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Pure Strategy Solutions in the Progressive Discrete Silent Duel with Quadratic Accuracy Symmetry and Shooting Uniform Jitter

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Abstract. A generalized class of the discrete game of timing is solved, where possible shooting moments are uniformly jittered. This is a finite zero-sum game defined on a symmetric lattice of the unit square. The game is a progressive discrete silent duel whose kernel is skew-symmetric, and the duelist having a single bullet shoots with quadratic accuracy. As the duel starts, possible shooting moments become denser by a geometric progression, where every following moment, apart from the duel beginning and end moments, is the partial sum of the respective geometric series. Due to the skew-symmetry, both the duelists have the same optimal strategies and the game optimal value is 0. The 3×3 duel always has a pure strategy solution, whichever the jitter is. As the duel becomes bigger, an open interval of pure strategy solution non-existence appears. The endpoints of the open interval are irrational. The 4×4 duel has three jitter intervals, within which it has a pure strategy solution, whose optimal strategies can be only either a jittered middle or three-quarters of the duel time span, and the duel end moment. Bigger duels have two jitter intervals, within which a single pure strategy solution exists, but a jittered middle of the duel time span is never optimal. The 4×4 duel has two open intervals of the jitter, within which it does not have a pure strategy solution. Bigger duels have just a single open interval of the jitter, where no pure strategy solution exists. The left endpoint of this interval depends on the number of possible shooting moments.

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1. Progressive discrete silent duels

Lack of observability is a common drawback in building mathematical models based on real-world data. Another side of the lack arises when data access is restricted. Another one arises from informational delays typical for time-lagged systems like those in economics, ecosystems, jurisprudence [3, 5, 25, 32]. Making decisions under such conditions is modeled by games of timing, where competitive interaction processes involve two or more intelligent participants (players) [8, 9, 12, 14]. Such games consider a time span of the finite duration during which the player must make a finite number of decisions of acting [30, 32, 15, 27, 28]. The decision is alternatively called a shot or shooting, and the possibility to make a decision is often figuratively called a bullet [7, 8, 4, 32]. The time span is usually standardized to unit segment [0; 1].

The most common games of timing involve two players. Such two-person games are often referred to as duels, where the players are called duelists [7, 8, 15, 32, 2].

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Duels feature economic [32, 9], social [18], ecological [3, 4], sportive [26], juristic and other conflict-competitive processes [1, 6, 10, 11], where the matter is the timing of innovation, adoption, response [13, 14, 20]. Duels are classified as silent and noisy duels, depending on information the duelist learns as the adversary acts (shoots or fires a bullet) [30, 31, 26, 18, 4].

Silent duels are divided into subclasses considering the number of bullets (the number of maximum possible shots) at duelists and their accuracy. The most studied subclass is when each of two duelists has exactly one bullet, and they have the same accuracy functions of time [23]. Such a silent duel is a zero-sum game, so its every solution is an optimal equilibrium with the same optimal payoffs of the duelists [32, 6, 15]. Despite the accuracy symmetry, rendering the silent duel into a game with a skew-symmetric kernel, where the optimal payoffs of the duelists are 0, the duelists may possess non-symmetric optimal strategies at certain conditions [16, 21, 29].

The duelist's accuracy function is a nondecreasing function of time [32, 15, 3, 4]. In a silent duel, being more complicated to study in comparison with a noisy duel [7, 2], it is unknown to the duelist whether a bullet was fired by the other duelist or not until the duel ends [14, 26, 27]. The duelist may obtain a greater payoff by firing as late as possible, but then the loss likelihood increases due to the other duelist may shoot first. If both the duelists shoot simultaneously, the payoff of each of them is 0 [30, 32, 15, 3]. The accuracy nonlinearity is important to reflect the duelist's nonlinear efforts and tension as the duel progresses. Typically, it can be modeled by quadratic accuracy functions $p_X(x) = x^2$ and $p_Y(y) = y^2$ of the first and second duelists, respectively, so that

$$p_X(0) = p_Y(0) = 0$$

and

$$p_X(1) = p_Y(1) = 1$$

in a silent duel as a zero-sum game [7, 15, 17]

$$\langle X,\ Y,\ K(x,\ y)\rangle \tag{1}$$

with kernel

$$K(x,\ y) = x^2 - y^2 + x^2y^2 \operatorname{sign}(y - x) \tag{2}$$

defined on unit square

$$X \times Y = [0;\ 1] \times [0;\ 1] \tag{3}$$

being the Cartesian product of the duel unit-standardized time spans (i. e., the product is the square of the span), where $x \in X$, $y \in Y$. Kernel (2) is skew-symmetric, i. e.

$$K(y,\ x) = y^2 - x^2 + y^2x^2 \operatorname{sign}(x - y) = -K(x,\ y). \tag{4}$$

As the duelist has a single bullet in game (1) with kernel (2) on (3), there is no reason for considering solutions in mixed strategies, if any, with non-singleton supports [25, 13, 14, 18, 19].

Duel (1) on square (3) by kernel (2) is infinite, so it is not always possible to solve it, even approximately. To have an easy-to-implement optimal strategy for a duelist, a discrete silent duel is considered instead, in which the duelist can shoot only at specified time moments (pure strategies) whose number is finite [16, 17, 21]. Therefore, the kernel of the discrete silent duel is defined on a finite subset of unit square (3). The subset includes the moments of the duel beginning $x = y = 0$ and duel end $x = y = 1$, where the sets of pure strategies of the duelists are identical [32, 15, 21]:

$$X = \{x_i\}_{i=1}^{N} = Y = \{y_j\}_{j=1}^{N} = T = \{t_q\}_{q=1}^{N} \subset [0;\ 1]$$
$$\text{by } \ t_q < t_{q+1} \ \ \forall\, q = \overline{1,\ N-1} \ \text{ and } \ t_1 = 0, \ \ t_N = 1 \ \text{ for } \ N \in \mathbb{N} \backslash \{1\}\,.$$

The discrete silent duel is an $N \times N$ matrix game whose payoff matrix is skew-symmetric due to (4). Any solution of this matrix game is of finite supports only [15, 3, 4, 6], so any solution of the discrete silent duel is computed and implemented far easier and faster than that in the case of infinite game (1). A pure strategy solution is alternatively called a saddle point (with the reference to the respective row and column of the payoff matrix) or optimal situation in the matrix game.

Another option of nonlinearity in duels is how moments $\{t_q\}_{q=1}^{N}$ of possible shooting are specified. As the duelist approaches to the end moment $t_N = 1$, the space between consecutive moments t_q and t_{q+1}, $q = \overline{1,\ N-1}$, may shorten due to the growing tension, responsibility, and urgency to shoot first. In other words, the density of the duelist's pure strategies must grow as the duel progresses. One of the patterns of the growth is such that the density grows in the geometrical progression [21, 22]. In this case, apart from the duel beginning and end moments, every following moment is the partial sum of the respective geometric series:

$$t_q = \sum_{l=1}^{q-1} 2^{-l} = \frac{2^{q-1}-1}{2^{q-1}} \ \text{ for } \ q = \overline{2,\ N-1}. \tag{5}$$

However, due to finite accuracy in measuring the distance between neighboring moments of possible shooting, the precise assignment by (5) is not always realizable in practice. This is modeled [24] by adding a time jitter ξ so that still $\{t_q\}_{q=2}^{N-1} \subset (0;\ 1)$:

$$t_q = \xi + \sum_{l=1}^{q-1} 2^{-l} = \xi + \frac{2^{q-1}-1}{2^{q-1}} \ \text{ for } \ q = \overline{2,\ N-1} \ \text{ and } \ \xi \in \left(-\frac{1}{2};\ \frac{1}{2^{N-2}}\right).$$

Then game (1) by kernel (2) defined on a finite lattice

$$X \times Y = \{x_i\}_{i=1}^{N} \times \{y_j\}_{j=1}^{N} =$$
$$= \left\{0,\ \left\{\xi + \frac{2^{i-1}-1}{2^{i-1}}\right\}_{i=2}^{N-1},\ 1\right\} \times \left\{0,\ \left\{\xi + \frac{2^{j-1}-1}{2^{j-1}}\right\}_{j=2}^{N-1},\ 1\right\} \subset$$
$$\subset [0;\ 1] \times [0;\ 1] \tag{6}$$

is a progressive discrete silent duel (PDSD) with quadratic accuracy symmetry and shooting uniform jitter. It is obvious that this duel solution depends on N and ξ.

The case of $\xi = 0$ is the known PDSD whose solutions were studied in [21, 22]. The pure strategy solution is situation

$$\{x_3,\ y_3\} = \left\{\frac{3}{4},\ \frac{3}{4}\right\} \tag{7}$$

in 4×4 PDSDs and bigger, and optimal situation (7) is single. The single solution of the 3×3 PDSD is

$$\{x_3,\ y_3\} = \{1,\ 1\}\,. \tag{8}$$

The most trivial duel is the 2×2 PDSD, whose solution is

$$\{x_2,\ y_2\} = \{1,\ 1\} \tag{9}$$

and it would not depend on the jitter (the two only possible shooting moments at the duelist are the very beginning and end of the duel, which are not affected by any specification of moments $\{t_q\}_{q=2}^{N-1}$).

Hence, the goal is to study pure strategy solutions in the PDSD on finite lattice (6) by

$$\xi \in \left(-\frac{1}{2};\ 0\right) \cup \left(0;\ \frac{1}{2^{N-2}}\right) = \left(-\frac{1}{2};\ \frac{1}{2^{N-2}}\right) \setminus \{0\}\,. \tag{10}$$

In addition, there should be determined all ξ from the open intervals in (10) such that no pure strategy solution exists. To achieve the goal, the most important preliminary remarks are first stated in Section 2 to simplify further localization of (pure strategy) saddle points and corresponding inferences. The trivial case with three possible actions (shooting moments) at the duelist is next studied in Section 3. The non-optimality of the moment following the very beginning of the PDSD is substantiated in Section 4. Separately from bigger duels, Section 5 first studies 4×4 PDSDs, and then bigger PDSDs are studied in Section 6. Finally, Section 7 summarizes the results of pure strategy solution existence in PDSDs with quadratic accuracy symmetry and shooting uniform jitter, whereupon the study is concluded along with its contribution and an outlook for expanding the subject.

2. Preliminary remarks

In fact, the PDSD with quadratic accuracy functions is a matrix game

$$\left\langle \{x_i\}_{i=1}^{N},\ \{y_j\}_{j=1}^{N},\ \mathbf{K}_N \right\rangle \tag{11}$$

by the duelists' pure strategy sets

$$X = \{x_i\}_{i=1}^{N} = \left\{0,\ \left\{\xi + \frac{2^{i-1}-1}{2^{i-1}}\right\}_{i=2}^{N-1},\ 1\right\} \subset [0;\ 1] \tag{12}$$

and

$$Y = \{y_j\}_{j=1}^{N} = \left\{0,\ \left\{\xi + \frac{2^{j-1}-1}{2^{j-1}}\right\}_{j=2}^{N-1},\ 1\right\} \subset [0;\ 1] \tag{13}$$

for

$$\xi \in \left(-\frac{1}{2};\ \frac{1}{2^{N-2}}\right),$$

and skew-symmetric payoff matrix

$$\mathbf{K}_N = [k_{ij}]_{N \times N} \quad \text{by}$$
$$k_{ij} = K(x_i,\ y_j) = x_i^2 - y_j^2 + x_i^2 y_j^2 \,\mathrm{sign}\,(y_j - x_i)\,. \tag{14}$$

The skew-symmetry of matrix (14) implies that

$$k_{ij} = -k_{ji} \ \ \forall\, i = \overline{1,\ N} \ \text{ and } \ \forall\, j = \overline{1,\ N}. \tag{15}$$

In the further consideration, the case with $\xi > 0$ will be called a positive jitter, and the case with $\xi < 0$ will be called a negative jitter. Time moment

$$t_q = \xi + \frac{2^{q-1} - 1}{2^{q-1}} \ \text{at} \ q \in \left\{\overline{2,\ N-1}\right\} \tag{16}$$

will be called positively ξ-jittered moment and negatively $|\xi|$-jittered moment by $\xi > 0$ and $\xi < 0$, respectively. If the sign of jitter ξ is uncertain, time moment (16) will be called just a ξ-jittered moment. In particular, moment $t_2 = \xi + \frac{1}{2}$ following the very beginning of the PDSD is called a ξ-jittered middle of the duel time span.

Whichever number N is, the first row of matrix (14) contains a negative entry:

$$K(x_1,\ y_N) = K(0,\ 1) = -1 = -K(1,\ 0)\,. \tag{17}$$

Therefore, the minimum of the first row does not exceed $-1 < 0$ and thus the game optimal value (which is 0) cannot be reached in this row. So, the first row of matrix (14) does not contain saddle points. Due to the skew-symmetry of matrix (14), the stated inference is immediately followed by that the first column does not contain saddle points either. Therefore, the duelist in any PDSD does not have an optimal strategy at the very beginning of the duel.

As only a zero entry of matrix (14) can be a saddle point, then a row containing a negative entry does not contain saddle points; neither does the respective column containing the positive entry. In the further consideration, only the inferences on saddle points in definite rows of matrix (14), which imply the same inferences on saddle points in respective columns, will be stated. Inasmuch as the case of $\xi = 0$ was exhaustively studied in [21], only saddle points in PDSDs for

$$\xi \in \left(-\frac{1}{2};\ \frac{1}{2^{N-2}}\right) \setminus \{0\}$$

will be studied.

Meanwhile, it is obvious that a nonnegative row contains a saddle point on the main diagonal of matrix (14). Furthermore, if a row contains only positive entries, except for the main diagonal entry, the zero entry on the main diagonal in this row is a single saddle point in the duel due to all the other $N-1$ rows of the respective column (which shares the zero entry with the row) contain negative entries and thus cannot contain saddle points.

3. Triviality

First, consider the trivial case with three possible actions (shooting moments) at the duelist. Herein, the duelist is allowed to act (shoot, make a decision, or fire one's single bullet) at the very beginning of the PDSD, at a ξ-jittered middle of the duel time span, and at the very end of the PDSD.

Theorem 1. In a 3×3 PDSD (11) by (12)—(14) and (10), situation

$$\{x_2,\ y_2\} = \left\{\xi + \frac{1}{2},\ \xi + \frac{1}{2}\right\} \tag{18}$$

is solely optimal by

$$\xi \in \left(\frac{\sqrt{2}-1}{2};\ \frac{1}{2} \right) \tag{19}$$

and situation

$$\{x_3,\ y_3\} = \{1,\ 1\} \tag{20}$$

is solely optimal only by

$$\xi \in \left(-\frac{1}{2};\ \frac{\sqrt{2}-1}{2} \right). \tag{21}$$

The PDSD at

$$\xi = \frac{\sqrt{2}-1}{2} \tag{22}$$

has four optimal situations: (18), (20),

$$\{x_2,\ y_3\} = \left\{ \xi + \frac{1}{2},\ 1 \right\}, \tag{23}$$

$$\{x_3,\ y_2\} = \left\{ 1,\ \xi + \frac{1}{2} \right\}. \tag{24}$$

Proof. As it was mentioned above, situation

$$\{x_1,\ y_1\} = \{0,\ 0\}$$

is never optimal in the duel. The respective payoff matrix of a PDSD (11) by (12) — (14) and (10) is

$$\mathbf{K}_3 = [k_{ij}]_{3\times 3} = \begin{bmatrix} 0 & -\left(\frac{1}{2}+\xi\right)^2 & -1 \\ \left(\frac{1}{2}+\xi\right)^2 & 0 & 2\cdot\left(\frac{1}{2}+\xi\right)^2 - 1 \\ 1 & -2\cdot\left(\frac{1}{2}+\xi\right)^2 + 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & -\xi^2 - \xi - \frac{1}{4} & -1 \\ \xi^2 + \xi + \frac{1}{4} & 0 & 2\xi^2 + 2\xi - \frac{1}{2} \\ 1 & -2\xi^2 - 2\xi + \frac{1}{2} & 0 \end{bmatrix}. \tag{25}$$

It is clearly seen from (25) that situation (18) is solely optimal if

$$k_{23} = K(x_2,\ y_3) = K\left(\xi + \frac{1}{2},\ 1 \right) = 2\xi^2 + 2\xi - \frac{1}{2} > 0. \tag{26}$$

Inequality (26) is true when

$$\xi \in \left(-\infty;\ \frac{-\sqrt{2}-1}{2} \right) \cup \left(\frac{\sqrt{2}-1}{2};\ \infty \right). \tag{27}$$

Since here $\frac{1}{2^{N-2}} = \frac{1}{2}$ and

$$\frac{-\sqrt{2}-1}{2} < -1 < -\frac{1}{2} < 0 < \frac{\sqrt{2}-1}{2} < \frac{1}{2}, \tag{28}$$

situation (18) is solely optimal by (19). If (22) is true, then $k_{23} = 0 = k_{32}$, payoff matrix (25) is

$$\mathbf{K}_3 = \begin{bmatrix} 0 & -\frac{1}{2} & -1 \\ \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{29}$$

and there are four optimal situations (18), (20), (23), (24) in this PDSD. Using (27) and (28), it is easy to see that if (21) is true, then $k_{23} < 0$ and $k_{32} > 0$, that is (20) is the single optimal situation. □

4. Non-optimality of the jittered middle

Theorem 2. Situation (18) is not optimal in an $N \times N$ PDSD (11) by (12)—(14) and (10) for $N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}$.

Proof. In the second row of matrix (14), the last column entry

$$k_{2N} = K\left(x_2,\ y_N\right) = K\left(\xi + \frac{1}{2},\ 1\right) = 2\xi^2 + 2\xi - \frac{1}{2} < 0 \tag{30}$$

if

$$\xi \in \left(\frac{-\sqrt{2}-1}{2};\ \frac{\sqrt{2}-1}{2} \right). \tag{31}$$

Using (28) and (10), inequality (30) by (31) means that situation (18) is not optimal if

$$-\frac{1}{2} < \xi < \frac{\sqrt{2}-1}{2} \tag{32}$$

by

$$\frac{\sqrt{2}-1}{2} \leqslant \frac{1}{2^{N-2}} \tag{33}$$

and situation (18) is not optimal if

$$-\frac{1}{2} < \xi < \frac{1}{2^{N-2}} \tag{34}$$

by

$$\frac{\sqrt{2}-1}{2} > \frac{1}{2^{N-2}}. \tag{35}$$

Inequality (33) is false for $N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}$. From inequality (35) it follows that

$$\sqrt{2} - 1 > 2^{3-N},$$

$$\log_2\left(\sqrt{2}-1\right) > 3 - N,$$

$$N > 3 - \log_2\left(\sqrt{2}-1\right). \tag{36}$$

Inasmuch as

$$5 > 3 - \log_2\left(\sqrt{2} - 1\right) > 4,$$

inequality (36) holds for $N \geqslant 5$ implying the non-optimality of situation (18). □

Obviously, Theorem 2 does not exclude the optimality of situation (18) in 4×4 PDSDs, nor does it confirm situation (18) is optimal in such duels. Nevertheless, Theorem 2 prompts to consider 4×4 PDSDs separately from bigger ones.

5. The 4×4 PDSD

Theorem 3. In a 4×4 PDSD (11) by (12) — (14) and (10), situation (18) is solely optimal by

$$\xi \in \left[\frac{\sqrt{2} - 1}{2};\ \frac{1}{4}\right). \tag{37}$$

Proof. Situation (18) is solely optimal in a 4×4 PDSD if the second row of the respective payoff matrix contains only positive entries, except for the main diagonal entry. In the second row of matrix (14) by $N = 4$, its first (column) entry

$$k_{21} = \left(\frac{1}{2} + \xi\right)^2 > 0, \tag{38}$$

and entries k_{23}, k_{24} are positive if

$$k_{2j} = K\left(x_2,\ y_j\right) = K\left(\xi + \frac{1}{2},\ y_j\right) =$$
$$= x_2^2 - y_j^2 + x_2^2 y_j^2 = x_2^2\left(1 + y_j^2\right) - y_j^2 > 0 \ \text{ for } \ j = 3 \ \text{ and } \ j = 4. \tag{39}$$

From inequality (39) it follows that

$$x_2^2 > \frac{y_j^2}{1 + y_j^2} = 1 - \frac{1}{1 + y_j^2}, \tag{40}$$

which means that if inequality (40) holds for $j = 4$ (a greater value of y_j), it holds for $j = 3$ (a lesser value of y_j) as well. At $j = 4$ pure strategy $y_4 = 1$ and inequality (40) turns into just

$$x_2^2 > \frac{1}{2},$$

i. e.

$$\left(\frac{1}{2} + \xi\right)^2 > \frac{1}{2},$$
$$\frac{1}{2} + \xi > \frac{1}{\sqrt{2}},$$
$$\xi > \frac{1}{\sqrt{2}} - \frac{1}{2}. \tag{41}$$

Since here $\dfrac{1}{2^{N-2}} = \dfrac{1}{4}$ and (10) must hold, condition (41) is written as

$$\xi \in \left(\frac{\sqrt{2} - 1}{2};\ \frac{1}{4}\right) \tag{42}$$

by which situation (18) is solely optimal.

If the jitter value is (22), then

$$k_{24} = K\left(x_2,\ y_4\right) = K\left(\xi + \frac{1}{2},\ 1\right) =$$

$$= x_2^2 - 1 + x_2^2 = 2x_2^2 - 1 = 2 \cdot \left(\frac{\sqrt{2}-1}{2} + \frac{1}{2}\right)^2 - 1 = 0 \tag{43}$$

and

$$k_{23} = K\left(x_2,\ y_3\right) = K\left(\xi + \frac{1}{2},\ \xi + \frac{3}{4}\right) = K\left(\frac{\sqrt{2}}{2},\ \frac{2\sqrt{2}+1}{4}\right) =$$

$$= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{2\sqrt{2}+1}{4}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 \cdot \left(\frac{2\sqrt{2}+1}{4}\right)^2 =$$

$$= \frac{1}{2} - \frac{8+4\sqrt{2}+1}{16} + \frac{1}{2} \cdot \frac{8+4\sqrt{2}+1}{16} = \frac{7-4\sqrt{2}}{32} > 0, \tag{44}$$

whence situation (18) is still optimal. Due to (44), $k_{32} = -k_{23} < 0$ and the third row does not contain saddle points. Next,

$$k_{34} = K\left(x_3,\ y_4\right) = K\left(\xi + \frac{3}{4},\ 1\right) =$$

$$= x_3^2 - 1 + x_3^2 = 2x_3^2 - 1 = 2 \cdot \left(\frac{\sqrt{2}-1}{2} + \frac{3}{4}\right)^2 - 1 =$$

$$= 2 \cdot \left(\frac{2\sqrt{2}+1}{4}\right)^2 - 1 = \frac{8+4\sqrt{2}+1}{8} - 1 = \frac{4\sqrt{2}+1}{8} > 0. \tag{45}$$

Due to (45), $k_{43} = -k_{34} < 0$ and the fourth row does not contain saddle points. Therefore, situation (18) remains solely optimal by (22) as well. □

As it will turn out, there exists a class of 4×4 PDSDs which do not have pure strategy solutions. The open interval of positive jitter values at which no pure strategy solution exists is revealed in the following assertion.

Theorem 4. The 4×4 PDSD (11) by (12) — (14) and (10) with a jitter value of

$$\xi \in \left(\xi_{23}^{(2)};\ \frac{\sqrt{2}-1}{2}\right), \tag{46}$$

where $\xi_{23}^{(2)}$ is irrational and it is the greater root of the two roots of equation

$$\xi^4 + \frac{5}{2}\xi^3 + \frac{37}{16}\xi^2 + \frac{7}{16}\xi - \frac{11}{64} = 0, \tag{47}$$

does not have pure strategy solutions.

Proof. It is clearly seen from (39) — (41) that $k_{24} < 0$ by

$$\xi < \frac{\sqrt{2}-1}{2}. \tag{48}$$

So, situation (18) is not optimal by (48) that includes (46). Entry

$$k_{23} = K(x_2,\ y_3) = K\left(\xi + \frac{1}{2},\ \xi + \frac{3}{4}\right) =$$
$$= \left(\xi + \frac{1}{2}\right)^2 - \left(\xi + \frac{3}{4}\right)^2 + \left(\xi + \frac{1}{2}\right)^2 \cdot \left(\xi + \frac{3}{4}\right)^2 =$$
$$= \xi^4 + \frac{5}{2}\xi^3 + \frac{37}{16}\xi^2 + \frac{7}{16}\xi - \frac{11}{64}. \qquad (49)$$

Denote the last term in (49) by a function

$$\varphi(\xi) = \xi^4 + \frac{5}{2}\xi^3 + \frac{37}{16}\xi^2 + \frac{7}{16}\xi - \frac{11}{64} \text{ by } -\infty < \xi < \infty. \qquad (50)$$

The first derivative of function (50) is

$$\frac{d\varphi(\xi)}{d\xi} = 4\xi^3 + \frac{15}{2}\xi^2 + \frac{37}{8}\xi + \frac{7}{16} \qquad (51)$$

and

$$4\xi^3 + \frac{15}{2}\xi^2 + \frac{37}{8}\xi + \frac{7}{16} = 0$$

only if

$$\xi = \xi_* = \frac{\sqrt[3]{864 + 3\sqrt{82941}} \cdot \left(\sqrt[3]{864 + 3\sqrt{82941}} - 15\right) + 3}{24\sqrt[3]{864 + 3\sqrt{82941}}} \qquad (52)$$

where

$$-0.1146 < \frac{\sqrt[3]{864 + 3\sqrt{82941}} \cdot \left(\sqrt[3]{864 + 3\sqrt{82941}} - 15\right) + 3}{24\sqrt[3]{864 + 3\sqrt{82941}}} < -0.1145.$$

The second derivative of function (50) is

$$\frac{d^2\varphi(\xi)}{d\xi^2} = 12\xi^2 + 15\xi + \frac{37}{8} \qquad (53)$$

and function (53) is positive at point (52):

$$3.1 > \left.\frac{d^2\varphi(\xi)}{d\xi^2}\right|_{\xi=\xi_*} > 3 > 0.$$

This means that (52) is the global minimum point being the single extremum of function (50). The value of function (50) at minimum point (52) is negative:

$$-0.196 < \varphi(\xi_*) < -0.195.$$

Meanwhile,

$$0.203 < \varphi(-1) < 0.204$$

and

$$6.078 < \varphi(1) < 6.079.$$

Consequently,

$$\varphi(\xi) < 0 \text{ by } -1 < \xi_{23}^{(1)} < \xi < \xi_{23}^{(2)} < 1,$$

where $\xi_{23}^{(1)}$ and $\xi_{23}^{(2)}$ are points of ξ at which function (50) turns into zero, $\xi_{23}^{(1)} < \xi_{23}^{(2)}$, i. e. they are the lesser and greater roots of equation (47).

By the rational root theorem, every rational root of polynomial (50) must be among the numbers

$$\left\{\pm\frac{1}{64},\ \pm\frac{1}{32},\ \pm\frac{1}{16},\ \pm\frac{1}{8},\ \pm\frac{1}{4},\ \pm\frac{1}{2},\ \pm 1,\right.$$
$$\left.\pm\frac{11}{64},\ \pm\frac{11}{32},\ \pm\frac{11}{16},\ \pm\frac{11}{8},\ \pm\frac{11}{4},\ \pm\frac{11}{2},\ \pm 11\right\}. \tag{54}$$

However, none of numbers (54) satisfies equality (47). Consequently, the two roots $\xi_{23}^{(1)}$ and $\xi_{23}^{(2)}$ of (47) are irrational. They can be accurately estimated by using, for instance, the bisection method [33, 34]. Since $\varphi(-1) > 0$ and $\varphi(1) > 0$ by $-1 < \xi_* < 1$, then the roots are within interval $(-1;\ 1)$. Due to

$$\varphi(0) = -\frac{11}{64} < 0,$$

lesser root $\xi_{23}^{(1)}$ is iteratively sought within interval $(-1;\ 0)$, whereupon

$$-0.62451173365014 < \xi_{23}^{(1)} < -0.62451173365013. \tag{55}$$

Greater root $\xi_{23}^{(2)}$ is iteratively sought within interval $(0;\ 1)$, whereupon

$$0.181665998459103 < \xi_{23}^{(2)} < 0.181665998459104. \tag{56}$$

So,

$$\xi_{23}^{(1)} < -\frac{1}{2} < 0 < \xi_{23}^{(2)} < \frac{\sqrt{2}-1}{2}. \tag{57}$$

Inequality (57) means that, locally,

$$\varphi(\xi) < 0 \ \text{ by } \ \xi \in \left(-\frac{1}{2};\ \xi_{23}^{(2)}\right) \subset \left(-\frac{1}{2};\ \frac{\sqrt{2}-1}{2}\right) \tag{58}$$

and $\varphi(\xi) > 0$ by (46). Inequality (58) implies that

$$k_{23} < 0 \ \text{ by } \ \xi \in \left(-\frac{1}{2};\ \xi_{23}^{(2)}\right) \tag{59}$$

and

$$k_{32} = -k_{23} > 0 \ \text{ by } \ \xi \in \left(-\frac{1}{2};\ \xi_{23}^{(2)}\right). \tag{60}$$

Inequality $\varphi(\xi) > 0$ by (46) implies that $k_{23} > 0$ and $k_{32} = -k_{23} < 0$ by (46), i. e. the third row of payoff matrix $\mathbf{K}_4$ does not contain saddle points.

In the fourth row, entry

$$k_{43} = K(x_4,\ y_3) = K\left(1,\ \xi + \frac{3}{4}\right) = 1 - \left(\xi + \frac{3}{4}\right)^2 - \left(\xi + \frac{3}{4}\right)^2 =$$
$$= -2\xi^2 - 3\xi - \frac{1}{8} > 0 \ \text{ by } \ \xi \in \left(-\frac{2\sqrt{2}+3}{4};\ \frac{2\sqrt{2}-3}{4}\right), \tag{61}$$

where

$$-\frac{2\sqrt{2}+3}{4} < -1 < -0.05 < \frac{2\sqrt{2}-3}{4} < 0. \tag{62}$$

So, $k_{43} < 0$ by

$$\xi > \frac{2\sqrt{2}-3}{4}. \tag{63}$$

This also means that $k_{43} < 0$ by (46) and the fourth row of payoff matrix $\mathbf{K}_4$ does not contain saddle points by (46) as well. □

As the moving-to-the-left jitter reaches irrational value $\xi_{23}^{(2)}$, at which the jitter is still positive, the 4×4 PDSD takes back the existence of a pure strategy solution. The respective solution is expectedly single within a half-interval, which is shown below.

Theorem 5. The 4×4 PDSD (11) by (12)—(14) and (10) with a jitter value of

$$\xi \in \left(\frac{2\sqrt{2}-3}{4};\ \xi_{23}^{(2)} \right], \tag{64}$$

where $\xi_{23}^{(2)}$ is irrational and it is the greater root of the two roots of equation (47), has the single pure strategy solution

$$\{x_3,\ y_3\} = \left\{ \xi + \frac{3}{4},\ \xi + \frac{3}{4} \right\}. \tag{65}$$

Proof. With a jitter value of (64), the second row of payoff matrix $\mathbf{K}_4$ does not contain saddle points due to $k_{24} < 0$ by (48), where (57) is true by

$$-0.05 < \frac{2\sqrt{2}-3}{4} < 0 < \xi_{23}^{(2)} < \frac{\sqrt{2}-1}{2}. \tag{66}$$

Due to $k_{43} < 0$ by (63), the fourth row of payoff matrix $\mathbf{K}_4$ does not contain saddle points by (64) as well. In the third row, $k_{34} = -k_{43} > 0$ by (63), $k_{32} > 0$ due to (60), and

$$k_{31} = K(x_3,\ y_1) = K\left(\xi + \frac{3}{4},\ 0 \right) = \left(\xi + \frac{3}{4} \right)^2 > 0. \tag{67}$$

So, the third row by

$$\xi \in \left(\frac{2\sqrt{2}-3}{4};\ \xi_{23}^{(2)} \right) \tag{68}$$

contains only positive entries, except for the main diagonal entry, and situation (65) is solely optimal by (68). At $\xi = \xi_{23}^{(2)}$ the third row contains another zero entry, $k_{32} = k_{23} = 0$, but the second row does not contain saddle points due to $k_{24} < 0$ by (48), where (57) is true by (66). □

Does situation (65) still remain optimal when the moving-to-the-left jitter reaches the left endpoint of half-interval (64)? The assertion below answers this question.

Theorem 6. The 4×4 PDSD (11) by (12)—(14) and (10) with a jitter value of

$$\xi = \frac{2\sqrt{2}-3}{4} \tag{69}$$

has four optimal situations: (65),

$$\{x_3,\ y_4\} = \left\{ \xi + \frac{3}{4},\ 1 \right\}, \tag{70}$$

$$\{x_4,\ y_3\} = \left\{1,\ \xi + \frac{3}{4}\right\}, \tag{71}$$

$$\{x_4,\ y_4\} = \{1,\ 1\}. \tag{72}$$

Proof. At (69), still $k_{24} < 0$, $k_{31} > 0$ due to (67), $k_{32} > 0$ due to (60), $k_{34} = -k_{43} = 0$, $k_{41} = -k_{14} = 1$, and $k_{42} = -k_{24} > 0$. So, the third and fourth rows of payoff matrix $\mathbf{K}_4$ have positive entries in the first and second columns, and zero entries in the third and fourth columns. This implies optimality of situations (65), (70) — (72). □

In fact, value (69) is a marginal jitter, at which the optimality of shooting at the third moment remains, but also drags in the end moment of the 4×4 PDSD as another optimal decision moment. As the jitter drops below its marginal value (69), the optimality of situation (65) vanishes along with the multiplicity of the best decision.

Theorem 7. The 4×4 PDSD (11) by (12) — (14) and (10) with a jitter value of

$$\xi \in \left(-\frac{1}{2};\ \frac{2\sqrt{2}-3}{4}\right) \tag{73}$$

has the single pure strategy solution (72).

Proof. With a jitter value of (73), the second row of payoff matrix $\mathbf{K}_4$ does not contain saddle points due to $k_{24} < 0$ by (48), and $k_{43} > 0$ due to (61), whence $k_{34} = -k_{43} < 0$ and the third row does not contain saddle points either. Besides, in the fourth row, $k_{41} = -k_{14} = 1$ and $k_{42} = -k_{24} > 0$. So, the fourth row contains only positive entries, except for the main diagonal entry, and situation (72) is solely optimal by (73). □

Thus, Theorems 3 — 7 have completely covered the case of when the duelist possesses four possible moments to shoot. PDSDs with greater number of possible shooting moments are considered below.

6. Bigger PDSDs with quadratic accuracy

6.1. The third moment optimality.

Theorem 8. In an $N \times N$ PDSD (11) by (12) — (14) and (10) for $N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\}$, situation (65) is solely optimal by

$$\xi \in \left[\frac{2\sqrt{2}-3}{4};\ \frac{1}{2^{N-2}}\right). \tag{74}$$

Proof. According with Theorem 2, situation (18) cannot be optimal here. In the third row of matrix (14), $k_{31} > 0$ due to (67). Next, inequality (56) holds and $\frac{1}{2^{N-2}} < \xi_{23}^{(2)}$ for $N \geqslant 5$, so $k_{32} > 0$ due to (60) ever in an $N \times N$ PDSD for $N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\}$. The last column entry

$$k_{3N} = K(x_3,\ y_N) = K\left(\xi + \frac{3}{4},\ 1\right) =$$

$$= 2x_3^2 - 1 = 2 \cdot \left(\xi + \frac{3}{4}\right)^2 - 1 > 0 \tag{75}$$

if

$$\xi \in \left(-\infty;\ -\frac{2\sqrt{2}+3}{4}\right) \cup \left(\frac{2\sqrt{2}-3}{4};\ \infty\right) \tag{76}$$

by using (61). So, $k_{3N} > 0$ by

$$\xi \in \left(\frac{2\sqrt{2}-3}{4};\ \frac{1}{2^{N-2}}\right). \tag{77}$$

Overall, the entries in the third row above the main diagonal are

$$k_{3j} = K\left(x_3,\ y_j\right) = x_3^2 - y_j^2 + x_3^2 y_j^2 = y_j^2\left(x_3^2 - 1\right) + x_3^2 \ \text{ for } \ j = \overline{4,\ N}. \tag{78}$$

As $x_3^2 - 1 < 0$, it is seen from (78) that entry k_{3j} is a decreasing function of y_j, where

$$k_{3j} > k_{3N} > 0 \ \text{ for } \ j = \overline{4,\ N-1}.$$

So, the third row contains only positive entries, except for the main diagonal entry, and situation (65) is solely optimal by (77).

At marginal value (69) of the jitter, the third row contains two zero entries — k_{33} and k_{3N}. In the last row, entry

$$k_{N,N-1} = K\left(x_N,\ y_{N-1}\right) = K\left(1,\ \xi + \frac{2^{N-2}-1}{2^{N-2}}\right) = K\left(1,\ \frac{2\sqrt{2}-3}{4} + \frac{2^{N-2}-1}{2^{N-2}}\right) =$$

$$= 1 - 2y_{N-1}^2 = 1 - 2\cdot\left(\frac{2\sqrt{2}-3}{4} + \frac{2^{N-2}-1}{2^{N-2}}\right)^2 < 0$$

if

$$y_{N-1}^2 = \left(\frac{2\sqrt{2}-3}{4} + \frac{2^{N-2}-1}{2^{N-2}}\right)^2 > \frac{1}{2},$$

whence

$$\frac{2\sqrt{2}-3}{4} + \frac{2^{N-2}-1}{2^{N-2}} > \frac{\sqrt{2}}{2},$$

$$\frac{2^{N-2}-1}{2^{N-2}} > \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}-3}{4} = \frac{2\sqrt{2}-2\sqrt{2}+3}{4} = \frac{3}{4},$$

which holds for $N \geqslant 5$ owing to

$$-\frac{1}{2^{N-2}} > -\frac{1}{4},\ \ \frac{1}{2^{N-2}} < \frac{1}{4},\ \ 2^{N-2} > 4,\ \ N > 4.$$

Consequently, $k_{N,N-1} < 0$ and the last row does not contain saddle points implying that situation (65) remains solely optimal by (69) as well. □

According to (66), value (69) is a negative jitter, and thus situation (65) remains solely optimal within relatively narrow half-interval (74) of negative and positive jitter. As the PDSD gets bigger, this half-interval gets narrower from the right side. It is inherently expected that a pure strategy optimal situation, if any, will tend to the duel end moment as the negative jitter tends to its maximum.

6.2. The end moment optimality.

Theorem 9. In an $N \times N$ PDSD (11) by (12) — (14) and (10) for $N \in \mathbb{N} \backslash \{1, 2, 3, 4\}$, situation

$$\{x_N, y_N\} = \{1, 1\} \tag{79}$$

is solely optimal by

$$\xi \in \left(-\frac{1}{2}; \frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}}\right]. \tag{80}$$

Proof. In the last row of payoff matrix $\mathbf{K}_N$, entry

$$k_{Nj} = K(x_N, y_j) = 1 - 2y_j^2 \text{ for } j = \overline{1, N-1} \tag{81}$$

is a decreasing function of y_j, where

$$k_{Nj} > 0 \ \forall j = \overline{1, N-2} \text{ if } k_{N,N-1} > 0. \tag{82}$$

Thus, entry

$$k_{N,N-1} = K(x_N, y_{N-1}) = K\left(1, \xi + \frac{2^{N-2}-1}{2^{N-2}}\right) =$$

$$= 1 - 2y_{N-1}^2 = 1 - 2 \cdot \left(\xi + \frac{2^{N-2}-1}{2^{N-2}}\right)^2 > 0 \tag{83}$$

if

$$y_{N-1}^2 = \left(\xi + \frac{2^{N-2}-1}{2^{N-2}}\right)^2 < \frac{1}{2},$$

whence

$$\xi + \frac{2^{N-2}-1}{2^{N-2}} < \frac{\sqrt{2}}{2},$$

$$\xi < \frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}}, \tag{84}$$

where

$$-\frac{1}{2} < \frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}} < \frac{2\sqrt{2}-3}{4} < 0 \text{ for } N \in \mathbb{N} \backslash \{1, 2, 3, 4\}. \tag{85}$$

Using (85), entry $k_{N,N-1} > 0$ by

$$\xi \in \left(-\frac{1}{2}; \frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}}\right) \text{ for } N \in \mathbb{N} \backslash \{1, 2, 3, 4\} \tag{86}$$

and the last row contains only positive entries, except for the main diagonal entry, and situation (79) is solely optimal by (86).

When

$$\xi = \frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}}, \tag{87}$$

entry

$$k_{N,N-1} = 0 = k_{N-1,N},$$

while still

$$k_{Nj} > 0 \ \forall j = \overline{1, N-2} \tag{88}$$

and there are no saddle points in the first $N-2$ rows of payoff matrix $\mathbf{K}_N$ by (88). Besides, in the $(N-1)$-th row, entry

$$k_{N-1,N-2} = K\left(x_{N-1},\ y_{N-2}\right) =$$
$$= K\left(\frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}} + \frac{2^{N-2}-1}{2^{N-2}},\ \frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}} + \frac{2^{N-3}-1}{2^{N-3}}\right) =$$
$$= K\left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2} - \frac{1}{2^{N-2}}\right) = = x_{N-1}^2 - y_{N-2}^2 - x_{N-1}^2 y_{N-2}^2 =$$
$$= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2} - \frac{1}{2^{N-2}}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 \cdot \left(\frac{\sqrt{2}}{2} - \frac{1}{2^{N-2}}\right)^2 =$$
$$= \frac{1}{2} - \frac{3}{2} \cdot \left(\frac{\sqrt{2}}{2} - \frac{1}{2^{N-2}}\right)^2 = \frac{1}{2} - \frac{3}{2} y_{N-2}^2 < 0 \tag{89}$$

if

$$y_{N-2}^2 = \left(\frac{\sqrt{2}}{2} - \frac{1}{2^{N-2}}\right)^2 > \frac{1}{3},$$

whence

$$\frac{\sqrt{2}}{2} - \frac{1}{2^{N-2}} > \frac{1}{\sqrt{3}},$$
$$\frac{1}{2^{N-2}} < \frac{\sqrt{2}}{2} - \frac{1}{\sqrt{3}},$$
$$\frac{1}{2^{N-2}} < \frac{\sqrt{6}-2}{2\sqrt{3}},$$
$$2^{N-2} \geqslant 8 > \frac{2\sqrt{3}}{\sqrt{6}-2} \quad \text{for} \quad N \geqslant 5.$$

Consequently, $k_{N-1,N-2} < 0$ and the $(N-1)$-th row does not contain saddle points implying that situation (79) remains solely optimal by (87) as well. □

Now, the existence of pure strategy solutions in PDSDs bigger than the 4×4 one remains to be ascertained by

$$\xi \in \left(\frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}};\ \frac{2\sqrt{2}-3}{4}\right) \quad \text{for} \quad N \in \mathbb{N} \setminus \{1,\ 2,\ 3,\ 4\}. \tag{90}$$

It is better to start considering bigger PDSDs by (90) with the 5×5 PDSD.

6.3. Pure strategy solution non-existence in 5×5 PDSD.

Theorem 10. The 5×5 PDSD (11) by (12)—(14) and (10) with a jitter value of

$$\xi \in \left(\frac{4\sqrt{2}-7}{8};\ \frac{2\sqrt{2}-3}{4}\right) \tag{91}$$

does not have pure strategy solutions.

Proof. Using (75) and (76), entry $k_{3N} < 0$ by (91). So, using additionally Theorem 2, payoff matrix $\mathbf{K}_5$ does not contain saddle points in its first three rows. Using (83) and (84) for $N = 5$, where

$$\frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}} = \frac{4\sqrt{2}-7}{8} < \xi,$$

entry $k_{N,N-1} = k_{54} < 0$ by (91) and the last row does not contain saddle points as well.

In the fourth row, entry

$$k_{43} = K(x_4,\ y_3) = K\left(\xi + \frac{7}{8},\ \xi + \frac{3}{4}\right) =$$
$$= \left(\xi + \frac{7}{8}\right)^2 - \left(\xi + \frac{3}{4}\right)^2 - \left(\xi + \frac{7}{8}\right)^2 \cdot \left(\xi + \frac{3}{4}\right)^2 =$$
$$= -\xi^4 - \frac{13}{4}\xi^3 - \frac{253}{64}\xi^2 - \frac{241}{128}\xi - \frac{233}{1024}. \tag{92}$$

Denote the last term in (92) by a function

$$\psi(\xi) = -\xi^4 - \frac{13}{4}\xi^3 - \frac{253}{64}\xi^2 - \frac{241}{128}\xi - \frac{233}{1024} \ \text{ by } \ -\infty < \xi < \infty. \tag{93}$$

The first derivative of function (93) is

$$\frac{d\psi(\xi)}{d\xi} = -4\xi^3 - \frac{39}{4}\xi^2 - \frac{253}{32}\xi - \frac{241}{128} \tag{94}$$

and

$$-4\xi^3 - \frac{39}{4}\xi^2 - \frac{253}{32}\xi - \frac{241}{128} = 0$$

only if

$$\xi = \xi_{**} = \frac{\sqrt[3]{3456 + 3\sqrt{1327101}} \cdot \left(\sqrt[3]{3456 + 3\sqrt{1327101}} - 39\right) + 3}{48\sqrt[3]{3456 + 3\sqrt{1327101}}} \tag{95}$$

where

$$-0.4124 < \frac{\sqrt[3]{3456 + 3\sqrt{1327101}} \cdot \left(\sqrt[3]{3456 + 3\sqrt{1327101}} - 39\right) + 3}{48\sqrt[3]{3456 + 3\sqrt{1327101}}} < -0.4123.$$

The second derivative of function (93) is

$$\frac{d^2\psi(\xi)}{d\xi^2} = -12\xi^2 - \frac{39}{2}\xi - \frac{253}{32} \tag{96}$$

and function (96) is negative at point (95):

$$-1.91 < \left.\frac{d^2\psi(\xi)}{d\xi^2}\right|_{\xi=\xi_{**}} < -1.9 < 0.$$

This means that (95) is the global maximum point being the single extremum of function (93). The value of function (93) at maximum point (95) is positive:

$$0.075 < \psi(\xi_{**}) < 0.076.$$

Meanwhile,

$$-0.048 < \psi(-1) < -0.047$$

and

$$-10.32 < \psi(1) < -10.31.$$

Consequently,

$$\psi(\xi) > 0 \ \text{by} \ -1 < \xi_{43}^{(1)} < \xi < \xi_{43}^{(2)} < 1,$$

where $\xi_{43}^{(1)}$ and $\xi_{43}^{(2)}$ are points of ξ at which function (93) turns into zero, $\xi_{43}^{(1)} < \xi_{43}^{(2)}$, i. e. they are the lesser and greater roots of equation

$$\psi(\xi) = 0. \tag{97}$$

By the rational root theorem, every rational root of polynomial (93) must be among the numbers

$$\left\{ \pm\frac{1}{1024}, \pm\frac{1}{512}, \pm\frac{1}{256}, \pm\frac{1}{128}, \pm\frac{1}{64}, \pm\frac{1}{32}, \pm\frac{1}{16}, \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1, \right. \\ \pm\frac{233}{1024}, \pm\frac{233}{512}, \pm\frac{233}{256}, \pm\frac{233}{128}, \pm\frac{233}{64}, \pm\frac{233}{32}, \\ \left. \pm\frac{233}{16}, \pm\frac{233}{8}, \pm\frac{233}{4}, \pm\frac{233}{2}, \pm 233 \right\}. \tag{98}$$

However, none of numbers (98) satisfies equality (97). Consequently, the two roots $\xi_{43}^{(1)}$ and $\xi_{43}^{(2)}$ of (97) are irrational. They are accurately estimated again by using the bisection method [33, 34]. Since $\psi(-1) < 0$ and $\psi(1) < 0$ by $-1 < \xi_{**} < 1$, then the roots are within interval $(-1;\ 1)$. Due to

$$\psi(-0.5) = \frac{71}{1024} > 0,$$

lesser root $\xi_{43}^{(1)}$ is iteratively sought within interval $(-1;\ -0.5)$, whereupon

$$-0.8124389649602 < \xi_{43}^{(1)} < -0.8124389649601. \tag{99}$$

Greater root $\xi_{43}^{(2)}$ is iteratively sought within interval $(-0.5;\ 1)$, whereupon

$$-0.17842575816112 < \xi_{43}^{(2)} < -0.17842575816111. \tag{100}$$

So,

$$\xi_{43}^{(1)} < -\frac{1}{2} < \xi_{43}^{(2)} < \frac{4\sqrt{2}-7}{8} < \frac{2\sqrt{2}-3}{4}. \tag{101}$$

Inequality (101) means that, locally,

$$\psi(\xi) < 0 \ \text{by} \ \xi \in \left(\xi_{43}^{(2)};\ \frac{2\sqrt{2}-3}{4} \right) \tag{102}$$

and entry $k_{43} < 0$ by (91), i. e. payoff matrix $\mathbf{K}_5$ does not contain saddle points in its fourth row. Consequently, $\mathbf{K}_5$ does not contain saddle points at all. □

6.4. Pure strategy solution non-existence in $N \times N$ PDSD. To proceed further, one needs to prove the following lemma.

Lemma 1. Entry

$$k_{i,i+1} = K\left(x_i,\ y_{i+1}\right) = x_i^2 - y_{i+1}^2 + x_i^2 y_{i+1}^2 \tag{103}$$

of payoff matrix (14) as a function of $i = \overline{3,\ N-2}$ is an increasing function.

Proof. Obviously,

$$y_{i+1} - x_i = \xi + \frac{2^i - 1}{2^i} - \left(\xi + \frac{2^{i-1} - 1}{2^{i-1}}\right) = \frac{1}{2^i}, \tag{104}$$

so denote $a = \frac{1}{2^i}$ and take

$$y_{i+1} = x_{i+1} = x_i + a \tag{105}$$

to simplify conversion. By having

$$y_{i+2} - x_{i+1} = \xi + \frac{2^{i+1} - 1}{2^{i+1}} - \left(\xi + \frac{2^i - 1}{2^i}\right) = \frac{1}{2^{i+1}} = \frac{a}{2} \tag{106}$$

it can be also written that

$$y_{i+2} = x_{i+1} + \frac{a}{2} = x_i + \frac{3a}{2}, \tag{107}$$

whence

$$\begin{aligned} k_{i+1,i+2} - k_{i,i+1} &= K\left(x_{i+1},\ y_{i+2}\right) - K\left(x_i,\ y_{i+1}\right) = \\ &= x_{i+1}^2 - y_{i+2}^2 + x_{i+1}^2 y_{i+2}^2 - \left(x_i^2 - y_{i+1}^2 + x_i^2 y_{i+1}^2\right) = \\ &= \left(x_i + a\right)^2 - \left(x_i + \frac{3a}{2}\right)^2 + \left(x_i + a\right)^2 \left(x_i + \frac{3a}{2}\right)^2 - x_i^2 + \left(x_i + a\right)^2 - x_i^2 \left(x_i + a\right)^2 \\ &= \frac{9}{4}a^4 + \frac{15}{2}x_i a^3 + \frac{33}{4}x_i^2 a^2 + 3x_i^3 a - \frac{a^2}{4} + x_i a > 0 \end{aligned} \tag{108}$$

due to

$$-\frac{a^2}{4} + x_i a = a\left(x_i - \frac{a}{4}\right) > 0$$

by

$$x_i > \frac{1}{4} > \frac{a}{4} = \frac{1}{2^{i+2}} \quad \text{for} \quad i \geqslant 3.$$

As (104)—(107) are altogether correct by $i = \overline{1,\ N-3}$, inequality (108) holds for $i = \overline{3,\ N-3}$, which means that entry (103) for $i = \overline{3,\ N-2}$ is an increasing function of i. □

In fact, Lemma 1 means that, starting from the third row and ending up by the $(N-2)$-th row of payoff matrix $\mathbf{K}_N$, the entries right above the main diagonal increase. Due to the skew-symmetry of $\mathbf{K}_N$, this directly implies that

$$k_{i+1,i} = K\left(x_{i+1},\ y_i\right) = x_{i+1}^2 - y_i^2 - x_{i+1}^2 y_i^2 \tag{109}$$

is a decreasing function of i for $i = \overline{3,\ N-2}$, i. e. the entries right under the main diagonal decrease starting from the fourth row and ending up by the $(N-1)$-th row.

Theorem 11. An $N \times N$ PDSD (11) by (12)—(14) and (10) for $N \in \mathbb{N} \backslash \left\{\overline{1, 5}\right\}$ with a jitter value of

$$\xi \in \left(\frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}};\ \frac{2\sqrt{2}-3}{4} \right) \tag{110}$$

does not have pure strategy solutions.

Proof. Once again, using (75) and (76), entry $k_{3N} < 0$ by (110), and payoff matrix $\mathbf{K}_N$ does not contain saddle points in its first three rows. Using (83) and (84), entry $k_{N,N-1} < 0$ by (110) and the last row of $\mathbf{K}_N$ does not contain saddle points as well. In the fourth row, the last column entry

$$k_{4N} = K\left(x_4,\ y_N\right) = K\left(\xi + \frac{7}{8},\ 1\right) =$$
$$= 2x_4^2 - 1 = 2 \cdot \left(\xi + \frac{7}{8}\right)^2 - 1 \leqslant 0 \tag{111}$$

if

$$\xi \in \left[-\frac{4\sqrt{2}+7}{8};\ \frac{4\sqrt{2}-7}{8} \right]. \tag{112}$$

So, $k_{4N} \leqslant 0$ by

$$\xi \in \left(\frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}};\ \frac{4\sqrt{2}-7}{8} \right]. \tag{113}$$

Meanwhile, $k_{43} < 0$ by (91) using (101) and (102). Therefore, the fourth row contains at least one negative entry and thus it does not contain saddle points.

In the fifth row, entry

$$k_{54} = K\left(x_5,\ y_4\right) = K\left(\xi + \frac{15}{16},\ \xi + \frac{7}{8}\right) =$$
$$= \left(\xi + \frac{15}{16}\right)^2 - \left(\xi + \frac{7}{8}\right)^2 - \left(\xi + \frac{15}{16}\right)^2 \cdot \left(\xi + \frac{7}{8}\right)^2 =$$
$$= -\xi^4 - \frac{29}{8}\xi^3 - \frac{1261}{256}\xi^2 - \frac{2917}{1024}\xi - \frac{9169}{16384}. \tag{114}$$

Denote the last term in (114) by a function

$$\vartheta\left(\xi\right) = -\xi^4 - \frac{29}{8}\xi^3 - \frac{1261}{256}\xi^2 - \frac{2917}{1024}\xi - \frac{9169}{16384} \quad \text{by} \quad -\infty < \xi < \infty. \tag{115}$$

The first derivative of function (115) is

$$\frac{d\vartheta\left(\xi\right)}{d\xi} = -4\xi^3 - \frac{87}{8}\xi^2 - \frac{1261}{128}\xi - \frac{2917}{1024} \tag{116}$$

and

$$-4\xi^3 - \frac{87}{8}\xi^2 - \frac{1261}{128}\xi - \frac{2917}{1024} = 0$$

only if

$$\xi = \xi_{***} = \frac{\sqrt[3]{13824 + 3\sqrt{21233661}} \cdot \left(\sqrt[3]{13824 + 3\sqrt{21233661}} - 87\right) + 3}{96\sqrt[3]{13824 + 3\sqrt{21233661}}} \tag{117}$$

where

$$-0.591 < \frac{\sqrt[3]{13824+3\sqrt{21233661}}\cdot\left(\sqrt[3]{13824+3\sqrt{21233661}}-87\right)+3}{96\sqrt[3]{13824+3\sqrt{21233661}}} < -0.59.$$

The second derivative of function (115) is

$$\frac{d^2\vartheta\left(\xi\right)}{d\xi^2} = -12\xi^2 - \frac{87}{4}\xi - \frac{1261}{128} \tag{118}$$

and function (118) is negative at point (117):

$$-1.2 < \left.\frac{d^2\vartheta\left(\xi\right)}{d\xi^2}\right|_{\xi=\xi_{***}} < -1.19 < 0.$$

This means that (117) is the global maximum point being the single extremum of function (115). The value of function (115) at maximum point (117) is positive:

$$0.0297 < \vartheta\left(\xi_{***}\right) < 0.0298.$$

Meanwhile,

$$-0.012 < \vartheta\left(-1\right) < -0.011$$

and

$$-0.56 < \vartheta\left(0\right) < -0.55.$$

Consequently,

$$\vartheta\left(\xi\right) > 0 \;\text{ by }\; -1 < \xi_{54}^{(1)} < \xi < \xi_{54}^{(2)} < 0,$$

where $\xi_{54}^{(1)}$ and $\xi_{54}^{(2)}$ are points of ξ at which function (115) turns into zero, $\xi_{54}^{(1)} < \xi_{54}^{(2)}$, i. e. they are the lesser and greater roots of equation

$$\vartheta\left(\xi\right) = 0. \tag{119}$$

By the rational root theorem, every rational root of polynomial (115) must be among the numbers

$$\left\{\pm\frac{1}{16384}, \pm\frac{1}{8192}, \pm\frac{1}{4096}, \pm\frac{1}{2048}, \pm\frac{1}{1024}, \pm\frac{1}{512}, \pm\frac{1}{256}, \pm\frac{1}{128},\right.$$
$$\pm\frac{1}{64}, \pm\frac{1}{32}, \pm\frac{1}{16}, \pm\frac{1}{8}, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm1,$$
$$\pm\frac{53}{16384}, \pm\frac{53}{8192}, \pm\frac{53}{4096}, \pm\frac{53}{2048}, \pm\frac{53}{1024}, \pm\frac{53}{512}, \pm\frac{53}{256}, \pm\frac{53}{128},$$
$$\pm\frac{53}{64}, \pm\frac{53}{32}, \pm\frac{53}{16}, \pm\frac{53}{8}, \pm\frac{53}{4}, \pm\frac{53}{2}, \pm53,$$
$$\pm\frac{173}{16384}, \pm\frac{173}{8192}, \pm\frac{173}{4096}, \pm\frac{173}{2048}, \pm\frac{173}{1024}, \pm\frac{173}{512}, \pm\frac{173}{256}, \pm\frac{173}{128},$$
$$\pm\frac{173}{64}, \pm\frac{173}{32}, \pm\frac{173}{16}, \pm\frac{173}{8}, \pm\frac{173}{4}, \pm\frac{173}{2}, \pm173,$$
$$\pm\frac{9169}{16384}, \pm\frac{9169}{8192}, \pm\frac{9169}{4096}, \pm\frac{9169}{2048}, \pm\frac{9169}{1024}, \pm\frac{9169}{512}, \pm\frac{9169}{256}, \pm\frac{9169}{128},$$
$$\left.\pm\frac{9169}{64}, \pm\frac{9169}{32}, \pm\frac{9169}{16}, \pm\frac{9169}{8}, \pm\frac{9169}{4}, \pm\frac{9169}{2}, \pm9169\right\}. \tag{120}$$

However, none of numbers (120) satisfies equality (119). Consequently, the two roots $\xi_{54}^{(1)}$ and $\xi_{54}^{(2)}$ of (119) are irrational. They are accurately estimated again by using the bisection method [33, 34]. Since $\vartheta(-1) < 0$ and $\vartheta(0) < 0$ by $-1 < \xi_{***} < 0$, then the roots are within interval $(-1;\ 0)$. Due to

$$\vartheta(-0.5) = \frac{391}{16384} > 0,$$

lesser root $\xi_{54}^{(1)}$ is iteratively sought within interval $(-1;\ -0.5)$, whereupon

$$-0.9062423706064 < \xi_{54}^{(1)} < -0.9062423706063. \tag{121}$$

Greater root $\xi_{54}^{(2)}$ is iteratively sought within interval $(-0.5;\ 0)$, whereupon

$$-0.40495044954854 < \xi_{54}^{(2)} < -0.40495044954853. \tag{122}$$

So,

$$\xi_{54}^{(1)} < -\frac{1}{2} < \xi_{54}^{(2)} < \frac{\sqrt{2}-2}{2} + \frac{1}{2^{N-2}} < \frac{4\sqrt{2}-7}{8} < \frac{2\sqrt{2}-3}{4}. \tag{123}$$

Inequality (123) means that, locally,

$$\vartheta(\xi) < 0 \text{ by } \xi \in \left(\xi_{54}^{(2)};\ \frac{2\sqrt{2}-3}{4}\right) \tag{124}$$

and entry $k_{54} < 0$ by (110), i. e. payoff matrix $\mathbf{K}_N$ does not contain saddle points in its fifth row. Due to function (109) is decreasing, this entry is the greatest one among the entries right under the main diagonal, starting from the fifth row and ending up by the $(N-1)$-th row. So, each of these rows contains at least one negative entry. Therefore, every row of $\mathbf{K}_N$ contains at least one negative entry by (110), which implies that the $N \times N$ PDSD for $N \in \mathbb{N} \backslash \left\{\overline{1,\ 5}\right\}$ does not have a pure strategy solution by (110). □

An important corollary from unifying Theorem 10 and Theorem 11 is that the $N \times N$ PDSD for $N \in \mathbb{N} \backslash \{1,\ 2,\ 3,\ 4\}$ does not have a pure strategy solution by (110).

7. Discussion and conclusion

The 3×3 PDSD always has a pure strategy solution, whichever the jitter is. According with Theorem 1, a ξ-jittered middle of the duel time span is the single optimal strategy of the duelist in a 3×3 PDSD if the jitter is positive and it is higher than marginal value (22). Otherwise, the very end of the PDSD becomes an optimal strategy. It is single if the jitter drops below marginal value (22). When the jitter is exactly equal to marginal value (22), the 3×3 PDSD has four optimal situations (18), (20), (23), (24): the first two respectively include a ξ-jittered middle of the duel time span and the very end of the PDSD, whereas the second two are non-symmetric including both moments. Nevertheless, marginal value (22) is an irrational number, so falling on such a jitter is far less likely in practice.

This number is also a marginal value in the 4×4 PDSD, but it separates half-interval (37) of pure strategy optimality from open interval (46), within which no pure strategy solution exists. Another marginal value of the jitter is (69), at which the 4×4 PDSD has four optimal situations: symmetric situations (65), (72), including a ξ-jittered three-quarters of the duel time span and the very end of the PDSD, respectively, and non-symmetric situations (70), (71), including both moments (Theorem 6). Nevertheless, marginal value (69) is an irrational number, so the case with non-symmetric optimal situations is practically unlikely.

The 4×4 PDSD has three jitter intervals, within which it has a pure strategy solution, and open interval (46), within which the duel does not have pure strategy optimality. Within half-interval (37) the single optimal strategy of the duelist is a ξ-jittered middle of the duel time span (Theorem 3). Then goes the open interval of pure strategy solution non-existence (Theorem 4). The endpoints of this interval are positive irrational numbers, where the right endpoint is marginal value (22), and the left endpoint is the greater root of the two roots of fourth-degree-polynomial equation (47). Then goes half-interval (64), whose left and right endpoints are marginal value (69) and the mentioned irrational root, respectively, within which the single optimal strategy of the duelist is a ξ-jittered three-quarters of the duel time span (Theorem 5). If the negative jitter drops below marginal value (69), the optimal strategy of the duelist in the 4×4 PDSD is to shoot at the very end of the duel (Theorem 7).

In PDSDs bigger than the 4×4 one, a ξ-jittered middle of the duel time span is never optimal (Theorem 2). Such duels, unlike PDSDs with four possible shooting moments, have two jitter intervals, within which a single pure strategy solution exists, and open interval (110), within which the $N \times N$ PDSD does not have pure strategy optimality (Theorem 11). The single pure strategy solution can be either a ξ-jittered three-quarters of the duel time span or the very end of the PDSD. If the jitter is not lower than marginal value (69), then the single optimal strategy of the duelist is a ξ-jittered three-quarters of the duel time span (Theorem 8). Within open interval (110), whose left endpoint depends on the size of the duel (the number of possible shooting moments), the $N \times N$ PDSD by $N \in \mathbb{N} \backslash \left\{\overline{1,\ 4}\right\}$ does not have a pure strategy solution. If the jitter is not higher than that left endpoint, then the optimal strategy of the duelist in the $N \times N$ PDSD is to shoot at the very end of the duel (Theorem 9).

Another contribution to the games of timing consists in determining a series of jitter marginal values

$$\frac{\sqrt{2}-1}{2}, \quad \frac{2\sqrt{2}-3}{4}, \quad \frac{4\sqrt{2}-7}{8}$$

(the last one is from the 5×5 PDSD by Theorem 10 being a partial case of Theorem 11), across which either pure strategy solution structure changes or pure strategy solution existence and non-existence shift. In addition, the study has ascertained that the greater root of the two roots of fourth-degree-polynomial equation (47) is irrational and satisfies inequality (56), and this root is another jitter marginal value across which pure strategy solution non-existence in the 4×4 PDSD is shifted by the single pure strategy optimal situation (65). This root, however, does not matter for bigger duels.

The study can be expanded into non-uniform jitter, where each of possible shooting moments (5) can have its own jitter. Besides, the duelist's quadratic accuracy function can be scaled with a positive coefficient, which particularly is 1 in kernel (2).

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