds]d\sigma dt \geq 0.$$

By few calculations we deduce

$$\begin{aligned} &\int_0^T \int_{\Gamma_u} (-p_\varepsilon(x,t))(u_{0\varepsilon}^*(x) - v_{0\varepsilon}(x))d\sigma dt + \\ &+ \int_0^T \int_{\Gamma_u} (q_\varepsilon^*(x,s) - q_\varepsilon(x,s)) \int_s^T (-p_\varepsilon(x,t))dt d\sigma ds \geq 0, \ \forall (v_{0\varepsilon}, q_\varepsilon) \in U_T. \end{aligned} \tag{70}$$

From this last relationship we conclude that

$$\left(-p_\varepsilon(x,t), \int_s^T (-p_\varepsilon(x,t))dt\right) \in \partial I_{U_T}(u_{0\varepsilon}^*, q_\varepsilon^*) = N_{U_T}(u_0^*, q_\varepsilon^*). \tag{71}$$

Hence, if $q_\varepsilon(x,s) = q_\varepsilon^*(x,s)$ it follows that

$$-p_\varepsilon \in \partial I_{K_0}(u_{0\varepsilon}^*) = N_{K_0}(u_{0\varepsilon}^*)$$

where

$$K_0 = \{y \in L^\infty(\Gamma_u); -R(x,0) \leq y(x) \leq 0 \text{ a.e. on } \Gamma_u\}.$$

It follows that

$$\begin{cases} -p_\varepsilon(x,t)) < 0 \text{ on } \Sigma_u & \text{if } u^*_{0\varepsilon} = -R(x,0), \\ -p_\varepsilon(x,t) = 0 \text{ on } \Sigma_u & \text{if } u^*_{0\varepsilon} \in (-R(x,0),0), \\ -p_\varepsilon(x,t) > 0 \text{ on } \Sigma_u & \text{if } u^*_{0\varepsilon} = 0. \end{cases}$$

From here we get the optimal initial data for the control as

$$\begin{cases} u^*_{0\varepsilon}(x) = -R(x,0) & \text{on } [p_\varepsilon(x_1,x_2,0,t) > 0] \\ u^*_{0\varepsilon}(x) \in [-R(x,0),0] & \text{on } [p_\varepsilon(x_1,x_2,0,t) = 0] \\ u^*_{0\varepsilon}(x) = 0 & \text{on } [p_\varepsilon(x_1,x_2,0,t) < 0]. \end{cases} \tag{72}$$

Then, if $v_{0\varepsilon} = u^*_{0\varepsilon}$, we deduce from (70) that

$$\int_s^T (-p_\varepsilon(x,t))dt \in \partial I_{K_T}(q^*_\varepsilon) = N_{K_T}(q^*_\varepsilon), \tag{73}$$

where

$$K_T = \{q \in L^2(\Sigma_u); a(x) \leq q(x,t) \leq b(x) \text{ a.e. on } \Sigma_u\}.$$

Therefore we have

$$\begin{cases} \displaystyle\int_s^T (-p_\varepsilon(x,t))dt < 0 & \text{on } \Sigma_u & \text{if } q^*_\varepsilon(x,s) = a(x), \\ \displaystyle\int_s^T (-p_\varepsilon(x,t))dt = 0 & \text{on } \Sigma_u & \text{if } q^*_\varepsilon(x,s) \in (a(x),b(x)), \\ \displaystyle\int_s^T (-p_\varepsilon(x,t))dt > 0 & \text{on } \Sigma_u & \text{if } q^*_\varepsilon(x,s) = b(x). \end{cases}$$

Eventually we obtain from (54) that on Σ_u we have

$$\begin{cases} q^*_\varepsilon(x,t) = a(x) & \text{on } \left[\displaystyle\int_t^T p_\varepsilon(x,s)ds > 0\right], \\ q^*_\varepsilon(x,t) \in [a(x),b(x)] & \text{on } \left[\displaystyle\int_t^T p_\varepsilon(x,s)ds = 0\right], \\ q^*_\varepsilon(x,t) = b(x) & \text{on } \left[\displaystyle\int_t^T p_\varepsilon(x,s)ds < 0\right]. \end{cases} \tag{74}$$

Correspondingly, we compute u^*_ε from (72) and (74).

By (72) and the uniqueness of the Cauchy problem (61)-(64) it follows that the set $\{(x_1,x_2,t); p_\varepsilon(x_1,x_2,0,t)\}$ is susceptible to have an empty interior. Roughly speaking this means that the initial optimal control $u^*_{0\varepsilon}$ might follows to be almost *bang-bang*.

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