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Remarks on the Hilbert Depth of Powers of the Maximal Graded Ideal

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Abstract. Let $\mathbf{m} = (x_1, \dots, x_n)$ be the maximal graded ideal of $S := K[x_1, \dots, x_n]$. We present a new method for computing the Hilbert depth of $\mathbf{m}^t$, using the polarization and a combinatorial characterization of the Hilbert depth of a quotient of squarefree monomial ideals.

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1. Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i]$ as a $\mathbb{Z}^n$-graded K-vector space, where $m_i \in M$ is homogeneous with respect to $\mathbb{Z}^n$-grading, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i] = \{u m_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$-submodule of M. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\dots,r} |Z_i|$ and

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) |\ \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vladoiu and Zheng show in [7] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. In [1], J. Apel restated a conjecture firstly given by Stanley in [9], namely that

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M),$$

for any $\mathbb{Z}^n$-graded S-module M. This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $0 \neq I \subset J \subset S$ are monomial ideals, see [6], but remains open for $M = I$. For a friendly introduction in the thematic of Stanley depth, we refer to [8].

Stanley depth is an important combinatorial invariant and deserves a thorough study. The explicit computation of the Stanley depth is a difficult task, even in some, seemingly, very simple cases as the maximal graded ideal $\mathbf{m} = (x_1, \dots, x_n)$ of S, see [2]. Let $t \geq 1$ be an integer. In [5, Theorem 2.2] it was proved that $\operatorname{sdepth}(\mathbf{m}^t) \leq \left\lceil \frac{n}{t+1} \right\rceil$. Also, in [5] it was conjectured that $\operatorname{sdepth}(\mathbf{m}^t) = \left\lceil \frac{n}{t+1} \right\rceil$ for

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any $t \geq 1$. This conjecture holds for $t = 1$, see [2, Theorem 2.2], and, also, for $t \geq n-1$, but it is open in general.

Let M be a finitely generated graded S-module. The Hilbert depth of M, denoted by $\operatorname{hdepth}(M)$, is the maximal depth of a finitely generated graded S-module N with the same Hilbert series as M. Bruns et al. [3] proved that $\operatorname{hdepth}(\mathbf{m}^t) = \left\lceil \frac{n}{t+1} \right\rceil$ for any $t \geq 1$. In [4] we proved a new formula for the Hilbert depth of a quotient J/I of two squarefree monomial ideals $I \subset J \subset S$, see Theorem 2.1. Also, we extended this method, through polarization, to a quotient J/I of two arbitrary monomial ideals, see Proposition 2.2. The aim of this note is to study the Hilbert depth of $\mathbf{m}^t$, where $t \geq 1$ is an integer, from this new perspective.

In Theorem 3.7 we prove that $\operatorname{hdepth}(\mathbf{m}^t) \leq \left\lceil \frac{n}{t+1} \right\rceil$ for any $t \geq 1$. We deduce that $\operatorname{hdepth}(\mathbf{m}) = \operatorname{sdepth}(\mathbf{m}) = \left\lceil \frac{n}{2} \right\rceil$, see Corollary 3.8. In Theorem 3.9 we prove that $\operatorname{hdepth}(\mathbf{m}^t) = 1$ for $t \geq n-1$ and $\operatorname{hdepth}(\mathbf{m}^2) = \left\lceil \frac{n}{3} \right\rceil$. Also, in Theorem 3.15 we show that $\operatorname{hdepth}(\mathbf{m}^t) = \left\lceil \frac{n}{t+1} \right\rceil$, if $n \leq (t+1)(t+3)$.

2. Preliminaries

First, we fix some notations and we recall the main result of [4].

Let K be an infinite field and $S := K[x_1, \ldots, x_n]$, the ring of polynomials in n variables over K. Let $I \subsetneq J \subset S$ be two square free monomial ideals. We consider the nonnegative integers

$$\alpha_k(J/I) := \#\{u \in S \;:\; u \text{ squarefree, with } u \in J \setminus I \text{ and } \deg(u) = k\},\ 0 \leq k \leq n.$$

For all $0 \leq d \leq n$ and $0 \leq k \leq d$, we consider the integers

$$\beta_k^d(J/I) := \sum_{j=0}^{k} (-1)^{k-j} \binom{d-j}{k-j} \alpha_j(J/I). \tag{1}$$

Note that, using an inverse formula, from (1) we deduce that

$$\alpha_k(J/I) := \sum_{j=0}^{k} \binom{d-j}{k-j} \beta_j^d(J/I). \tag{2}$$

With the above notations we have the following result:

Theorem 2.1. ([4, Theorem 2.4]) The Hilbert depth of J/I is

$$\operatorname{hdepth}(J/I) := \max\{d \;:\; \beta_k^d(J/I) \geq 0 \text{ for all } 0 \leq k \leq d\}.$$

Theorem 2.1 can be applied, indirectly, via polarization, in the non squarefree case, as follows. If $I \subsetneq J \subset S$ are two monomial ideals, then we consider their polarizations $I^p \subset J^p \subset R$, where R is a new ring of polynomials obtained from S by adding N new variables:

Proposition 2.2. The *Hilbert depth* of J/I is the number

$$\operatorname{hdepth}(J/I) := \operatorname{hdepth}(J^p/I^p) - N.$$

Also, as the Hilbert depth is an upper bound of the Stanley depth, in particular we have:

$$\operatorname{sdepth}(J/I) \leq \operatorname{hdepth}(J/I). \tag{3}$$

3. Main results

Let $n \geq 2$ and $S = K[x_1, \ldots, x_n]$. Let $\mathbf{m} = (x_1, \ldots, x_n) \subset S$ be the maximal graded ideal and $t \geq 1$ and integer. We denote by I_t, the polarization of the ideal $\mathbf{m}^t$, in the ring $R_t := K[x_1, \ldots, x_{nt}]$, that is

$$I_t = \left(\prod_{j=1}^{n} x_j x_{n+j} \cdots x_{n(i_j-1)+j} \ : \ i_1, \ldots, i_j \geq 0 \text{ with } i_1 + \cdots + i_n = t \right).$$

In other words, the polarization of x_j^a is $x_j x_{n+j} \cdots x_{n(a-1)+j}$, for any $1 \leq j \leq n$ and $a \geq 0$.

Lemma 3.1. With the above notations, we have

$$\alpha_k(R_t/I_t) = \sum_{j=0}^{t-1} \binom{nt-n-j}{k-j} \binom{n+j-1}{j} \text{ for all } 0 \leq k \leq nt.$$

Proof. Let $u = x_{i_1} x_{i_2} \cdots x_{i_k}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq nt$ be a (squarefree) monomial. If $i_1 \geq n+1$, i.e. $\{i_1, \ldots, i_k\} \subset \{n+1, \ldots, nt\}$, then $u \notin I_t$. Note that there are $\binom{nt-n}{k}$ such monomials. On the other hand, if $k \geq t$, $t \leq n$ and $i_t \leq n$, i.e. $\{i_1, \ldots, i_t\} \subset \{1, 2, \ldots, n\}$, then $u \in I_t$.

Now, assume that $\{i_1, \ldots, i_s\} \subset \{1, \ldots, n\}$ and $\{i_{s+1}, \ldots, i_k\} \subset \{n+1, \ldots, nt\}$, where $1 \leq s \leq t-1$. Note that $u \notin I_t$ if and only if there exists some nonnegative integers $a_1, \ldots, a_s$ such that $0 \leq \ell = a_1 + \cdots + a_s \leq t-1-s$,

$$L_{a_1,\ldots,a_s} := \bigcup_{j=1}^{s} \{i_j + n, i_j + 2n, \ldots, i_j + (a_j-1)n\} \subset \{i_{s+1}, \ldots, i_k\}$$
$$\text{and } \{i_{s+1}, \ldots, i_k\} \setminus L_{a_1,\ldots,a_s} \subset \{n, \ldots, nt\} \setminus \left(L_{a_1,\ldots,a_s} \cup \{i_1 + na_1, \ldots, i_s + na_s\}\right).$$

It follows that the number of such monomials $u \notin I_t$ is

$$\binom{n}{s} \cdot \sum_{\ell=0}^{t-1-s} \sum_{\substack{a_1,\ldots,a_s \geq 0 \\ a_1+\cdots+a_s=\ell}} \binom{nt-n-s-\ell}{k-s-\ell} = \binom{n}{s} \cdot \sum_{\ell=0}^{t-1-s} \binom{s+\ell-1}{s-1} \binom{nt-n-s-\ell}{k-s-\ell}.$$

Using the first part of the proof, it follows that

$$\alpha_k(R_t/I_t) = \binom{nt-n}{k} + \sum_{s=1}^{t-1} \binom{n}{s} \sum_{\ell=0}^{t-1-s} \binom{s+\ell-1}{s-1} \binom{nt-n-s-\ell}{k-s-\ell}. \tag{4}$$

Denoting $j = s + \ell$ in (4), we get

$$\alpha_k(R_t/I_t) = \binom{nt-n}{k} + \sum_{s=1}^{t-1} \binom{n}{s} \sum_{j=s}^{t-1} \binom{j-1}{s-1} \binom{nt-n-j}{k-j} = \binom{nt-n}{k} +$$
$$+ \sum_{j=1}^{t-1} \binom{nt-n-j}{k-j} \sum_{s=1}^{j} \binom{j-1}{s-1} \binom{n}{s} = \binom{nt-n}{k} + \sum_{j=1}^{t-1} \binom{nt-n-j}{k-j} \binom{n+j-1}{j},$$

as required. ∎

We state the following combinatorial formulas

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{n}{j} = \binom{n-d+k-1}{k} \text{ for all } 0 \leq k \leq d \leq n, \tag{5}$$

$$\sum_{\ell=0}^{k}(-1)^{k-\ell}\binom{n+\ell-1}{\ell}\binom{d}{k-\ell} = \binom{n-d+k-1}{k} \text{ for all } k,d,n \geq 0, \tag{6}$$

which can be easily deduced from the Chu-Vandermonde identity.

Lemma 3.2. For any $0 \leq k \leq d \leq nt$, we have that

$$\beta_k^d(R_t/I_t) = \sum_{\ell=0}^{t-1}\binom{n+\ell-1}{\ell}\binom{tn-n-d+k-\ell-1}{k-\ell}.$$

Proof. From Lemma 3.1 and (1) it follows that

$$\begin{aligned}\beta_k^d(R_t/I_t) &= \sum_{j=0}^{k}(-1)^{k-j}\binom{d-j}{k-j}\sum_{\ell=0}^{t-1}\binom{tn-n-\ell}{j-\ell}\binom{n+\ell-1}{\ell}\\ &= \sum_{\ell=0}^{t-1}\binom{n+\ell-1}{\ell}\sum_{j=\ell}^{k}(-1)^{k-j}\binom{d-j}{k-j}\binom{tn-n-\ell}{j-\ell}. \end{aligned}\tag{7}$$

Using the substitution $s = j-\ell$ in (7) and applying (5), we obtain

$$\begin{aligned}\beta_k^d(R_t/I_t) &= \sum_{\ell=0}^{t-1}\binom{n+\ell-1}{\ell}\sum_{s=0}^{k-\ell}(-1)^{k-\ell-s}\binom{d-\ell-s}{k-\ell-s}\binom{tn-n-\ell}{s} =\\ &= \sum_{\ell=0}^{t-1}\binom{n+\ell-1}{\ell}\binom{tn-n-d+k-\ell-1}{k-\ell},\end{aligned}$$

as required. □

Remark 3.3. As $\dim(S/\mathbf{m}^t) = 0$, from the definition of the Hilbert depth we have that $\operatorname{hdepth}(S/\mathbf{m}^t) = 0$. We mention that this result can be deduce also directly from Lemma 3.2: Indeed, if $d = tn-n$, according to Lemma 3.2 we have that

$$\beta_k^{tn-n}(R_t/I_t) = \sum_{\ell=0}^{t-1}\binom{n+\ell-1}{\ell}\binom{k-\ell-1}{k-\ell} = \begin{cases}\binom{n+k-1}{k}, & 0 \leq k \leq t-1\\ 0, & t \leq k \leq nt\end{cases}.$$

On the other hand, we have that

$$\beta_t^{tn-n+1}(R_t/I_t) = -\binom{n+t-2}{t-1} < 0.$$

Thus, $\operatorname{hdepth}(R_t/I_t) = nt-n$ and, therefore, $\operatorname{hdepth}(S/\mathbf{m}^t) = 0$.

Note that $\operatorname{depth}(S/\mathbf{m}^t) = \operatorname{sdepth}(S/\mathbf{m}^t) = 0$, since $\mathbf{m}$ is an associated prime to $S/\mathbf{m}^t$.

Proposition 3.4. For any $0 \leq k \leq d \leq nt$, we have that

$$\beta_k^d(I_t) = \binom{nt-d+k-1}{k} - \sum_{\ell=0}^{t-1}\binom{n+\ell-1}{\ell}\binom{tn-n-d+k-\ell-1}{k-\ell}.$$

Proof. Since $\alpha_j(I_t) = \binom{nt}{j} - \alpha_j(R_t/I_t)$ for all $0 \leq j \leq nt$, from (5) we get

$$\beta_k^d(I_t) = \binom{nt-d+k-1}{k} - \beta_k^d(R_t/I_t).$$

Hence, the conclusion follows from Lemma 3.2. ☐

Proposition 3.5. With the above notations, we have that:

(1) $\beta_{t+1}^{nt-n+\lceil \frac{n}{t+1} \rceil+1}(I_t) = \binom{n-\lceil \frac{n}{t+1} \rceil+t-1}{t+1} + \sum\limits_{\ell=0}^{t-1}(-1)^{t-\ell}\binom{n+\ell-1}{\ell}\binom{\lceil \frac{n}{t+1} \rceil+1}{t+1-\ell}$.

(2) $\beta_k^{nt-n+\lceil \frac{n}{t+1} \rceil}(I_t) = \binom{n-\lceil \frac{n}{t+1} \rceil+k-1}{k} - \sum\limits_{\ell=0}^{t-1}(-1)^{k-\ell}\binom{n+\ell-1}{\ell}\binom{\lceil \frac{n}{t+1} \rceil}{k-\ell}$, for all $t+1 \leq k \leq nt-n+\left\lceil \frac{n}{t+1} \right\rceil$.

Proof. (1) Let $d := nt - n + \left\lceil \frac{n}{t+1} \right\rceil + 1$. From Proposition 3.4 it follows that

$$\beta_{t+1}^d(I_t) = \binom{n - \left\lceil \frac{n}{t+1} \right\rceil + t - 1}{t+1} - \sum_{\ell=0}^{t-1} \binom{n+\ell-1}{\ell}\binom{-\left\lceil \frac{n}{t+1} \right\rceil - 1 + t - \ell}{t+1-\ell}. \quad (8)$$

On the other hand, we have that

$$\binom{-\left\lceil \frac{n}{t+1} \right\rceil - 1 + t - \ell}{t+1-\ell} = (-1)^{t-1-\ell}\binom{\left\lceil \frac{n}{t+1} \right\rceil + 1}{t+1-\ell}. \quad (9)$$

From (8) and (9) it follows that

$$\beta_{t+1}^d(I_t) = \binom{n - \left\lceil \frac{n}{t+1} \right\rceil + t - 1}{t+1} - \sum_{\ell=0}^{t-1}(-1)^{t+1-\ell}\binom{n+\ell-1}{\ell}\binom{\left\lceil \frac{n}{t+1} \right\rceil + 1}{t+1-\ell},$$

as required.

(2) The proof is similar to the proof of (1). ☐

Remark 3.6. In order to prove that $\operatorname{hdepth}(I_t) = nt - n + \left\lceil \frac{n}{t+1} \right\rceil$, it suffice to show that

$$\beta_{t+1}^{nt-n+\lceil \frac{n}{t+1} \rceil+1}(I_t) < 0 \text{ and } \beta_k^{nt-n+\lceil \frac{n}{t+1} \rceil}(I_t) \geq 0 \text{ for all } t+1 \leq k \leq nt-n+\left\lceil \frac{n}{t+1} \right\rceil. \quad (10)$$

Indeed, from $\beta_{t+1}^{nt-n+\lceil \frac{n}{t+1} \rceil+1}(I_t)$ it follows that $\operatorname{hdepth}(I_t) \leq nt - n + \left\lceil \frac{n}{t+1} \right\rceil$. Also, since $\beta_k^{nt-n+\lceil \frac{n}{t+1} \rceil}(I_t) = 0$ for $k \leq t-1$ and $\beta_t^{nt-n+\lceil \frac{n}{t+1} \rceil}(I_t) = \alpha_t(I_t) > 0$, (10) implies that $\operatorname{hdepth}(I_t) \geq nt - n + \left\lceil \frac{n}{t+1} \right\rceil$. Also, $\operatorname{hdepth}(I_t) = nt - n + \left\lceil \frac{n}{t+1} \right\rceil$ implies that $\operatorname{hdepth}(\mathbf{m}^t) = \left\lceil \frac{n}{t+1} \right\rceil$, since $I_t \subset R_t$ is obtained from $\mathbf{m}^t \subset S$ via polarization and $R_t = S[x_{n+1}, x_{n+2}, \ldots, x_{nt}]$.

Theorem 3.7. We have that $\operatorname{hdepth}(\mathbf{m}^t) \leq \left\lceil \frac{n}{t+1} \right\rceil$.

Proof. From (6), it follows that

$$\sum_{\ell=0}^{t+1}(-1)^{t+1-\ell}\binom{n-\ell+1}{\ell}\binom{\left\lceil \frac{n}{t+1}\right\rceil+1}{t+1-\ell}=\binom{n-\left\lceil \frac{n}{t+1}\right\rceil+t}{t+1}. \tag{11}$$

From (11) and Proposition 3.5(1) we henceforth get

$$\beta_{t+1}^{nt-n+\left\lceil \frac{n}{t+1}\right\rceil+1}(I_t)=-\binom{n-\left\lceil \frac{n}{t+1}\right\rceil+t-1}{t}-\binom{n+t-1}{t}\left(\left\lceil \frac{n}{t+1}\right\rceil+1\right)+\binom{n+t}{t+1}. \tag{12}$$

On the other hand, we have that

$$\binom{n+t}{t+1}-\binom{n+t-1}{t}\left(\left\lceil \frac{n}{t+1}\right\rceil+1\right)=\binom{n+t-1}{t}\left(\frac{n-1}{t+1}-\left\lceil \frac{n}{t+1}\right\rceil\right)<0. \tag{13}$$

From (12) and (13) it follows that

$$\beta_{t+1}^{nt-n+\left\lceil \frac{n}{t+1}\right\rceil+1}(I_t)<0,$$

and, therefore, as in Remark 3.6, it follows that $\operatorname{hdepth}(\mathbf{m}^t)\leq\left\lceil \frac{n}{t+1}\right\rceil$, as required. □

Corollary 3.8. We have that $\operatorname{hdepth}(\mathbf{m})=\left\lceil \frac{n}{2}\right\rceil$.

Proof. From Theorem 3.7 it follows that $\operatorname{hdepth}(\mathbf{m})\leq\left\lceil \frac{n}{2}\right\rceil$. On the other hand, from (3) and [2, Theorem 2.2], it follows that $\operatorname{hdepth}(\mathbf{m})\geq\operatorname{sdepth}(\mathbf{m})=\left\lceil \frac{n}{2}\right\rceil$. Hence, we are done. □

Theorem 3.9. With the above notations, we have that:
(1) $\operatorname{hdepth}(\mathbf{m}^t)=1$ for $t\geq n-1$.
(2) $\operatorname{hdepth}(\mathbf{m}^2)=\left\lceil \frac{n}{3}\right\rceil$.

Proof. First, note that, using Theorem 3.7, it suffice to show the $\geq$ inequality.

(1) Since $t\geq n-1$, that is $\left\lceil \frac{n}{t+1}\right\rceil=1$, from Proposition 3.5(2) it follows that

$$\beta_k^{nt-n+1}(I_t)=\binom{n+k-2}{k},\text{ for all }k\geq t+1. \tag{14}$$

From (14) and the fact that $\beta_k^{nt-n+1}(I_t)=0$ for $k\leq t-1$ and $\beta_t^{nt-n+1}(I_t)=\alpha_t(I_t)>0$, it follows that $\operatorname{hdepth}(I_t)\geq nt-n+1$. Therefore, $\operatorname{hdepth}(I_t)=nt-n$ and, as in Remark 3.6, this implies $\operatorname{hdepth}(\mathbf{m}^t)\geq 1$, as required.

(2) Since $t=2$, from Proposition 3.5(2) and the fact that $n-\left\lceil \frac{n}{3}\right\rceil=\left\lfloor \frac{2n}{3}\right\rfloor$, it follows that

$$\beta_k^{n+\left\lceil \frac{n}{3}\right\rceil}(I_2)=\binom{\left\lfloor \frac{2n}{3}\right\rfloor+k-1}{k}-(-1)^k\binom{\left\lceil \frac{n}{3}\right\rceil}{k}+(-1)^k n\binom{\left\lceil \frac{n}{3}\right\rceil}{k-1} \tag{15}$$

If $k\geq\left\lceil \frac{n}{3}\right\rceil+1$ then, from (15), it follows that

$$\beta_k^{n+\left\lceil \frac{n}{3}\right\rceil}(I_2)=\binom{\left\lfloor \frac{2n}{3}\right\rfloor+k-1}{k}>0.$$

Also, if $k=\left\lceil \frac{n}{3}\right\rceil$ then, from (15) and the fact that $n\geq 2$, it follows that

$$\beta_k^{n+\left\lceil \frac{n}{3}\right\rceil}(I_2)=\binom{n-1}{\left\lceil \frac{n}{3}\right\rceil}+(-1)^{\left\lceil \frac{n}{3}\right\rceil}\geq 0.$$

Now, assume that $k \leq \left\lceil \frac{n}{3} \right\rceil - 1$. From (15) we get

$$\beta_k^{n+\left\lceil \frac{n}{3} \right\rceil}(I_2) = \binom{\left\lfloor \frac{2n}{3} \right\rfloor + k - 1}{k} + (-1)^k \cdot \frac{nk - \left\lceil \frac{n}{3} \right\rceil + k - 1}{\left\lceil \frac{n}{3} \right\rceil - k + 1} \binom{\left\lceil \frac{n}{3} \right\rceil}{k}. \tag{16}$$

If k is even then, from (16) it follows that $\beta_k^{n+\left\lceil \frac{n}{3} \right\rceil}(I_2) > 0$, hence, the only case needed to be considered is k is odd and $3 \leq k \leq \left\lceil \frac{n}{3} \right\rceil - 1$. If $n \leq 9$ then there is nothing to prove, so we can assume that $n \geq 10$. In order to show that $\beta_k^{n+\left\lceil \frac{n}{3} \right\rceil}(I_2) \geq 0$, by (16), it suffice to prove that

$$\left(\left\lfloor \frac{2n}{3} \right\rfloor + k - 1\right)\left(\left\lfloor \frac{2n}{3} \right\rfloor + k - 2\right) \cdots \left\lfloor \frac{2n}{3} \right\rfloor \geq \left(nk - \left\lceil \frac{n}{3} \right\rceil + k - 1\right)\left\lceil \frac{n}{3} \right\rceil \cdots \left(\left\lceil \frac{n}{3} \right\rceil - k + 2\right). \tag{17}$$

In order to prove (17), we use induction on $k \geq 3$. If $k = 3$, then (17) became

$$\left(\left\lfloor \frac{2n}{3} \right\rfloor + 2\right)\left(\left\lfloor \frac{2n}{3} \right\rfloor + 1\right)\left\lfloor \frac{2n}{3} \right\rfloor \geq \left(3n + 2 - \left\lceil \frac{n}{3} \right\rceil\right)\left\lceil \frac{n}{3} \right\rceil \left(\left\lceil \frac{n}{3} \right\rceil - 1\right). \tag{18}$$

We consider three cases:

(i) $n = 3p$. Equation (18) is equivalent to

$$(2p + 2)(2p + 1)2p \geq (8p + 2)p(p - 1) \Leftrightarrow 8p^3 + 12p^2 + 4p \geq 8p^3 + 10p^2 - 2p,$$

which is obviously true.

(ii) $n = 3p + 1$. Equation (18) is equivalent to

$$(2p + 2)(2p + 1)2p \geq (8p + 4)(p + 1)p \Leftrightarrow 8p^3 + 12p^2 + 4p \geq 8p^3 + 12p^2 4p,$$

which is also true.

(iii) $n = 3p + 2$. Equation (18) is equivalent to

$$(2p+3)(2p+2)(2p+1) \geq (8p+6)(p+1)p \Leftrightarrow 8p^3 + 24p^2 + 22p + 6 \geq 8p^3 + 14p^2 + 6p,$$

which is again true.

Hence, the initial step of the induction is done. In order to prove the induction step, assume (17) holds for k. We have to show that it holds also for $k+2$. In order to do that, it suffice to prove that

$$\left(\left\lfloor \frac{2n}{3} \right\rfloor + k + 1\right)\left(\left\lfloor \frac{2n}{3} \right\rfloor + k\right)\left(nk - \left\lceil \frac{n}{3} \right\rceil + k - 1\right) \geq$$
$$\geq \left(\left\lceil \frac{n}{3} \right\rceil - k + 1\right)\left(\left\lceil \frac{n}{3} \right\rceil - k\right)\left(nk + 2n - \left\lceil \frac{n}{3} \right\rceil + k + 1\right).$$

This can be proved, by straightforward computations, in a similar manner as (18).

Now, from all the above considerations, it follows that

$$\beta_k^{n+\left\lceil \frac{n}{3} \right\rceil}(I_2) \geq 0 \text{ for all } 0 \leq k \leq n + \left\lceil \frac{n}{3} \right\rceil,$$

and, therefore, $\operatorname{hdepth}(I_2) \geq n + \left\lceil \frac{n}{3} \right\rceil$. Thus, $\operatorname{hdepth}(\mathbf{m}^2) \geq \left\lceil \frac{n}{3} \right\rceil$, as required. □

Proposition 3.10. The following are equivalent:

(1) $\operatorname{hdepth}(\mathbf{m}^t) = \left\lceil \frac{n}{t+1} \right\rceil$.

(2) $\sum\limits_{j=0}^{k-t} (-1)^j \binom{n+k-j-1}{k-j} \binom{\left\lceil \frac{n}{t+1} \right\rceil}{j} \geq 0$ for all $t + 1 \leq k \leq nt - n + \left\lceil \frac{n}{t+1} \right\rceil$.

(3) $\sum_{j=0}^{k}(-1)^j\binom{k+t}{j}\binom{n-j}{m-j} \geq 0$ for all $t,k,m,n\geq 1$ such that

$m(t+1)+k-1 \leq n \leq (m+1)(t+1)+k-2$ and $1\leq k\leq nt-n-t+m$.

Proof. (1) $\Leftrightarrow$ (2). Note that, according to Theorem 3.7, we have that $\operatorname{hdepth}(\mathbf{m}^t) \leq \left\lceil \frac{n}{t+1} \right\rceil$.

From Proposition 3.5(2) and (6), using the substitution $j=\ell-t$, it follows that

$$\beta_k^{nt-n+\lceil \frac{n}{t+1}\rceil}(I_t) = \sum_{\ell=t}^{k}(-1)^{k-\ell}\binom{n+\ell-1}{\ell}\binom{\lceil \frac{n}{t+1}\rceil}{k-\ell}$$
$$= \sum_{j=0}^{k-t}(-1)^{j}\binom{n+k-j-1}{k-j}\binom{\lceil \frac{n}{t+1}\rceil}{j},$$

for all $t+1\leq k\leq nt-n+\left\lceil \frac{n}{t+1}\right\rceil$. Hence, the equivalence follows as in Remark 3.6.

(2) $\Rightarrow$ (3). It is clear that $m=\left\lceil \frac{n}{t+1}\right\rceil$, if and only if

$$m(t+1)\leq n\leq (m+1)(t+1)-1. \tag{19}$$

Now, let $n'=n+k-1$, $k'=k-t$. From (2) it follows that

$$\sum_{j=0}^{k-t}(-1)^{j}\binom{n+k-j-1}{k-j}\binom{\lceil \frac{n}{t+1}\rceil}{j} = \sum_{j=0}^{k'}(-1)^j\binom{n'-j}{k'+t-j}\binom{m}{j}. \tag{20}$$

On the other hand, we have that

$$\binom{n'-j}{k'+t-j}\binom{m}{j} = \frac{(n'-j)!m!}{(k'+t-j)!(n'-k-t)!j!(m-j)!} = \frac{(n'-m)!m!}{(k'+t)!(n'-k'-t)!}\times$$
$$\times\frac{(n'-j)!(k'+t)!}{(n'-m)!(m-j)!(k'+t-j)!j!} = \frac{\binom{n'}{k'+t}}{\binom{n'}{n-m}}\cdot\binom{k'+t}{j}\binom{n'-j}{n'-m}. \tag{21}$$

From (19), (20) and (21), be renaming n' with n and k' with k, we get the required conclusion. (3) $\Rightarrow$ (2). The proof is similar. $\square$

For $n,m,k,t\geq 1$, we denote

$$b(n,m,t,k) := \sum_{j=0}^{k}(-1)^j\binom{k+t}{j}\binom{n-j}{m-j}. \tag{22}$$

Corollary 3.11. Let $n,t\geq 1$ and $m=\left\lceil \frac{n}{t+1}\right\rceil$ such that

$$b(n+k-1,m,t,k)\geq 0 \text{ for all } 1\leq k\leq nt-n-t+m.$$

Then $\operatorname{hdepth}(\mathbf{m}^t)=m$.

Proof. It follows from Remark 3.6 and the proof of Proposition 3.10. $\square$

Lemma 3.12. We have that

$$b(n,m,t,k)=\binom{n-t-k}{m}, \text{ for all } 1\leq m\leq k.$$

Proof. Since $m \leq k$, according to (5), we have that

$$b(n,m,t,k) = (-1)^m \sum_{j=0}^{m} (-1)^{m-j} \binom{k+t}{k} \binom{n-j}{m-j} = \\ = (-1)^m \binom{k+t-n+m-1}{m} = \binom{n-t-k}{m},$$

as required. □

Let $n,m,t,k \geq 1$ and $0 \leq j \leq k$, such that $m \geq k+1$. We denote

$$f(n,m,t,k,j) := \binom{k+t}{j} \binom{n-j}{m-j}.$$

By straightforward computations, we get:

$$\frac{f(n,m,t,k,j)}{f(n,m,t,k,j+1)} = \frac{(n-k+j+1)(j+1)}{(m-k+j+1)(k+t-j)}. \tag{23}$$

From (23), it follows that

$$f(n,m,t,k,j) \geq f(n,m,t,k,j+1) \text{ if and only if} \\ n \geq m+k+t-2j-1+\frac{(m-k)(k+t+1)}{j+1}. \tag{24}$$

Since the function $\varphi(j) = m+k+t-2j-1+\frac{(m-k)(k+t+1)}{j+1}$ is decreasing, from (24) it follows that for $n \geq \varphi(0)$ we have that $f(n,m,t,k,j) \geq f(n,m,t,k,j+1)$ for all $0 \leq j \leq k-1$. This allows us to prove the following:

Lemma 3.13. Let $n,m,k,t \geq 1$ such that $n \geq m(t+1)+k-1$. Then:
(1) $b(n,m,t,1) \geq 0$.
(2) If $m \leq k+t$ then $b(n,m,t,k) \geq 0$.

Proof. First, note that

$$\varphi(0) = m+k+t-1+(m-k)(k+t+1) = m(t+1)+k-1+(k-1)(m-t-k).$$

Hence, since $n \geq m(t+1)+k-1$, we have that $n \geq \varphi(0)$ for $k=1$ or $m \geq k+t$. On the other hand, if $n \geq \varphi(0)$ then, according to a previous remark, we have that $f(n,m,t,k,j) \geq f(n,m,t,k,j+1)$, for all $0 \leq j \leq k-1$, and therefore

$$b(n,m,t,k) = (f(n,m,t,k,0) - f(n,m,t,k,1)) + (f(n,m,t,k,2) - f(n,m,t,k,3)) + \cdots \\ \geq 0.$$

□

Lemma 3.14. Let $n,m \geq 1$, $t \geq 3$ such that $m \geq t+3$ and $m(t+1)+1 \leq n \leq (m+1)(t+1)$. Then $b(n,m,t,2) \geq 0$.

Proof. We have that

$$b(n,m,t,2) = \binom{n-2}{m-2} \left(\frac{n(n-1)}{m(m-1)} - (t+2)\frac{n}{m} + \binom{t+2}{2} \right). \tag{25}$$

From hypothesis, we have that $\frac{n-1}{m-1} > \frac{n}{m} \geq t+1$ and $\frac{n}{m} \leq t+2$. From (25) it follows that

$$b(n,m,t,2) \geq \binom{n-2}{m-2} \cdot \left((t+1)^2 - (t+2)^2 + \frac{(t+2)(t+1)}{2} \right)$$
$$= \binom{n-2}{m-2} \cdot \left(\frac{1}{2}t^2 - \frac{1}{2}t - 2 \right).$$

Therefore, $b(n,m,t,2) \geq 0$, since $t \geq 3$. □

Now, we are able to prove the following result:

Theorem 3.15. Let $n,t \geq 1$ such that $n \leq (t+1)(t+3)$. Then $\operatorname{hdepth}(\mathbf{m}^t) = \left\lceil \frac{n}{t+1} \right\rceil$.

Proof. If $t=1$ then the conclusion follows from Corollary 3.8. Also, if $t=2$ then the conclusion follows from Theorem 3.9(2). Hence, we can assume that $t \geq 3$. Let $m = \left\lceil \frac{n}{t+1} \right\rceil$. Note that, $n \leq (t+1)(t+3)$ implies $m \leq t+3$. Also, $m(t+1) \leq n \leq m(t+1)+t$.

From Lemma 3.12 it follows that

$$b(n+k-1,m,t,k) = \binom{n-t-1}{k} \geq 0 \text{ for all } m \leq k \leq nt-n-t+m. \tag{26}$$

Now, suppose that $k < m$. From Lemma 3.13(1) we have that

$$b(n,m,t,1) \geq 0. \tag{27}$$

Also, from Lemma 3.14 we have that

$$b(n+1,m,t,2) \geq 0. \tag{28}$$

Hence, we can assume that $3 \leq k \leq m-1$. Since $m = t+3$ and $k \geq 3$ it follows that $m \leq k+t$. Therefore, from Lemma 3.13(2) it follows that

$$b(n-k+1,m,t,k) \geq 0 \text{ for all } 3 \leq k \leq m-1. \tag{29}$$

The conclusion follows from (26), (27), (28), (29) and Corollary 3.11. □

4. Conclusion

In [4] we introduced a new combinatorial method to compute the Hilbert depth of a quotient of two squarefree monomial ideals $I \subset J \subset S$. Also, we noted that, if I and J are not squarefree, we can reduce to the squarefree case via polarization. The aim of our paper is to illustrate this method in order to compute $\operatorname{hdepth}(\mathbf{m}^t)$, where $\mathbf{m} = (x_1,\ldots,x_n)$ is the maximal graded ideal of S and $t \geq 1$ is an integer. Although the formula for $\operatorname{hdepth}(\mathbf{m}^t)$ was already known in literature, see [3], out method is original and can be adapted to other classes of monomial ideals. Also, the description of the minimal set of monomial generators of the polarization of $\mathbf{m}^t$ is new.

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