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Existence Results for Singular Double Phase Problems by Topological Degree

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Abstract. In this paper we consider singular elliptic problems directed by the double phase operator, including a gradient-dependent reaction term as well as Dirichlet boundary conditions. Using topological degree methods for a class of non-continuous operators based on the abstract Hammerstein equation, we prove the existence of weak solutions in the Musielak-Orlicz-Sobolev space $W_0^{1,\mathcal{E}}(\mathfrak{O})$. Our assumptions are appropriate and different from those discussed above.

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1. Introduction and main result

In the 1980s, to provide models for strongly anisotropic materials, Zhikov [22, 23] introduced a class of functionals that involve the nonautonomous operator

$$\mathfrak{F}(w) = -\operatorname{div}\left(|\nabla w|^{\varrho-2}\nabla w + \lambda(\cdot)|\nabla w|^{\tau-2}\nabla w\right).$$

Zhikov devised $\mathfrak{F}(w)$ to illustrate this phenomenon, using integrands that adjust their ellipticity rate based on the characteristics of the point. The function $\lambda(\cdot)$ helps to regulate the mixture of two different materials, each characterised by power hardening rates ϱ and τ respectively (see Ref [23] for more details). $\mathfrak{F}(w)$ falls into the category of integral functionals with a non-standard growth condition, according to Marcellini's terminology. Recently, Baroni, Colombo and Mingione explored the regularity theory for minimizers of $\mathfrak{F}(w)$, obtaining precise results for $\tau > \varrho$ and $\lambda(\cdot) \geq 0$ (see references [3, 4] for detailed information).

Recently, Hästö-Ok [17] extended this study to double phase functionals with a weight on each phase, i.e., functionals involving the following operator

$$w \longmapsto -\operatorname{div}\left(\mu(z)|\nabla w|^{\varrho-2}\nabla w + \lambda(z)|\nabla w|^{\tau-2}\nabla w\right).$$

In addition, other recent studies have also investigated the existence of non-trivial solutions for double phase problems with a weight on the τ phase only. This type of problem is characterised by the following system

$$\begin{cases} -\operatorname{div}\left(|\nabla w|^{\varrho-2}\nabla w + \lambda(z)|\nabla w|^{\tau-2}\nabla w\right) = \zeta k(z,w), & z \in \mathfrak{O}, \\ w = 0, & z \in \partial\mathfrak{O}, \end{cases}$$

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These studies have been carried out under special conditions, namely $\zeta \in \mathbb{R}$, $1 < \varrho < \tau < N$, $\frac{\tau}{\varrho} < 1 + \frac{1}{N}$, $N \geq 2$, where $\mathfrak{O} \subset \mathbb{R}^N$ represents a bounded domain with a Lipschitz boundary. In addition, $\lambda : \bar{\mathfrak{O}} \to [0, \infty)$ is Lipschitz continuous, and $k : \mathfrak{O} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition. In the case where $N = 1$, a separate study by the authors in [18] investigated the existence, non-existence and multiplicity of positive solutions for double-phase problems with a weight on each phase. The corresponding system is given by

$$\begin{cases} -\left(\alpha(t)\varpi_\varrho(w') + \lambda(t)\varpi_\varrho(w')\right)' = \zeta f(w)h(t), \quad t \in (0,1), \\ w(0) = 0 = w(1), \end{cases}$$

such that $\lambda > 0$, $1 < \varrho < \tau < \infty$, and $\varpi_\varepsilon(r) := |r|^{\varepsilon-2}r$. Certain conditions are also imposed on α, ζ, f and h.

The aim of this paper is to prove the existence of non-trivial weak solutions for the double-phase problem with Hardy potential.

$$(\mathcal{P}) \begin{cases} -\operatorname{div}\left(|\nabla w|^{\varrho-2}\nabla w + \kappa(z)|\nabla w|^{\tau-2}\nabla w\right) = |w|^{\varrho-2}w \\ \qquad +\kappa(z)|w|^{\tau-2}w + \frac{|w|^{\varrho-2}w}{|z|^\varrho} + \kappa(z)\frac{|w|^{\tau-2}w}{|z|^\tau} + \varphi(z, w, \nabla w) & \text{in} \quad \mathfrak{O}, \\ w = 0 & \text{on} \quad \partial\mathfrak{O}, \end{cases}$$

where $\mathfrak{O} \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open set containing the origin with smooth boundary $\partial\mathfrak{O}$ and $1 < \varrho < \tau < N$

$$\frac{\tau}{\varrho} < 1 + \frac{1}{N}, \text{ the weight function } \kappa : \bar{\mathfrak{O}} \to [0, \infty) \text{ is Lipschitz continuous}$$

$$\text{such that } \kappa(\lambda z) \leqslant \kappa(z) \text{ for any } \lambda \in (0, 1] \text{ and any } z \in \bar{\mathfrak{O}} \tag{1}$$

and $\varphi : \mathfrak{O} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function which satisfies the following growth condition

$$(\mathcal{H}_\varphi) \qquad \text{There exists } \varepsilon \in L^{\frac{s}{s-1}}(\Omega),\ 1 < s < \varrho \ \text{ and } \ C > 0, \text{ such that}$$
$$|\varphi(z, \gamma, \xi)| \leqslant C\left(\varepsilon(z) + |\gamma|^{s-1} + |\xi|^{\varrho\frac{s-1}{s}}\right),$$

for a.a. $z \in \mathfrak{O}$, for all $\gamma \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$.

Furthermore, Browder, in [5], extended the notion of topological degree to operators belonging to the class (S_+) in reflexive Banach spaces. Additional insights into these concepts, along with detailed information and examples, can be found in classic works such as [9].

In this article, we investigate singular elliptic problems characterized by the double-phase operator, incorporating a gradient-dependent reaction term and subject to Dirichlet boundary conditions. Using compactness methods described in [9, 15], we also prove the existence of weak solutions for problem $(\mathcal{P})$ in Musielak-Orlicz spaces. We reformulate it as a new problem governed by a Hammerstein equation. Specifically, using the topological degree theory introduced in Section 3, we establish the existence of weak solutions for the given problem. This result heavily relies on appropriate assumptions.

First we define the operator $\mathcal{N}$, which operates from $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$ into $\left(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\right)^*$, as follows

$$\langle \mathcal{N}w, \vartheta \rangle = -\int_{\mathfrak{O}} \left(|w|^{\varrho-2}w + \kappa(z)|w|^{\tau-2}w + \varphi(z, w, \nabla w)\right)\vartheta dz, \tag{2}$$

for each $\vartheta \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$.
In sequel, let us consider the following Euler-Lagrange functional $\mathcal{J} : \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \to \mathbb{R}$, given by

$$\mathcal{J}(w) := \frac{1}{\varrho}\|\nabla w\|_\varrho^\varrho + \frac{1}{\tau}\|\nabla w\|_{\tau,\kappa}^\tau - \Big(\frac{1}{\varrho}\int_{\mathfrak{O}} \frac{|w|^\varrho}{|z|^\varrho}dz + \frac{1}{\tau}\int_{\mathfrak{O}} \kappa(z)\frac{|w|^\tau}{|z|^\tau}dz\Big),$$

it is well known that $\mathcal{J}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $w \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$ is the functional $\mathcal{J}'(w) \in \Big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\Big)^*$ setting by

$$\langle \mathcal{J}'(w)\,,\, v\rangle = \langle \mathcal{D}w\,,\, \vartheta\rangle, \qquad \text{for all } w, v \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}),$$

where the operator $\mathcal{D}$ acting from $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$ to its dual $\Big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\Big)^*$ is defined by

$$\langle \mathcal{D}w\,,\, \vartheta\rangle = \int_{\mathfrak{O}} \Big(|\nabla w|^{\varrho-2} + \kappa(z)|\nabla w|^{\tau-2}\Big)\nabla w \nabla\vartheta dz - \int_{\mathfrak{O}} \Big(\frac{|w|^{\varrho-2}w}{|z|^\varrho} + \kappa(z)\frac{|w|^{\tau-2}w}{|z|^\tau}\Big)\vartheta dz \quad (3)$$

for all $w, \vartheta \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$.
Following that, we proceed to define weak solutions for problem $(\mathcal{P})$

Definition 1.1. A measurable function w is called to be a weak solution of $(\mathcal{P})$, if $w \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$ such that

$$\langle \mathcal{D}w, \vartheta\rangle = -\langle \mathcal{N}w, \vartheta\rangle, \qquad \text{for all} \quad \vartheta \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}). \quad (4)$$

We are now ready to unveil our primary outcome.

Theorem 1.1. *Let φ satisfy $(\mathcal{H}_\varphi)$. Then, the problem $(\mathcal{P})$ has a weak solution w in $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$.*

The following sections of this paper are structured as follows. In Section 2, we introduce the notation and provide essential results related to the Musielak-Orlicz-Sobolev space $W_0^{1,\mathcal{E}}(\mathfrak{O})$ to enhance the understanding of the paper. Section 3 is devoted to establishing the variational framework associated with the problem $(\mathcal{P})$, along with the presentation of key lemmas crucial for the proofs of Theorems 1.1. The proofs of the theorems 1.1 are presented in Section 4.

2. Preliminaries

In this section, we summarize the relevant material on the Musielak-Orlicz space $L^{\mathcal{E}}(\mathfrak{O})$ and $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$. For more detail, please see references [8, 20].
Let us denote by $\mathbb{R}^+ = [0, +\infty)$, the function $\mathcal{E} : \mathfrak{O} \times \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\mathcal{E}(z,t) = t^\varrho + \kappa(z)t^\tau \text{ for each } z \in \Omega \text{ and } t \in \mathbb{R}^+,$$

with $1 < \varrho < \tau$ and $0 \leq \kappa(\cdot) \in L^1(\mathfrak{O})$, is a generalized N-function, and

$$\mathcal{E}(z, 2t) \leqslant 2^\tau \mathcal{E}(z,t) \text{ for a.e. } z \in \Omega \text{ and } t \in \mathbb{R}^+,$$

which is called condition $(\triangle_2)$. The Musielak-Orlicz space $L^{\mathcal{E}}(\mathfrak{O})$ is defined by

$$L^{\mathcal{E}}(\mathfrak{O}) = \{w : \mathfrak{O} \to \mathbb{R} \text{ measurable } : \rho_{\mathcal{E}}(w) < +\infty\},$$

and it can be equipped with the norm

$$\|w\|_{\mathcal{E}} = \inf\Big\{\lambda > 0 : \rho_{\mathcal{E}}\Big(\frac{w}{\lambda}\Big) \leqslant 1\Big\},$$

where $\rho_{\mathcal{E}}(w) := \int_{\mathfrak{O}} \mathcal{E}(z,|w|)dz = \int_{\mathfrak{O}} |w|^{\varrho} + \kappa(z)|w|^{\tau} dz$ is called $\mathcal{E}$-modular. The space $L^{\mathcal{E}}(\mathfrak{O})$ is a separable, uniformly convex Banach space. We denote by $\|\cdot\|_{\varrho}$ the norm in $L^{\varrho}(\mathfrak{O})$ and by

$$L^{\tau}(\mathfrak{O},\kappa) := \Big\{w : \mathfrak{O} \to \mathbb{R} \text{ measurable } : \|w\|_{\tau,\kappa} := \Big(\int_{\mathfrak{O}} \kappa(z)|w|^{\tau} dz\Big)^{\frac{1}{\tau}} < +\infty\Big\}.$$

It is straightforward to verify that

$$L^{\tau}(\mathfrak{O}) \hookrightarrow L^{\mathcal{E}} \hookrightarrow L^{\varrho}(\mathfrak{O}) \cap L^{\tau}(\mathfrak{O},\kappa)$$

are continuous. The relationship between the norm and the $\mathcal{E}$-modular is as follows

Proposition 2.1. *The $\mathcal{E}$-modular has the following properties:*

(i) *for* $w \neq 0$, $\|w\|_{\mathcal{E}} = \kappa \Leftrightarrow \rho_{\mathcal{E}}\Big(\frac{w}{\kappa}\Big) = 1$;
(ii) $\|w\|_{\mathcal{E}} < 1$ (*resp.* $= 1; > 1$) $\Leftrightarrow \rho_{\mathcal{E}}(w) < 1$ (*resp.* $= 1; > 1$) ;
(iii) $\|w\|_{\mathcal{E}} < 1 \Rightarrow \|w\|_{\mathcal{E}}^{\tau} \leqslant \rho_{\mathcal{E}}(w) \leqslant \|w\|_{\mathcal{E}}^{\varrho}$; $\|w\|_{\mathcal{E}} > 1 \Rightarrow \|w\|_{\mathcal{E}}^{\varrho} \leqslant \rho_{\mathcal{E}}(w) \leqslant \|w\|_{\mathcal{E}}^{\tau}$;
(iv) $\|w\|_{\mathcal{E}} \to 0 \Leftrightarrow \rho_{\mathcal{E}}(w) \to 0$; $\|w\|_{\mathcal{E}} \to +\infty \Leftrightarrow \rho_{\mathcal{E}}(w) \to +\infty$.

The space $\mathcal{W}^{1,\mathcal{E}}(\mathfrak{O})$ is defined by

$$\mathcal{W}^{1,\mathcal{E}}(\mathfrak{O}) := \{w \in L^{\mathcal{E}}(\mathfrak{O}) : |\nabla w| \in L^{\mathcal{E}}(\mathfrak{O})\},$$

equipped with the norm

$$\|w\|_{1,\mathcal{E}} := \|w\|_{\mathcal{E}} + \|\nabla w\|_{\mathcal{E}}. \tag{5}$$

We represent $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$ as the closure of $C_0^{\infty}(\mathfrak{O})$ in $\mathcal{W}^{1,\mathcal{E}}$, equipped with the norm

$$\|w\| := \|\nabla w\|_{\mathcal{E}},$$

which is equivalent to the norm defined in (5), as indicated in [8].

Proposition 2.2. [8]

(i) $\mathcal{W}^{1,\mathcal{E}}(\mathfrak{O})$ *and* $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$ *are separable reflexive Banach space.*
(ii) *If* $\varrho \neq N$, *then* $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \hookrightarrow L^s(\mathfrak{O})$ *for all* $s \in [1,\ \varrho^*]$.
If $\varrho = N$, *then* $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \hookrightarrow L^s(\mathfrak{O})$ *for all* $s \in [1,\ +\infty]$.
(iii) *If* $\varrho \leq N$, *then* $W_0^{1,\mathcal{E}}(\mathfrak{O}) \hookrightarrow\hookrightarrow L^s(\mathfrak{O})$ *for all* $s \in [1,\ \varrho^*)$.
If $\varrho > N$, *then* $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \hookrightarrow\hookrightarrow L^{\infty}(\mathfrak{O})$.
(iv) *If condition* (1) *holds, then there exists a constant* $C > 0$ *such that*

$$\|w\|_{\mathcal{E}} \leqslant C\|\nabla w\|_{\mathcal{E}}, \qquad \text{for all } w \in W_0^{1,\mathcal{E}}(\mathfrak{O}).$$

Making use of this additional notation, the following result gives Hardy inequalities for the space $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$. The proof is inspired by [12, Lemma 2.1].

Lemma 2.3. *Assume that* (1) *holds true. Then, for any* $w \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$ *we have*

$$\mathsf{C}_{\varrho}\|w\|_{\mathsf{C}_{\varrho}}^{\varrho} := \mathsf{C}_{\varrho}\int_{\mathfrak{O}} \frac{|w|^{\varrho}}{|z|^{\varrho}} dz \leqslant \|\nabla w\|_{\varrho}^{\varrho}; \text{ and } \mathsf{C}_{\tau}\|w\|_{\mathsf{C}_{\tau,\kappa}}^{\tau} := \mathsf{C}_{\tau}\int_{\mathfrak{O}} \kappa(z)\frac{|w|^{\tau}}{|z|^{\tau}} dz \leqslant \|\nabla w\|_{\tau,\kappa}^{\tau},$$

where $\mathsf{C}_m := \big(\frac{m}{N-m}\big)^{-m}$ *when* $m = \varrho$ *and* $m = \tau$.

Now, let's delve into the theory of topological degree, which constitutes a pivotal tool for our results. To begin, we will introduce some classes of mappings. Consider a real separable reflexive Banach space Γ with its dual space denoted as Γ^*. This setup includes a continuous dual pairing denoted by $\langle\cdot,\cdot\rangle$, with Γ^* appearing before Γ in the pairing order. The symbol $\rightharpoonup$ signifies weak convergence.

Definition 2.1. Let $\mathcal{E}$ be another real Banach space. A operator $B : \mathfrak{O} \subset \Gamma \to \mathcal{E}$ is said to be
(1) bounded, if it takes any bounded set into a bounded set.
(2) demicontinuous, if for any sequence $(w_n) \subset \mathfrak{O}, w_n \to w$ implies $Bw_n \rightharpoonup \mathcal{B}w$.
(3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 2.2. A mapping $B : \mathfrak{O} \subset \Gamma \to \Gamma^*$ is said to be
(1) of type (S_+), if for any sequence $(w_n) \subset \mathfrak{O}$ with $w_n \rightharpoonup w$ and $\limsup_{n\to\infty}\langle Bw_n, w_n - w\rangle \leqslant 0$, we have $w_n \to w$.
(2) quasimonotone, if for any sequence $(w_n) \subset \mathfrak{O}$ with $w_n \rightharpoonup w$, we have $\limsup_{n\to\infty}\langle Bw_n,\ w_n - w\rangle \geq 0$.

Definition 2.3. Let $\mathcal{T} : \mathfrak{O}_1 \subset \Gamma \to \Gamma^*$ be a bounded operator such that $\mathfrak{O} \subset \mathfrak{O}_1$. For any operator $B : \mathfrak{O} \subset \Gamma \to \Gamma$, we say that
(1) B satisfies condition $(S_+)_{\mathcal{T}}$, if for any sequence $(w_n) \subset \mathfrak{O}$ with $w_n \rightharpoonup w$, $y_n := \mathcal{T}w_n \rightharpoonup y$ and $\limsup_{n\to\infty}\langle Bw_n,\ y_n - y\rangle \leq 0$, we have $w_n \to w$.
(2) B has the property $(QM)_{\mathcal{T}}$, if for any sequence $(w_n) \subset \mathfrak{O}$ with $w_n \rightharpoonup w$, $y_n := \mathcal{T}w_n \rightharpoonup y$, we have $\limsup_{n\to\infty}\langle Bw_n, y - y_n\rangle \geq 0$.

In the following, consider $\mathcal{O}$ as the set of all bounded open sets in Γ. For any $\mathfrak{O} \subset \Gamma$, we investigate the subsequent classes of operators

$$\mathcal{B}_1(\mathfrak{O}) := \{B : \mathfrak{O} \to \Gamma^* \ \mid\ B \text{ is bounded, demicontinuous and of type}(S_+)\},$$
$$\mathcal{B}_{\mathcal{T},B}(\mathfrak{O}) := \{B : \mathfrak{O} \to \Gamma \ \mid\ B \text{ is bounded, demicontinuousand of type}(S_+)_{\mathcal{T}}\},$$
$$\mathcal{B}_{\mathcal{T}}(\mathfrak{O}) := \{B : \mathfrak{O} \to \Gamma \ \mid\ B \text{ is demicontinuous and of type}(S_+)_{\mathcal{T}}\},$$
$$\mathcal{B}_B(\Gamma) := \{B \in \mathcal{B}_{\mathcal{T},B}(\overline{\mathfrak{O}}) \ \mid\ \mathfrak{O} \in \mathcal{O},\ \mathcal{T} \in \mathcal{B}_1(\overline{\mathfrak{O}})\}.$$

Lemma 2.4. [2] *Let $\mathcal{T} \in \mathcal{B}_1(\overline{\mathfrak{O}})$ be continuous and $\mathcal{S} : D_{\mathcal{S}} \subset \Gamma^* \to \Gamma$ be demicontinuous such that $\mathcal{T}(\overline{\mathfrak{O}}) \subset D_{\mathcal{S}}$, where $\mathfrak{O}$ is a bounded open set in a real reflexive Banach space Γ. Then the following statements are true :*
(1) *If $\mathcal{S}$ is quasimonotone, then $\mathcal{I} + \mathcal{S} \circ \mathcal{T} \in \mathcal{B}_{\mathcal{T}}(\overline{\mathfrak{O}})$, where $\mathcal{I}$ denotes the identity operator.*
(2) *If $\mathcal{S}$ is of class (S_+), then $\mathcal{S} \circ \mathcal{T} \in \mathcal{B}_{\mathcal{T}}(\overline{\mathfrak{O}})$.*

Definition 2.4. Suppose that $\mathfrak{O}$ is bounded open subset of a real reflexive Banach space Γ, $T \in \mathcal{B}_1(\overline{\mathfrak{O}})$ be continuous and let $B,\ \mathcal{S} \in \mathcal{B}_{\mathcal{T}}(\overline{\mathfrak{O}})$. The affine homotopy $\mathcal{H} : [0,1] \times \overline{\mathfrak{O}} \to \Gamma$ defined by

$$\mathcal{H}(t,w) := (1-t)Bu + t\mathcal{S}w, \quad \text{for} \qquad (t,w) \in [0,1] \times \overline{\mathfrak{O}}$$

is called an admissible affine homotopy with the common continuous essential inner map $\mathcal{T}$.

Remark 2.1. [2] The affine homotopy described above meets the condition $(S_+)_\mathcal{T}$.

Following that, we provide the Berkovits topological degree applicable to the class $\mathcal{B}_B(\Gamma)$, for further details, refer to [2].

Theorem 2.5. *Let*

$$M = \{(B, \mathfrak{O}, h) \mid \mathfrak{O} \in \mathcal{O},\ \mathcal{T} \in \mathcal{B}_1(\overline{\mathfrak{O}}),\ B \in \mathcal{B}_{\mathcal{T},B}(\overline{\mathfrak{O}}),\ h \notin B(\partial\mathfrak{O})\}.$$

Then, there exists a unique degree function $\delta : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:

1. *For all $h \in \mathfrak{O}$, we obtain $\delta(\mathcal{I}, \mathfrak{O}, h) = 1$. (**Normalization**)*
2. *If $\mathcal{H} : [0,1] \times \overline{\mathfrak{O}} \to \Gamma$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to \Gamma$ is a continuous path in Γ such that $h(t) \notin \mathcal{H}(t, \partial\mathfrak{O})$ for every $t \in [0,1]$, then the value of $\delta(\mathcal{H}(t,\cdot), \mathfrak{O}, h(t))$ is constant for every $t \in [0,1]$. (**Homotopy invariance**)*
3. *If $\delta(B, \mathfrak{O}, h) \neq 0$, then the equation $Bw = h$ has a solution in $\mathfrak{O}$. (**Existence**)*

3. Proof of Theorem 1.1

The purpose of this section is the demonstration of theorem1.1. Initially, we establish the following proposition

Proposition 3.1. *The nonlinear operator $\mathcal{D} : \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \to \left(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\right)^*$ is*

(i) *continuous, bounded and strictly monotone operators,*
(ii) *of type (S_+).*

Proof. (i) The continuity of $\mathcal{D}$ is obvious because $\mathcal{D} = \mathcal{J}'$ and $\mathcal{J} \in C^1$. We will now show that $\mathcal{D}$ is bounded. To make the proof straightforward, let us define $\phi := \|w\|$ and $\gamma := \|\vartheta\|$. Applying Hölder's inequality, Young's inequality and Hardy's inequality, we obtain :

$$\begin{aligned}
&\Big|\Big\langle \frac{\mathcal{D}w}{\phi}, \frac{\vartheta}{\gamma}\Big\rangle\Big| \\
&= \int_\Omega \Big(|\nabla w|^{\varrho-2} + \kappa(z)|\nabla w|^{\tau-2}\Big)\nabla w \nabla\vartheta dz - \int_\mathfrak{O} \Big(\frac{|w|^{\varrho-2}w}{|z|^\varrho} + \kappa(z)\frac{|w|^{\tau-2}w}{|z|^\tau}\Big)\vartheta dz \\
&\leqslant \Big(\int_\mathfrak{O} \Big|\frac{\nabla w}{\phi}\Big|^\varrho dz\Big)^{\frac{\varrho-1}{\varrho}} \Big(\int_\mathfrak{O} \Big|\frac{\nabla\vartheta}{\gamma}\Big|^\varrho dz\Big)^{\frac{1}{\varrho}} + \Big(\int_\mathfrak{O} \kappa(z)\Big|\frac{\nabla w}{\phi}\Big|^\tau dz\Big)^{\frac{\tau-1}{\tau}} \Big(\int_\mathfrak{O} \kappa(z)\Big|\frac{\nabla\vartheta}{\gamma}\Big|^\tau dz\Big)^{\frac{1}{\tau}} \\
&+ \Big(\int_\mathfrak{O} \Big|\frac{w}{\phi\, z}\Big|^\varrho dz\Big)^{\frac{\varrho-1}{\varrho}} \Big(\int_\mathfrak{O} \Big|\frac{\vartheta}{\gamma\, z}\Big|^\varrho dz\Big)^{\frac{1}{\varrho}} + \Big(\int_\mathfrak{O} \kappa(z)\Big|\frac{w}{\phi\, z}\Big|^\tau dz\Big)^{\frac{\tau-1}{\tau}} \Big(\int_\mathfrak{O} \kappa(z)\Big|\frac{\vartheta}{\gamma\, z}\Big|^\tau dz\Big)^{\frac{1}{\tau}} \\
&\leqslant \frac{\varrho-1}{\varrho}\int_\mathfrak{O} \Big|\frac{\nabla w}{\phi}\Big|^\varrho dz + \frac{1}{\varrho}\int_\mathfrak{O} \Big|\frac{\nabla\vartheta}{\gamma}\Big|^\varrho dz + \frac{\tau-1}{\tau}\int_\mathfrak{O} \kappa(z)\Big|\frac{\nabla w}{\phi}\Big|^\tau dz + \frac{1}{\tau}\int_\mathfrak{O} \kappa(z)\Big|\frac{\nabla\vartheta}{\gamma}\Big|^\tau dz \\
&+ \frac{\varrho-1}{\varrho}\int_\mathfrak{O} \Big|\frac{w}{\phi\, z}\Big|^\varrho dz + \frac{1}{\varrho}\int_\mathfrak{O} \Big|\frac{\vartheta}{\gamma\, z}\Big|^\varrho dz + \frac{\tau-1}{\tau}\int_\mathfrak{O} \kappa(z)\Big|\frac{w}{\phi\, z}\Big|^\tau dz + \frac{1}{\tau}\int_\mathfrak{O} \kappa(z)\Big|\frac{\vartheta}{\gamma\, z}\Big|^\tau dz \\
&\leqslant \frac{\varrho-1}{\varrho}\int_\mathfrak{O} \Big|\frac{\nabla w}{\phi}\Big|^\varrho dz + \frac{1}{\varrho}\int_\mathfrak{O} \Big|\frac{\nabla\vartheta}{\gamma}\Big|^\varrho dz + \frac{\tau-1}{\tau}\int_\mathfrak{O} \kappa(z)\Big|\frac{\nabla w}{\phi}\Big|^\tau dz + \frac{1}{\tau}\int_\mathfrak{O} \kappa(z)\Big|\frac{\nabla\vartheta}{\gamma}\Big|^\tau dz \\
&+ C_1\int_\mathfrak{O} \Big|\frac{\nabla w}{\phi}\Big|^\varrho dz + C_2\int_\mathfrak{O} \Big|\frac{\nabla\vartheta}{\gamma}\Big|^\varrho dz + C_3\int_\mathfrak{O} \kappa(z)\Big|\frac{\nabla w}{\phi}\Big|^\tau dz + C_3\int_\mathfrak{O} \kappa(z)\Big|\frac{\frac{1}{\tau}\vartheta}{\gamma}\Big|^\tau dz
\end{aligned}$$

$$\leqslant C_1'\Big(\int_{\mathfrak{O}}\Big(\Big|\frac{\nabla w}{\phi}\Big|^{\varrho}+\kappa(z)\Big|\frac{\nabla w}{\phi}\Big|^{\tau}\Big)dz\Big)+C_2'\Big(\int_{\mathfrak{O}}\Big(\Big|\frac{\nabla \vartheta}{\gamma}\Big|^{\varrho}+\kappa(z)\Big|\frac{\nabla \vartheta}{\gamma}\Big|^{\tau}\Big)dz\Big)$$
$$\leqslant C_{max}$$

Hence, we have that

$$\|\mathcal{D}w\|_{\left(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\right)^*}=\sup_{\|\vartheta\|\le 1}\Big|\Big\langle \mathcal{D}w,\vartheta\Big\rangle\Big|\le C_{max}\|w\|,$$

leading to the conclusion that $\mathcal{D}$ is bounded.
The monotonicity of $\mathcal{D}$ can be readily inferred from the following inequalities (refer to [13, 16])

$$\begin{cases}(|\xi|^{\varrho-2}\xi-|\eta|^{\varrho-2}\eta)(\xi-\eta)\cdot(|\xi|^{\varrho}+|\eta|^{\varrho})^{\frac{2-\varrho}{\varrho}}\ge(\varrho-1)|\xi-\eta|^{\varrho} & \text{if } 1<\varrho<2,\\ (|\xi|^{\varrho-2}\xi-|\eta|^{\varrho-2}\eta)(\xi-\eta)\ge\left(\frac{1}{2}\right)^{\varrho}|\xi-\eta|^{\varrho} & \text{if } \varrho\ge 2.\end{cases}\tag{6}$$

(ii) Assume that $(w_n)\subset\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$, $w_n\rightharpoonup w$ and

$$\limsup_{n\to+\infty}\langle\mathcal{D}w_n-\mathcal{D}w,w_n-w\rangle\le 0.$$

Then

$$\liminf_{n\to+\infty}\langle\mathcal{D}w_n-\mathcal{D}w,w_n-w\rangle\ge 0,$$

since $\mathcal{D}$ is monotone. Thus

$$\lim_{n\to+\infty}\langle\mathcal{D}w_n-\mathcal{D}w,w_n-w\rangle=0,$$

that is,

$$\lim_{n\to+\infty}\Big(\int_{\mathfrak{O}}(|\nabla w_n|^{\varrho-2}\nabla w_n+\kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n-|\nabla w|^{\varrho-2}\nabla w-\kappa(z)|\nabla w|^{\tau-2}\nabla w)$$
$$\times\big(\nabla w_n-\nabla w\big)dz-\int_{\mathfrak{O}}\Big(\frac{|w_n|^{\varrho-2}w_n}{|z|^{\varrho}}+\kappa(z)\frac{|w_n|^{\tau-2}w_n}{|z|^{\tau}}-\frac{|w|^{\varrho-2}w}{|z|^{\varrho}}-\kappa(z)\frac{|w|^{\tau-2}w}{|z|^{\tau}}\Big)$$
$$\times\big(w_n-w\big)dz\Big)=0.\tag{7}$$

Hence, $(w_n)_n$ is bounded in $W_0^{1,\mathcal{E}}(\mathfrak{O})$. By Propositions 2.1-2.2, Lemma 2.3 [7, Theorem 4.9] and the reflexivity of $W_0^{1,\mathcal{E}}(\mathfrak{O})$, there exists a subsequence, still denoted by $(w_n)_n$, and $w\in W_0^{1,\mathcal{E}}(\mathfrak{O})$ such that

$$\begin{cases}w_n\rightharpoonup w \text{ in } W_0^{1,\mathcal{E}}(\mathfrak{O}), \qquad \nabla w_n\rightharpoonup\nabla w \text{ in } [L^{\mathcal{E}}(\mathfrak{O})]^N,\\ w_n\rightharpoonup w \text{ in } L^{\varrho}(\mathfrak{O},\ |z|^{-\varrho}), w_n\rightharpoonup w \text{ in } L^{\tau}\big(\mathfrak{O}\setminus\mathcal{A},\kappa(z)|z|^{-\tau}\big),\\ \|w_n-w\|^{\varrho}_{\mathsf{C}_{\varrho}}+\|w_n-w\|^{\tau}_{\mathsf{C}_{\tau,\kappa}}\to\ell,\\ w_n\to w \text{ in } L^t(\mathfrak{O}), w_n(z)\to w(z) \text{ a.e. in } \mathfrak{O}, |w_n(z)|\leqslant g(z) \text{ a.e. in } \mathfrak{O},\end{cases}\tag{8}$$

as $n\to\infty$, with $t\in[1,\varrho^*)$, $g\in L^{\tau}(\mathfrak{O})$ and $\mathcal{A}$ is the nodal set of weight a given by

$$\mathcal{A}:=\{z\in\mathfrak{O}:\kappa(z)=0\}.$$

In fact, given that κ is a continuous Lipschitz function as indicated by (1), it follows that $\mathfrak{O}\setminus\mathcal{A}$ is an open subset of $\mathbb{R}^N$. Moreover, by considering Proposition 2.2 and [7, Theorem 4.9], we ensure that $g\in L^{\tau}(\mathfrak{O})$ since $\tau<\varrho^*$ according to (1). Now, we claim that

$$\nabla w_n(z)\to\nabla w(z)\quad \text{a.e. in } \mathfrak{O},\ \text{as } n\to\infty.\tag{9}$$

Let $\psi \in C^\infty(\mathbb{R}^N)$ be a cut-off function with

$$\begin{cases} \psi(r) \equiv 1 & if \quad r \in B(0,1/2), \\ \psi(r) \equiv 0 & if \quad r \in B(0,1), \\ 0 \leq \psi \leq 1 & otherwise. \end{cases}$$

We then introduce the function $\phi_R(z) = 1 - \psi(z/R)$ for all $R > 0$, ensuring that $\phi_R \in C^\infty(\mathbb{R}^N)$ with

$$\begin{cases} \phi(r) \equiv 1 & if \quad r \in \mathbb{R}^N \setminus B(0,R), \\ \phi(r) \equiv 0 & if \quad r \in B(0,R/2), \\ 0 \leq \phi \leq 1 & otherwise. \end{cases}$$

The sequence $(\phi_R w_n)_n$ remains bounded in $W_0^{1,\mathcal{E}}(\mathfrak{O})$, as established by Proposition 2.1. We can easily deduce, for all $n \in \mathbb{N}$, that

$$\begin{aligned} \langle \mathcal{D}w_n, \phi_R(w_n - w)\rangle = &\int_{\mathfrak{O}} \phi_R(|\nabla w_n|^{\varrho-2}\nabla w_n + \kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n)(\nabla w_n - \nabla w)dz \\ &+ \int_{\mathfrak{O}} (|\nabla w_n|^{\varrho-2}\nabla w_n + \kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n)\nabla\phi_R(w_n - w)dz \\ &- \int_{\mathfrak{O}} \phi_R\Big(\frac{|w_n|^{\varrho-2}w_n}{|z|^\varrho} + \kappa(z)\frac{|w_n|^{\tau-2}w_n}{|z|^\tau}\Big)(w_n - w)dz. \end{aligned} \tag{10}$$

Certainly, all integrals in (10) vanish when $\overline{\mathfrak{O}} \subset B(0,R/2)$, given that $\phi_R \equiv 0$ in $B(0,R/2)$. Therefore, we focus on selecting $R > 0$ adequately small such that

$$\big[\mathbb{R}^N \backslash B(0,R/2)\big] \cap \overline{\mathfrak{O}} \neq \emptyset. \tag{11}$$

By Hölder inequality, (8), the facts that $\phi_R \in C^\infty(\mathbb{R}^N)$, a is continuous in $\overline{\mathfrak{O}}$ and $(w_n)_n$ is bounded in $W_0^{1,\mathcal{E}}(\mathfrak{O})$, we get

$$\begin{aligned} &\int_{\mathfrak{O}} (|\nabla w_n|^{\varrho-2}\nabla w_n + \kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n)\nabla\phi_R(w_n - w)dz \\ &\qquad \leqslant C(\|\nabla w_n\|_\varrho^{\varrho-1}\|w_n - w\|_\varrho + \|\nabla w_n\|_{\tau,\kappa}^{\tau-1}\|w_n - w\|_{\tau,\kappa}) \\ &\qquad \leqslant \overline{C}(\|w_n - w\|_\varrho + \|w_n - w\|_\tau) \to 0, \end{aligned} \tag{12}$$

as $n \to \infty$, for suitable C, $\overline{C}$. Furthermore, by (8) and [1, Proposition A.8], considering that $\kappa > 0$ in $\mathfrak{O} \setminus A$, we have

$$|w_n|^{\varrho-2}w_n \rightharpoonup |w|^{\varrho-2}w \text{ in } L^\varrho(\mathfrak{O}, |z|^{-\varrho}), \; |w_n|^{\tau-2}w_n \rightharpoonup |w|^{\tau-2}w \text{ in } L^{\tau'}(\mathfrak{O}\backslash\mathcal{A}, \kappa(z)|z|^{-\tau})$$

so that

$$\lim_{n\to\infty} \int_{\mathfrak{O}} \phi_R \frac{|w_n|^{\varrho-2}w_n}{|z|^\varrho} w dz = \int_{\mathfrak{O}} \phi_R \frac{|w|^\varrho}{|z|^\varrho} dz, \tag{13}$$

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathfrak{O}} \phi_R\kappa(z)\frac{|w_n|^{\tau-2}w_n}{|z|^\tau} w dz &= \lim_{n\to\infty} \int_{\mathfrak{O}\backslash\mathcal{A}} \phi_R\kappa(z)\frac{|w_n|^{\tau-2}w_n}{|z|^\tau} w dz \\ &= \int_{\mathfrak{O}\backslash\mathcal{A}} \phi_R\kappa(z)\frac{|w|^\tau}{|z|^\tau} dz = \int_{\mathfrak{O}} \phi_R\kappa(z)\frac{|w|^\tau}{|z|^\tau} dz. \end{aligned} \tag{14}$$

While, by (8) it follows that

$$\phi_R(z)\frac{|w_n(z)|^\varrho}{|z|^\varrho} \leqslant \Big(\frac{2}{\varrho}\Big)^\varrho |w_n(z)|^\varrho \leqslant \Big(\frac{2}{\varrho}\Big)^\varrho g^\varrho(z) \text{ a.e in } \mathfrak{O}\backslash B(0,R/2).$$

Therefore, given that $\phi_R \equiv 0$ in $B(0,R/2)$, the Dominated Convergence Theorem yields

$$\lim_{n\to\infty}\int_{\mathfrak{O}}\phi_R\frac{|w_n|^{\varrho}}{|z|^{\varrho}}dz=\lim_{n\to\infty}\int_{\mathfrak{O}\setminus B(0,R/2)}\phi_R\frac{|w_n|^{\varrho}}{|z|^{\varrho}}dz$$
$$=\int_{\mathfrak{O}\setminus B(0,R/2)}\phi_R\frac{|w|^{\varrho}}{|z|^{\varrho}}dz=\int_{\mathfrak{O}}\phi_R\frac{|w|^{\varrho}}{|z|^{\varrho}}dz. \quad (15)$$

Likewise, employing (1) as well, we obtain, for a suitable constant $K>0$

$$\phi_R(z)\kappa(z)\frac{|w_n(z)|^{\tau}}{|z|^{\tau}}\leqslant K\Big(\frac{2}{\tau}\Big)^{\tau}g^{\tau}(z) \text{ a.e in } \mathfrak{O}\setminus B(0,R/2),$$

which yields joint with the dominated convergence theorem

$$\lim_{n\to\infty}\int_{\mathfrak{O}}\phi_R\kappa(z)\frac{|w_n|^{\tau}}{|z|^{\tau}}dz=\int_{\mathfrak{O}}\phi_R\kappa(z)\frac{|w|^{\tau}}{|z|^{\tau}}dz. \quad (16)$$

Thus, by (7),(10), (12)-(16), we obtain

$$\lim_{n\to\infty}\int_{\mathfrak{O}}\phi_R\big(|\nabla w_n|^{\varrho-2}\nabla w_n+\kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n\big)(\nabla w_n-\nabla w)dz=0.$$

Applying Hölder's inequality and considering $\phi_R\leqslant 1$, we observe that the functional

$$G\,:\,h\in[L^{\mathcal{E}}(\mathfrak{O})]^N\mapsto\int_{\mathfrak{O}}\phi_R\big(|\nabla w|^{\varrho-2}\nabla w+\kappa(z)|\nabla w|^{\tau-2}\nabla w\big)hdz$$

is linear and bounded. Hence, by (8) we get

$$\lim_{n\to\infty}\int_{\mathfrak{O}}\phi_R\big(|\nabla w|^{\varrho-2}\nabla w+\kappa(z)|\nabla w|^{\tau-2}\nabla w\big)(\nabla w_n-\nabla w)dz=0.$$

Thus, defining $\mathfrak{O}_R:=\Big\{z\in\mathfrak{O}:\ |z|>R\Big\}$ for any $R>0$, we obtain

$$\lim_{n\to\infty}\int_{\mathfrak{O}_R}\big[|\nabla w_n|^{\varrho-2}\nabla w_n-|\nabla w|^{\varrho-2}\nabla w+\kappa(z)(|\nabla w_n|^{\tau-2}\nabla w_n-|\nabla w|^{\tau-2}\nabla w)\big]$$
$$\times(\nabla w_n-\nabla w)dz$$
$$\leqslant\lim_{n\to\infty}\int_{\mathfrak{O}}\phi_R\big[|\nabla w_n|^{\varrho-2}\nabla w_n-|\nabla w|^{\varrho-2}\nabla w+\kappa(z)(|\nabla w_n|^{\tau-2}\nabla w_n-|\nabla w|^{\tau-2}\nabla w)\big]$$
$$\times(\nabla w_n-\nabla w)dz=0. \quad (17)$$

Given that $\phi_R\equiv 1$ in $\mathbb{R}^N\backslash B(0,R)$. We can deduce, for $\varrho\geq 2$, utilizing (6), that

$$\int_{\mathfrak{O}_R}|\nabla w_n-\nabla w|^{\varrho}dz\leq M_{\varrho}\int_{\mathfrak{O}_R}(|\nabla w_n|^{\varrho-2}\nabla w_n-|\nabla w|^{\varrho-2}\nabla w)(\nabla w_n-\nabla w)dz, \quad (18)$$

with $M_{\varrho}>0$ a suitable constant.
On the other hand, for $1<\varrho<2$, using (6) and the Hölder inequality, we derive

$$\begin{aligned}\int_{\mathfrak{O}_R} |\nabla w_n - \nabla w|^{\varrho} dz &\leqslant M_{\varrho} \int_{\mathfrak{O}_R} \left[(|\nabla w_n|^{\varrho-2}\nabla w_n - |\nabla w|^{\varrho-2}\nabla w)(\nabla w_n - \nabla w)\right]^{\varrho/2} \\ &\qquad\qquad\qquad \times (|\nabla w_n|^{\varrho} + |\nabla w|^{\varrho})^{(2-\varrho)/2} dz \\ &\leqslant M_{\varrho}\Big[\int_{\mathfrak{O}_R} (|\nabla w_n|^{\varrho-2}\nabla w_n - |\nabla w|^{\varrho-2}\nabla w)(\nabla w_n - \nabla w) dz\Big]^{\varrho/2} \\ &\qquad\qquad\qquad \times (\|\nabla w_n\|_{\varrho}^{\varrho} + \|\nabla w\|_{\varrho}^{\varrho})^{(2-\varrho)/2} \\ &\leqslant \overline{M_{\varrho}}\Big[\int_{\mathfrak{O}_R} (|\nabla w_n|^{\varrho-2}\nabla w_n - |\nabla w|^{\varrho-2}\nabla w)(\nabla w_n - \nabla w) dz\Big]^{\varrho/2},\end{aligned}$$

where the last inequality follows by the boundedness of $(w_n)_n$ in $W_0^{1,\mathcal{E}}(\mathfrak{O})$ and Proposition 2.1 with a suitable new $\overline{M_{\varrho}} > 0$. Also, by convexity and since $\kappa(z) \geq 0$ a.e. in $\mathfrak{O}$ by (1), we have

$$\kappa(z)(|\nabla w_n|^{\tau-2}\nabla w_n - |\nabla w|^{\tau-2}\nabla w)(\nabla w_n - \nabla w) \geq 0 \quad \text{a.e. in } \mathfrak{O}. \tag{19}$$

Thus, combining (17), (18)-(19) we prove that $\nabla w_n \to \nabla w$ in $\left[L^{\varrho}(\mathfrak{O}_R)\right]^N$ as $n \to \infty$, whenever $R > 0$ satisfies (11). However, when $\overline{\mathfrak{O}} \subset B(0, R/2)$ we have $\mathfrak{O}_R = \emptyset$. Thus, for any $R > 0$ the sequence $\nabla w_n \to \nabla w$ in $\left[L^{\varrho}(\mathfrak{O}_R)\right]^N$ as $n \to \infty$, and by diagonalization we prove claim (9).
Since the sequence $\left(|\nabla w_n|^{\varrho-2}\nabla w_n\right)_n$ is bounded in $L^{\varrho'}(\mathfrak{O})$, by (9) we get

$$\lim_{n\to\infty} \int_{\mathfrak{O}} |\nabla w_n|^{\varrho-2}\nabla w_n \nabla w dz = \|\nabla w\|_{\varrho}^{\varrho}. \tag{20}$$

On the other hand, as $\left(|\nabla w_n|^{\tau-2}\nabla w_n\right)_n$ is bounded in $L^{\tau'}(\mathfrak{O} \setminus \mathcal{A}, \kappa(z))$, in light of (9) and [1, Proposition A.8], we have

$$\begin{aligned}\lim_{n\to\infty} \int_{\mathfrak{O}} \kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n \nabla w dz &= \lim_{n\to\infty} \int_{\mathfrak{O}\setminus\mathcal{A}} \kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n \nabla w dz \\ &= \|\nabla w\|_{\tau,\kappa}^{\tau}.\end{aligned} \tag{21}$$

Similarly, by reasoning as in (13), we can establish

$$\lim_{n\to\infty} \int_{\mathfrak{O}} \Big(\frac{|w_n|^{\varrho-2}w_n}{|z|^{\varrho}} w + \kappa(z)\frac{|w_n|^{\tau-2}w_n}{|z|^{\tau}} w\Big) dz = \|w\|_{\mathsf{C}_{\varrho}}^{\varrho} + \|w\|_{\mathsf{C}_{\tau,\kappa}}^{\tau}. \tag{22}$$

Furthermore, using (8), (9) and the Brézis-Lieb Lemma in [6, Theorem 1], we obtain

$$\begin{cases} \|\nabla w_n\|_{\varrho}^{\varrho} - \|\nabla w_n - \nabla w\|_{\varrho}^{\varrho} = \|\nabla w\|_{\varrho}^{\varrho} + o(1), \\ \|\nabla w_n\|_{\tau,\kappa}^{\tau} - \|\nabla w_n - \nabla w\|_{\tau,\kappa}^{\tau} = \|\nabla w\|_{\tau,\kappa}^{\tau} + o(1), \\ \|w_n\|_{\mathsf{C}_{\varrho}}^{\varrho} - \|w_n - w\|_{\mathsf{C}_{\varrho}}^{\varrho} = \|w\|_{\mathsf{C}_{\varrho}}^{\varrho} + o(1), \\ \|w_n\|_{\mathsf{C}_{\tau,\kappa}}^{\tau} - \|w_n - w\|_{\mathsf{C}_{\tau,\kappa}}^{\tau} = \|w\|_{\mathsf{C}_{\tau,\kappa}}^{\tau} + o(1), \end{cases} \tag{23}$$

as $n \to \infty$. Thus, by (7), (20)-(22), we get

$$\begin{aligned}o(1) = \langle \mathcal{D}w_n, w_n - w\rangle &= \int_{\mathfrak{O}} \Big(|\nabla w_n|^{\varrho-2}\nabla w_n + \kappa(z)|\nabla w_n|^{\tau-2}\nabla w_n\Big)(\nabla w_n - \nabla w) dz \\ &\quad - \int_{\mathfrak{O}} \Big(\frac{|w_n|^{\varrho-2}w_n}{|z|^{\varrho}} + \kappa(z)\frac{|w_n|^{\tau-2}w_n}{|z|^{\tau}}\Big)(w_n - w) dz\end{aligned}$$

$$= \|\nabla w_n\|_\varrho^\varrho - \|\nabla w\|_\varrho^\varrho + \|\nabla w_n\|_{\tau,\kappa}^\tau - \|\nabla w\|_{\tau,\kappa}^\tau - \Big(\|w_n\|_{\mathsf{C}_\varrho}^\varrho - \|w\|_{\mathsf{C}_\varrho}^\varrho + \|w_n\|_{\mathsf{C}_{\tau,\kappa}}^\tau - \|w\|_{\mathsf{C}_{\tau,\kappa}}^\tau\Big) + o(1)$$

as $n \to \infty$. Hence, by (23) it follows that

$$\|\nabla w_n - \nabla w\|_\varrho^\varrho + \|\nabla w_n - \nabla w\|_{\tau,\kappa}^\tau = \Big(\|w_n - w\|_{\mathsf{C}_\varrho}^\varrho + \|w_n - w\|_{\mathsf{C}_{\tau,\kappa}}^\tau\Big) + o(1) = \ell + o(1) \quad (24)$$

as $n \to \infty$. Now, assume for contradiction that $\ell > 0$. Then, from Lemma 2.3 and (23), we have

$$\begin{aligned}\lim_{n\to\infty} \|\nabla w_n - \nabla w\|_\varrho^\varrho + \lim_{n\to\infty} \|\nabla w_n - \nabla w\|_{\tau,\kappa}^\tau &\leqslant \lim_{n\to\infty} \|w_n - w\|_{\mathsf{C}_\varrho}^\varrho + \lim_{n\to\infty} \|w_n - w\|_{\mathsf{C}_{\tau,\kappa}}^\tau \\ &< \min\{\mathsf{C}_\varrho,\ \mathsf{C}_\tau\}\Big(\lim_{n\to\infty} \|w_n - w\|_{\mathsf{C}_\varrho}^\varrho + \lim_{n\to\infty} \|w_n - w\|_{\mathsf{C}_{\tau,\kappa}}^\tau\Big) \\ &\leqslant \lim_{n\to\infty} \|\nabla w_n - \nabla w\|_\varrho^\varrho + \lim_{n\to\infty} \|\nabla w_n - \nabla w\|_{\tau,\kappa}^\tau\end{aligned}$$

which is impossible. Therefore $\ell = 0$, so that by (24) we have $\nabla w_n \to \nabla w$ in $\big[L^\varrho(\mathfrak{O}) \cap L^\tau(\mathfrak{O}, \kappa)\big]^N$ as $n \to \infty$, implying that $w_n \to w$ in $W_0^{1,\mathcal{E}}(\mathfrak{O})$ thanks to (1) and Proposition 2.1. ☐

Lemma 3.2. *Under the condition* $(\mathcal{H}_\varphi)$, *the operator* $\mathcal{N} : \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \to \big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\big)^*$ *given in* (2) *is compact.*

Proof. We decompose the proof into three distinct steps.
First step: Let's introduce an operator $\mathcal{F} : \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \to L^{s'}(\mathfrak{O})$ defined as

$$\mathcal{F}w := -|w|^{\varrho-2}w + \kappa(z)|w|^{\tau-2}w \text{ for } \quad w \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \quad \text{and} \quad z \in \mathfrak{O}.$$

It is evident that $\mathcal{F}$ is continuous. We will now show that $\mathcal{F}$ is also bounded. Let $w \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$. Combine (1), Hölder's inequality and Proposition 2.2 to get

$$\begin{aligned}\|\mathcal{F}w\|_{s'}^{s'} &\leqslant 2^{s'-1} \int_\mathfrak{O} \big(|w|^{(\varrho-1)s'} + |\kappa^{s'}||w|^{(\tau-1)s'}\big)dz \\ &\leqslant 2^{s'-1}|\mathfrak{O}|^{\frac{\varrho'-s'}{\varrho'}}\Big(\int_\mathfrak{O} |w|^\varrho dz\Big)^{\frac{s'}{\varrho'}} + 2^{s'-1}|\mathfrak{O}|^{\frac{\tau'-s'}{\tau'}}\|\kappa^{s'}\Big(\int_\mathfrak{O} |w|^\tau dz\Big)^{\frac{s'}{\tau'}} \\ &\leqslant 2^{s'-1}|\mathfrak{O}|^{\frac{\varrho'-s'}{\varrho'}}\|w\|_\varrho^{(\varrho-1)s'} + 2^{s'-1}|\mathfrak{O}|^{\frac{\tau'-s'}{\tau'}}\|\kappa^{s'}\|_\infty\|w\|_\tau^{(\tau-1)s'} \\ &\leqslant C_1\|w\|^{(\varrho-1)s'} + C_2\|w\|^{(\tau-1)s'}.\end{aligned}$$

This implies that $\mathcal{F}$ is bounded on $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$.
Second step: Let $\phi : \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \to L^{\frac{s}{s-1}}(\mathfrak{O})$ be an operator defined as

$$\Phi w = -\varphi(z, w, \nabla w), \qquad \text{for any } w \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}).$$

We proceed to demonstrate that Φ is both bounded and continuous.
To establish boundedness, consider any $w \in \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$. By property $(\mathcal{H}_\varphi)$, we have

$$\|\Phi w\|_{s'}^{s'} = \int_\mathfrak{O} |\varphi(z, w(z), \nabla w(z))|^{s'} dz \leqslant C\big(\|\varepsilon\|_{s'}^{s'} + \|w\|_s^s + \|\nabla w\|_\varrho^\varrho\big), \qquad (25)$$

where $s' = \frac{s}{s-1}$. The continuous embedding $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \hookrightarrow L^s(\mathfrak{O})$ $(1 < s < \varrho < \varrho^*)$, $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \subset W_0^{1,\varrho}(\mathfrak{O})$ and (25) imply the estimate

$$|\Phi u|_{s'}^{s'} \leqslant C\big(|\varepsilon|_{s'}^{s'} + \|w\|^s + \|u\|^{\varrho}\big). \tag{26}$$

This shows that Φ is bounded on $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$.
To establish continuity, assume $w_n \to w$ in $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$. Then, $w_n \to w$ in $L^{\varrho}(\mathfrak{O})$ and $\nabla w_n \to \nabla w$ in $\big(L^{\varrho}(\mathfrak{O})\big)^N$. Consequently, there exists a subsequence $\{wn_k\}_{k=1}^{\infty}$ of $\{w_n\}_{n=1}^{\infty}$ and measurable functions $g \in L^{\varrho}(\mathfrak{O})$ and $h \in \big(L^{\varrho}(\mathfrak{O})\big)^N$ such that

$$\begin{cases} w_{n_k}(z) \to w(z) \text{ and } \nabla w_{n_k}(z) \to \nabla w(z), \text{ a.e. } z \in \mathfrak{O}, \text{ as } k \to +\infty; \\ \\ |w_{n_k}(z)| \leqslant g(z) \text{ and } |\nabla w_{n_k}(z)| \leqslant |h(z)|, \text{ a.e. } z \in \mathfrak{O} \text{ and all } k \in N. \end{cases} \tag{27}$$

Given that φ is Carathéodory function, we obtain

$$\varphi(z, w_{n_k}(z), \nabla w_{n_k}(z)) \to \varphi(z, w, \nabla w(z)), \text{ a.e. } z \in \mathfrak{O}, \text{ as } k \to +\infty. \tag{28}$$

Moreover, by (27), we get

$$|\varphi(z, w_{n_k}(z), \nabla w_{n_k}(z))| \leqslant C\big(\varepsilon(z) + |w(z)|^{s-1} + |\nabla w(z)|^{\varrho\frac{s-1}{s}}\big), \tag{29}$$

for all $k \in \mathbb{N}$ and a.e. $x \in \Omega$. Noting that $\varepsilon + |w|^{s-1} + |\nabla w|^{\frac{\varrho-1}{s}} \in L^{s'}(\mathfrak{O})$. Hence, by (28), (29), and the Dominated Convergence Theorem, we obtain that

$$\int_{\mathfrak{O}} \big|\varphi(z, w_{n_k}(z), \nabla w_{n_k}(z)) - \varphi(z, w, \nabla w(z))\big|^{s'} dz \to 0, \qquad \text{as } k \to +\infty;$$

that is

$$\|\Phi u_{n_k} - \Phi u\|_{s'} \to 0, \qquad \text{as } k \to +\infty.$$

Thus, the entire sequence Φw_n converges to Φu in $L^{s'}(\mathfrak{O})$.
Third step: We now demonstrate that the operator $\mathcal{N} : \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \to \big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\big)^*$ is compact. Recall that the embedding $i : \mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O}) \to L^s(\mathfrak{O})$ is compact. Therefore, we have that the adjoint operator $i^* : L^{s'}(\mathfrak{O}) \to \big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\big)^*$ is also compact. Hence, the compositions $i^* \circ \mathcal{F}$ and $i^* \circ \Phi$ are compact. This implies $\mathcal{N} = i^* \circ \mathcal{F} + i^* \circ \Phi$ is compact. The proof is complete. □

Now, let's establish the proof of Theorem 1.1.
Depending on the properties of the operator $\mathcal{D}$ as described in Proposition 3.1, and applying the Minty-Browder's Theorem on monotone operators (as in [21, Theorem 26 A]), we can conclude that the inverse operator $\mathcal{T} := \mathcal{D}^{-1}$, which maps from $\big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\big)^*$ to $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})$, is continuous, of type (S_+), and bounded. Additionally, by Lemma 3.2, the operator $\mathcal{N}$ is is bounded, continuous and quasimonotone.
Therefore, the equation (4) is equivalent to the abstract Hammerstein equation:

$$w = \mathcal{T}\vartheta \quad \text{and} \quad \vartheta + \mathcal{N} \circ \mathcal{T}\vartheta = 0. \tag{30}$$

We will use the theory of degrees introduced in section 3 to solve the above equation (30). To do this, we first establish the following lemma

Lemma 3.3. *The set*

$$\mathcal{G} := \Big\{\vartheta \in \big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{O})\big)^* \text{ such that } \vartheta + t\mathcal{N} \circ \mathcal{T}\vartheta = 0 \quad \text{for some} \quad t \in [0,1]\Big\}$$

is bounded.

Proof. Let $\vartheta \in \mathcal{G}$ and take $w = \mathcal{T}\vartheta$, then $\|\mathcal{T}\vartheta\| = \|\nabla w\|_{\mathcal{E}}$. We divide into two cases:
(*i*) If $\|\nabla w\|_{\mathcal{E}} \leqslant 1$, then $\|\mathcal{T}\vartheta\|$ is bounded.
(*ii*) If $\|\nabla w\|_{\mathcal{E}} > 1$, by Proposition 2.1, Proposition 2.2, Lemma 2.3, the condition $(\mathcal{H}_{\varphi})$, the Hölder inequality, and the Young inequality, then we get

$$\begin{aligned}
\|\mathcal{T}\vartheta\|^{\varrho} = \|w\|^{\varrho} &= \int_{\mathfrak{D}} \Big(|\nabla w|^{\varrho} + \kappa(z)|\nabla w|^{\tau}\Big)dz \\
&\leqslant \int_{\mathfrak{D}} \Big(|w|^{\varrho} + \kappa(z)|w|^{\tau}\Big)dz + \int_{\mathfrak{D}} \Big(\frac{|w|^{\varrho}}{|z|^{\varrho}} + \kappa(z)\frac{|w|^{\tau}}{|z|^{\tau}}\Big)dz + \int_{\mathfrak{D}} |\varphi(z, w, \nabla w) \cdot w| dz \\
&\leqslant \|w\|^{\tau} + \|w\|_{\mathcal{E}}^{\beta} + C_3 \int_{\Omega} \big(|\varepsilon(z)w(z)| + |w(z)|^{s} + |\nabla w(z)|^{\varrho\frac{s-1}{s}}|w|\big)dz \\
&\leqslant \|w\|^{\tau} + C_4\|w\|^{\beta} + C_3\Big(\|\varepsilon\|_{s'}\|w\|_{s} + \|w\|_{s}^{s} + \frac{\sigma}{s'}\|\nabla w\|_{\varrho}^{\varrho} + \frac{1}{s\sigma^{\frac{s}{s'}}}\|w\|_{s}^{s}\Big),
\end{aligned}$$

with $\beta = \varrho$ if $\|w\|_{\mathcal{E}} < 1$ and $\beta = \tau$ if $\|w\|_{\mathcal{E}} > 1$.
Thus, using the continuous embedding $\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{D}) \hookrightarrow L^{s}(\mathfrak{D})$, we deduce that

$$\|\mathcal{T}\vartheta\|^{\varrho} \leqslant \|w\|^{\tau} + C_4\|w\|^{\beta} + C_3\big(\|\varepsilon\|_{s'}\|w\| + \|w\|^{s} + \frac{1}{s\sigma^{\frac{s}{s'}}}\|w\|^{s}\big).$$

At this time, we can choose $\sigma > 0$ small enough, such that $\frac{C_3\sigma}{s'} < \frac{1}{2}$. Therefore, we obtain that

$$\|\mathcal{T}\vartheta\|^{\varrho} \leqslant C_{max}\big(\|\mathcal{T}\vartheta\|^{\tau} + C_4\|w\|^{\beta} + \|\varepsilon\|_{s'}\|\mathcal{T}\vartheta\| + \|\mathcal{T}\vartheta\|^{s} + \frac{1}{s\sigma^{\frac{s}{s'}}}\|\mathcal{T}\vartheta\|^{s}\big).$$

Note that $1 < s < \varrho$, and so $\|\mathcal{T}\vartheta\|$ is bounded. This proves that $\big\{\mathcal{T}\vartheta \mid \vartheta \in \mathcal{G}\big\}$ is bounded.
As $\mathcal{N}$ is bounded, and based on (30), we conclude that the set $\mathcal{G}$ is bounded in $\big(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{D})\big)^{*}$. □

Thanks to Lemma 3.3, we can determine a positive constant R such that

$$\|\vartheta\|_{\left(\mathcal{W}_0^{1,\mathcal{E}}(\mathfrak{D})\right)^{*}} < R, \qquad \text{for any } \vartheta \in \mathcal{G}.$$

As a result

$$\vartheta + t\mathcal{N} \circ \mathcal{T}\vartheta \neq 0 \qquad \text{for every } \vartheta \in \partial B_R(0) \quad \text{and each } t \in [0,1].$$

By applying Lemma 2.4, we establish that

$$\mathcal{I} + \mathcal{N} \circ \mathcal{T} \in \mathcal{B}_{\mathcal{T}}(\overline{B_R(0)}) \quad \text{ and } \quad \mathcal{I} = \mathcal{D} \circ \mathcal{T} \in \mathcal{B}_{\mathcal{T}}(\overline{B_R(0)}).$$

Since the operators $\mathcal{I}$, $\mathcal{N}$ and $\mathcal{T}$ are bounded, then $\mathcal{I} + \mathcal{N} \circ \mathcal{T}$ is also bounded. This says that

$$\mathcal{I} + \mathcal{N} \circ \mathcal{T} \in \mathcal{B}_{\mathcal{T},B}(\overline{B_R(0)}) \quad \text{ and } \quad \mathcal{I} \in \mathcal{B}_{\mathcal{T},B}(\overline{B_R(0)}).$$

Now, we can introduce the affine homotopy $\mathcal{H} : [0,1] \times \overline{B_R(0)} \to \mathcal{W}^{*}$ setting by

$$\mathcal{H}(t,\vartheta) := (1-t)\mathcal{I}\vartheta + t\big(\mathcal{I} + \mathcal{N} \circ \mathcal{T}\big)\vartheta \quad \text{ for } \quad (t,\vartheta) \in [0,1] \times \overline{B_R(0)}.$$

By means the properties of the degree, as established in Theorem 2.5, we conclude that

$$\delta(\mathcal{I} + \mathcal{N} \circ \mathcal{T}, B_R(0), 0) = \delta(\mathcal{I}, B_R(0), 0) = 1.$$

Hence, there exists a function $\vartheta \in B_R(0)$ such that

$$\vartheta + \mathcal{N} \circ \mathcal{T}\vartheta = 0.$$

Implying that $w = \mathcal{T}\vartheta$ is a weak solution of $(\mathcal{P})$. This concludes the proof.

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